STATIC KLEIN-GORDON-MAXWELL-PROCA SYSTEMS IN 4-DIMENSIONAL CLOSED MANIFOLDS. II

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Abstract. We discuss static Klein-Gordon-Maxwell-Proca systems in the critical case of 4-dimensional closed manifolds in the continuation of Hebey-Truong [10]. We prove phase stability for all possible phases in the nonpositively curved case, and in the positively curved case when the phase lies in some set that we show to be maximal in the case of the round sphere.

We investigate in this paper phase stability for the electrostatic Klein-Gordon-Maxwell-Proca system in 4-dimensional closed manifolds. The full Klein-Gordon-Maxwell-Proca system (in short KGMP) is a massive version of the more traditional Klein-Gordon-Maxwell system. It provides a dualistic model for the description of the interaction between a charged relativistic matter scalar field and the electromagnetic field that it generates. The external vector field (φ, A) in the system inherits a mass and is governed by the Proca action which generalizes that of Maxwell. Let (M, g) be a closed 4-dimensional Riemannian manifold. Writing the matter scalar field in polar form as $\psi(x,t) = u(x,t)e^{iS(x,t)}$, choosing the nonlinearity in the model to be pure and critical in terms of Sobolev embeddings, the full Klein-Gordon-Maxwell-Proca system is written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^3 + \left(\left(\frac{\partial S}{\partial t} + q\varphi \right)^2 - |\nabla S - qA|^2 \right) u \\ \frac{\partial}{\partial t} \left(\left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 \right) - \nabla . \left(\left(\nabla S - qA \right) u^2 \right) = 0 \\ -\nabla . \left(\frac{\partial A}{\partial t} + \nabla\varphi \right) + m_1^2 \varphi + q \left(\frac{\partial S}{\partial t} + q\varphi \right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial t} + \nabla\varphi \right) + m_1^2 A = q \left(\nabla S - qA \right) u^2 , \end{cases} \tag{0.1}$$

where $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator, $\overline{\Delta}_g = \delta d$ is half the Laplacian acting on forms, and δ is the codifferential. In its electrostatic form we assume A and φ do not depend on the time variable. Looking for standing waves solutions $\psi(x,t) = u(x)e^{-i\omega t}$, letting $\varphi = \omega v$, there necessarily holds that $A \equiv 0$ and the system reduces to the two following critical equations

$$\begin{cases} \Delta_g u + m_0^2 u = u^3 + \omega^2 (qv - 1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2 u^2 \right) v = q u^2 . \end{cases} \tag{0.2}$$

In the above, $m_0, m_1 > 0$ are masses (m_0 is the mass of the particle, m_1 is the Proca mass), and $q > 0$ is the electric charge of the particle. The Proca formalism comes with the assumption $m_1 > 0$. The system (0.2), in Proca form in closed manifolds, has been investigated in Druet and Hebey [5] and Hebey and Wei [11] in the case of 3-dimensional manifolds, and Hebey and Truong [10] in the critical dimension $n = 4$ (4 is the dimension for which the second equation in (0.2) is also

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critical by the u^2v term). A sequence $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$ of standing waves is said to have finite energy if $\|u_\alpha\|_{H^1} = O(1)$. We keep here the origin of the writing in polar form and always assume in the sequel that the amplitude u of a standing wave $ue^{i\omega t}$ is nonnegative. A priori bounds were proved in Hebey and Truong [10]. We aim here in proving phase stability with the objective of getting more phases and more geometries by relaxing the a priori bound compactness property in Hebey and Truong [10]. Given a sequence $(\omega_\alpha)_\alpha$ of phases, we consider the family of equations

$$\begin{cases}\Delta_g u + m_0^2 u = u^3 + \omega_\alpha^2 (qv-1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2u^2\right) v = qu^2\end{cases} \tag{0.3}$$

We define phase stability as follows.

Definition 0.1. *A phase $\omega \in (-m_0, +m_0)$ is said to be stable if for any sequence of phases $(\omega_\alpha)_\alpha$ converging to ω, any sequence $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$ of finite energy standing waves, $u_\alpha \geq 0$, and any sequence $(v_\alpha)_\alpha$ of gauge potentials, solutions of* (0.3) *for all α, there holds that, up to a subsequence, $u_\alpha \to u$ and $v_\alpha \to v$ in C^2 as $\alpha \to +\infty$, where u and v solve* (0.2) *with phase ω.*

When a phase ω is stable, (0.2) is automatically compact in the finite energy setting. But phase stability implies more and actually measures how much (0.2) is robust with respect to perturbations of ω. We let $\mathrm{Rg}(S_g)$ be the range interval

$$\mathrm{Rg}(S_g) = \left[\min_M S_g, \max_M S_g\right] , \tag{0.4}$$

where S_g is the scalar curvature of g. The first result of this paper,Theorem 0.1, establishes that phases are stable as long as they do not enter in resonance with the ambient geometry through the condition $m_0^2 - \omega^2 \notin \frac{1}{6}\mathrm{Rg}(S_g)$. The second result of the paper, Theorem 0.2, establishes that this condition is sharp.

Theorem 0.1. *Let (M,g) be a closed 4-dimensional manifold, and $m_0, m_1, q > 0$ be positive real numbers. Any phase $\omega \in (-m_0, +m_0)$ such that $m_0^2 - \omega^2 \notin \frac{1}{6}Rg(S_g)$ is a stable phase for* (0.2)*, where $Rg(S_g)$ is given by* (0.4)*. In particular, if g has nonpositive scalar curvature, then any phase $\omega \in (-m_0, +m_0)$ is stable.*

A priori bounds were established in Hebey and Truong [10] under the condition that m_0 and ω satisfy that $m_0^2 - \omega^2 < \frac{1}{6}\min_M S_g$, and thus that $m_0^2 - \omega^2$ sits on the left part of $\mathbb{R}^+ \backslash \frac{1}{6}\mathrm{Rg}(S_g)$. While a priori bounds are stronger than phase stability, the condition required to obtain such bounds is quite restrictive. It does not cover the full phases ω by missing the right part of $\mathbb{R}^+ \backslash \frac{1}{6}\mathrm{Rg}(S_g)$, and it implies that the background space has positive scalar curvature, which is not required by Theorem 0.1. Basically, Theorem 0.1 establishes that by relaxing the a priori bound property to the stable phase property, we gain more phases and more geometries with respect to what was handled in Hebey and Truong [10]. Theorem 0.1 coupled with Theorem 0.3 in Hebey and Truong [10] provide a complete description of the various compactness properties one can associate to (0.2). Theorem 0.2 below complements the above picture by showing that both Theorem 0.1 and Theorem 0.3 in Hebey and Truong [10] are optimal.

The second part of Theorem 0.1 obviously includes the model cases of flat torii $(\mathbb{T}^4, g)$ and compact hyperbolic spaces $(\mathbb{H}^4, g)$. Any phase $\omega \in (-m_0, +m_0)$ is stable for these manifolds. The model case of the round sphere $(\mathbb{S}^4, g)$ is contained in the first part of Theorem 0.1. In the case of the round sphere $(\mathbb{S}^4, g)$, $S_g = 12$ and

Theorem 0.1 asserts that any phase $\omega \in (-m_0, +m_0)$ is stable except, possibly, if $m_0 \geq \sqrt{2}$ and ω is one of the two solutions of $m_0^2 - \omega^2 \neq 2$. Theorem 0.2 asserts that these two solutions are indeed unstable phases, and thus that the set in Theorem 0.1 is maximal in the case of the round sphere.

Theorem 0.2. *Let $(\mathbb{S}^4, g)$ be the round 4-sphere, and $m_0, m_1, q > 0$ be positive real numbers with $m_0^2 \geq 2$. Let $\omega \in \mathbb{R}$ be such that $m_0^2 - \omega^2 = 2$. Then ω is an unstable phase for (0.2) which gives rise to resonant states in the sense that for such ω's there exist a sequence $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$, $u_\alpha \geq 0$, of standing waves, and a sequence $(v_\alpha)_\alpha$ of gauge potentials, solving (0.3) for all α, such that $\omega_\alpha \to \omega$ and $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.*

Theorems 0.1 and 0.2 complement each other. They are distinct in nature. Theorem 0.1 has to do with a priori estimates. Theorem 0.2 has to do with constructive approaches. They are proved by using two different methods. Theorem 0.1 is proved by passing through the C^0-theory for blow-up, the analysis of the range of influence, and then by getting a contradiction through a Pohozaev type identity. Theorem 0.2 is proved by going into the finite dimensional Lyapounov-Schmidt reduction method. We prove Theorem 0.1 in Sections 1 to 3. We prove Theorem 0.2 in Section 4. We refer to Hebey and Truong [10] for the physics origin of the problem and the building of the equations.

1. A priori L^∞-estimates

We let (M, g) be a smooth compact Riemannian 4-manifold, and $m_0, m_1, q > 0$ be positive real numbers. Following a very nice idea going back to Benci and Fortunato [1], we introduce the map $\Phi : H^1 \to H^1$ solution of

$$\Delta_g \Phi(u) + \left(m_1^2 + q^2 u^2\right) \Phi(u) = q u^2 \tag{1.1}$$

for all $u \in H^1$. We refer to Hebey and Truong [10], see also Section 4, for the existence and regularity of the map Φ in the critical dimension $n = 4$, in the Riemannian context. We let $\omega \in (-m_0, +m_0)$, $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$ be an arbitrary sequence of finite energy standing waves, $u_\alpha \geq 0$, and $(v_\alpha)_\alpha$ be an arbitrary sequence of gauge potentials, satisfying that

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^3 + \omega_\alpha^2 \left(q v_\alpha - 1\right)^2 u_\alpha \\ \Delta_g v_\alpha + \left(m_1^2 + q^2 u_\alpha^2\right) v_\alpha = q u_\alpha^2 \ . \end{cases} \tag{1.2}$$

for all α, and that $\omega_\alpha \to \omega$ in $\mathbb{R}$ as $\alpha \to +\infty$. Obviously, $v_\alpha = \Phi(u_\alpha)$, and by the maximum principle, there holds that $0 \leq v_\alpha \leq \frac{1}{q}$ in M for all α. We aim in proving that, passing to a subsequence, the u_α's and v_α's converge in C^2 as $\alpha \to \infty$. By standard elliptic regularity theory, it suffices to prove that, passing to a subsequence, $\|u_\alpha\|_{L^\infty} = O(1)$. We proceed by contradiction and assume that

$$\|u_\alpha\|_{L^\infty} \to +\infty \ . \tag{1.3}$$

The second equation in (1.2) is critical (of maximal homogeneity 3 in 4-space dimension) and it acts as an auto-inductive perturbation (the perturbation depends on the solution itself) of the first nonlinear Schrödinger equation through its potential term

$$\begin{aligned} h_\alpha &= m_0^2 - \omega_\alpha^2 \left(q v_\alpha - 1\right)^2 \\ &= m_0^2 - \omega_\alpha^2 \left(q \Phi(u_\alpha) - 1\right)^2 \ . \end{aligned} \tag{1.4}$$

Since $0 \le v_\alpha \le \frac{1}{q}$ for all α, the h_α's are bounded in L^∞. Noting that the v_α's are bounded in $H^1 \cap L^\infty$, they converge, up to a subsequence, in L^p for all p, and so do the h_α's. We let h be the limit of the h_α's, e.g. in L^2, so that

$$\|h_\alpha\|_{L^\infty} = O(1) \text{ and } h_\alpha \to h \text{ in } L^2 \tag{1.5}$$

as $\alpha \to +\infty$ (and, as we easily infer, the convergence in (1.4) holds in L^p for all $1 < p < +\infty$). No further control is a priori available on the h_α's, and this is going to be a serious issue in what follows. There holds that $h \in L^\infty$ and

$$h \ge m_0^2 - \omega^2 \tag{1.6}$$

in M. In particular, $\Delta_g + h$ is coercive, where h is as in (1.5), and by Robert [14], the C^0 and C^1-estimates on the Green's function of $\Delta_g + h$ hold true. Following standard terminology, we define a bubble $(B_\alpha)_\alpha$ as a sequence of functions in M given in dimension 4 by

$$B_\alpha(x) = \frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha,x)^2}{8}} \tag{1.7}$$

for all $x \in M$ and all α, where $(x_\alpha)_\alpha$ is a converging sequence of points in M, and $(\mu_\alpha)_\alpha$ is a sequence of positive real numbers converging to 0 as $\alpha \to +\infty$. We refer to the x_α's in (1.7) as the centers of $(B_\alpha)_\alpha$, and to the μ_α's as the weights of $(B_\alpha)_\alpha$. Then, see Druet-Hebey-Robert [6] and Hebey [9], the H^1-theory for blow-up, as developed in Struwe [15], can be applied. And, even more, the upper control of the C^0-theory developed in Druet-Hebey-Robert [6] (see also Druet-Hebey [4] and Hebey [9] for more recent expositions) also holds true. By the H^1-theory we get that there exist a solution $(u_\infty, \Phi(u_\infty))$ of (0.2) with phase ω, $k \in \mathbb{N}^\star$, and k-bubbles $(B_\alpha^i)_\alpha$, $i = 1, \dots, k$, such that, up to a subsequence,

$$u_\alpha = u_\infty + \sum_{i=1}^k B_\alpha^i + R_\alpha \tag{1.8}$$

in M for all α, where $\|R_\alpha\|_{H^1} \to 0$ as $\alpha \to +\infty$, and

$$\frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} + \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha}\mu_{j,\alpha}} \to +\infty \tag{1.9}$$

as $\alpha \to +\infty$, for all $i \ne j$, where the $x_{i,\alpha}$'s are the centers of $(B_\alpha^i)_\alpha$, and the $\mu_{i,\alpha}$'s are the weights of $(B_\alpha^i)_\alpha$. To state the C^0-theory, in the form we are going to use, we need to introduce some more notions. Given $i, j \in \{1, \dots, k\}$, $i \ne j$, we let $s_{i,j,\alpha}$ be given by

$$s_{i,j,\alpha}^2 = \frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} \frac{d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)^2}{8} + \mu_{i,\alpha}\mu_{j,\alpha} \tag{1.10}$$

for all α, and then we define the range of influence $r_{i,\alpha}$ of the blow-up point $x_{i,\alpha}$ by

$$r_{i,\alpha} = \begin{cases} \min\limits_{j \in \mathcal{A}_i} s_{i,j,\alpha} & \text{if } u_\infty \equiv 0 \\ \\ \min\left\{\min\limits_{j \in \mathcal{A}_i} s_{i,j,\alpha}; \sqrt{\mu_{i,\alpha}}\right\} & \text{if } u_\infty \not\equiv 0\,, \end{cases} \tag{1.11}$$

where

$$\mathcal{A}_i = \Big\{ j \in \{1, \dots, k\}\,,\ j \ne i \text{ s.t. } \mu_{i,\alpha} = O\left(\mu_{j,\alpha}\right) \Big\}\,. \tag{1.12}$$

If $\mathcal{A}_i = \emptyset$ and $u_\infty \equiv 0$, we adopt the convention that $r_{i,\alpha} = \frac{1}{2} i_g$, where i_g is the injectivity radius of (M, g), and if $\mathcal{A}_i = \emptyset$ and $u_\infty \not\equiv 0$, we adopt the convention that $r_{i,\alpha} = \sqrt{\mu_{i,\alpha}}$. From now on we order the blow-up points such that

$$\mu_{1,\alpha} \geq \cdots \geq \mu_{k,\alpha} \tag{1.13}$$

for all α. It follows from the structure equation (1.9) that

$$\frac{r_{i,\alpha}}{\mu_{i,\alpha}} \to +\infty \tag{1.14}$$

as $\alpha \to +\infty$. If $j \in \mathcal{A}_i$ and $i \in \mathcal{A}_j$, we let $\lambda_{i,j} > 0$ be given by

$$\lambda_{i,j} = \lim_{\alpha \to +\infty} \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} \ . \tag{1.15}$$

Given $i \in \{1, \ldots, k\}$, we define

$$\begin{aligned} \mathcal{B}_i &= \Big\{ j \in \{1, \ldots, k\}\,,\, j \neq i \text{ s.t. } d_g\left(x_{i,\alpha}, x_{j,\alpha}\right) = O\left(r_{i,\alpha}\right) \Big\} \text{ if } r_{i,\alpha} \to 0\ , \\ \mathcal{B}_i &= \Big\{ j \in \{1, \ldots, k\}\,,\, j \neq i \text{ s.t. } x_j \in B_{x_i}\left(\frac{1}{2} i_g\right) \Big\} \text{ if } r_{i,\alpha} \not\to 0 \end{aligned} \tag{1.16}$$

and, for $j \in \mathcal{B}_i$, we let $z_{i,j}$ be given by

$$z_{i,j} = \lim_{\alpha \to +\infty} r_{i,\alpha}^{-1} \exp_{x_{i,\alpha}}^{-1}\left(x_{j,\alpha}\right) \ . \tag{1.17}$$

All the limits involved in these definitions are assumed to exist, which is always possible passing to a subsequence. We let $\delta_i > 0$ be such that for any i and any $j \in \mathcal{B}_i$,

$$|z_{i,j}| \neq 0 \Rightarrow |z_{i,j}| \geq 10\delta_i \ . \tag{1.18}$$

We also define $\mathcal{C}_i$ to be the subset of $\mathcal{B}_i$ given by

$$\mathcal{C}_i = \Big\{ j \in \mathcal{B}_i \text{ s.t. } z_{i,j} = 0 \Big\} \cap \mathcal{A}_i^c\ , \tag{1.19}$$

where $\mathcal{A}_i^c$ is the complementary set of $\mathcal{A}_i$, and thus the set consisting of the j's which are such that $\mu_{j,\alpha} = o(\mu_{i,\alpha})$. Then, passing to a subsequence, and for any i, there exist a subset $\mathcal{D}_i$ of $\mathcal{C}_i$ and a family $\left(R_{i,j}\right)_{j \in \mathcal{D}_i}$ of positive real numbers such that for any $j_1, j_2 \in \mathcal{D}_i$, $j_1 \neq j_2$,

$$\frac{d_g\left(x_{j_1,\alpha}, x_{j_2,\alpha}\right)}{s_{j_1,i,\alpha}} \to +\infty \tag{1.20}$$

as $\alpha \to +\infty$, and such that for any $j \in \mathcal{C}_i$ there exists a unique $j' \in \mathcal{D}_i$ such that

$$\limsup_{\alpha \to +\infty} \frac{d_g\left(x_{j,\alpha}, x_{j',\alpha}\right)}{s_{j',i,\alpha}} \leq \frac{R_{i,j'}}{20} \text{ and } \limsup_{\alpha \to +\infty} \frac{s_{j,i,\alpha}}{s_{j',i,\alpha}} \leq \frac{R_{i,j'}}{20}\ , \tag{1.21}$$

where $\mathcal{C}_i$ is as in (1.19). We let $\mu_\alpha = \mu_{1,\alpha}$ be given by

$$\mu_\alpha = \max_{i=1,\ldots,k} \mu_{i,\alpha} \ . \tag{1.22}$$

As shown in Druet and Hebey [4], see also Hebey [9] for an exposition in book form, the C^0-theory then gives that for any $i \in \{1, \ldots, k\}$, there exists $C > 0$ and a sequence $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to zero such that, passing to a subsequence,

$$\begin{aligned} \left|u_\alpha - B_\alpha^i\right| &\leq C\left(H_\alpha^i + \varepsilon_\alpha\right) B_\alpha^i + C\Big(\mu_{i,\alpha} r_{i,\alpha}^{-2} + \sum_{j \in \mathcal{D}_i} B_\alpha^j\Big) \\ &\leq C B_\alpha^i \end{aligned} \tag{1.23}$$

in $B_{x_{i,\alpha}}(4\delta_i r_{i,\alpha})\backslash\bigcup_{j\in\mathcal{D}_i} B_{x_{j,\alpha}}\left(\frac{R_{i,j}}{10}s_{j,i,\alpha}\right)$, where x_i is the limit of the $x_{i,\alpha}$'s as $\alpha\to+\infty$, H^i_α is given by

$$H^i_\alpha(x)=\left(1+\ln\frac{1}{d_g(x_{i,\alpha},x)}\right)d_g(x_{i,\alpha},x)^2 \tag{1.24}$$

in $M\backslash\{x_{i,\alpha}\}$, and $H^i_\alpha(x_{i,\alpha})=0$ for all α and all i, δ_i is as in (1.18), the $\mathcal{D}_i$'s and $R_{i,j}$'s are as in (1.20)–(1.21), and $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11). When $r_{i,\alpha}\to 0$ as $\alpha\to+\infty$, it follows from (1.23) that there exist $C>0$ and a sequence $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to zero such that

$$\left|u_\alpha(x)-B^i_\alpha(x)\right|\le\varepsilon_\alpha B^i_\alpha(x)+C\Big(\mu_{i,\alpha}r_{i,\alpha}^{-2}+\sum_{j\in\mathcal{D}_i}B^j_\alpha(x)\Big) \tag{1.25}$$

for all $x\in\mathcal{N}_\alpha$ and all α. From (1.23) and standard elliptic theory we then get that for any $i\in\{1,\dots,k\}$, there exists $C>0$ such that, up to a subsequence,

$$|u_\alpha|\le C\mu_{i,\alpha}r_{i,\alpha}^{-2}\ \text{ and }\ |\nabla u_\alpha|\le C\mu_{i,\alpha}r_{i,\alpha}^{-3} \tag{1.26}$$

in $B_{x_{i,\alpha}}(2\delta_i r_{i,\alpha})\backslash B_{x_{i,\alpha}}\left(\frac{1}{2}\delta_i r_{i,\alpha}\right)$, where δ_i is as in (1.18), and $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11). We also get that there exists $C>0$ such that, up to a subsequence, for any $j\in\mathcal{D}_i$,

$$|u_\alpha|\le C\mu_{j,\alpha}s_{j,i,\alpha}^{-2}\ \text{ and }\ |\nabla u_\alpha|\le C\mu_{j,\alpha}s_{j,i,\alpha}^{-3} \tag{1.27}$$

in $B_{x_{j,\alpha}}(5R_{i,j}s_{j,i,\alpha})\backslash B_{x_{j,\alpha}}\left(\frac{1}{5}R_{i,j}s_{j,i,\alpha}\right)$, where the $\mathcal{D}_i$'s and $R_{i,j}$'s are given by (1.20)–(1.21). Still by the C^0-theory, we can prove that given $i\in\{1,\dots,k\}$, if $r_{i,\alpha}=o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, then, up to a subsequence,

$$r_{i,\alpha}^2\mu_{i,\alpha}^{-1}u_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}z)\right)\to 8\left(\frac{1}{|z|^2}+\mathcal{H}_i(z)\right) \tag{1.28}$$

in $C^2_{loc}(B_0(2\delta_i)\setminus\{0\})$ as $\alpha\to+\infty$, where

$$\mathcal{H}_i(z)=\sum_{j\in\mathcal{A}_i\cap\mathcal{B}_i,i\in\mathcal{A}_j}\frac{\lambda_{i,j}}{\left|z-z_{i,j}\right|^2}+X_i \tag{1.29}$$

is a smooth function in $B_0(2\delta_i)$ satisfying that $\mathcal{H}_i(0)\neq 0$, δ_i is as in (1.18), the $\lambda_{i,j}$'s are as in (1.15), the $z_{i,j}$'s are as in (1.17), and the X_i's are nonnegative real numbers given by

$$X_i=\left(\lim_{\alpha\to+\infty}r_{i,\alpha}^2\mu_{i,\alpha}^{-1}\right)u_\infty(x_i)+\sum_{j\in(\mathcal{A}_i\backslash\mathcal{B}_i)\cup\Theta_i}\left(\lim_{\alpha\to+\infty}\frac{r_{i,\alpha}}{s_{i,j,\alpha}}\right)^2, \tag{1.30}$$

where we adopt the convention that the first term in the right hand side of (1.30) is zero if $u_\infty\equiv 0$, that the second term is zero if $(\mathcal{A}_i\backslash\mathcal{B}_i)\cup\Theta_i=\emptyset$, and where $\Theta_i=\{j\in\mathcal{A}_i \text{ s.t. } i\notin\mathcal{A}_j\}$.

2. Sharp estimates on the range of influence

We let (M,g) be a smooth compact Riemannian 4-manifold, and $m_0,m_1,q>0$ be positive real numbers. We let $\omega\in(-m_0,+m_0)$, $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$ be an arbitrary sequence of finite energy standing waves, $u_\alpha\ge 0$, and $(v_\alpha)_\alpha$ be an arbitrary sequence of gauge potentials, satisfying that they solve (1.2) for all α, and that $\omega_\alpha\to\omega$ in $\mathbb{R}$ as $\alpha\to+\infty$. We assume that (1.3) holds true, and we want to get a contradiction.

We follow the analysis in Druet and Hebey [4] but we need to face the lost of C^1-control on the h_α's in (1.4) which is going to lead to quite serious difficulties. As in the preceding section, we order the blow-up points such that (1.13) holds true. We aim in proving that the following proposition holds true.

Proposition 2.1. *Let* $i \in \{1, \ldots, k\}$. *If* $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, *where* $r_{i,\alpha}$ *is the range of influence of* $x_{i,\alpha}$ *as in* (1.11), *and* μ_α *is as in* (1.22), *then, up to a subsequence,*

$$\left(m_0^2 - \omega^2 - \frac{1}{6}S_g(x_i) + o(1)\right) r_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} = 2\mathcal{H}_i(0) + o(1) , \tag{2.1}$$

where $\mathcal{H}_i$ *is as in* (1.29), ω *is the limit of the* ω_α*'s, and* x_i *is the limit of the* $x_{i,\alpha}$*'s. Moreover, there holds that* $\nabla\mathcal{H}_i(0) \equiv 0$ *if we assume in addition that* $m_0^2 - \omega^2 \notin \frac{1}{6}Rg(S_g)$. *In case* $r_{i,\alpha} \neq o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, *there holds that*

$$\left(m_0^2 - \omega^2 - \frac{1}{6}S_g(x_i) + o(1)\right) \mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} = O\left(\mu_{i,\alpha}\mu_\alpha\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) , \tag{2.2}$$

where μ_α *is as in* (1.22).

We prove Proposition 2.1 by finite inverse induction on i. We let $i \in \{1, \ldots, k\}$, and in case $i < k$, we assume that

$$\begin{aligned} &\text{for any } j = i+1, \ldots, k \text{ , (2.1) holds true} \\ &\text{for } j \text{ as soon as } \sqrt{\mu_\alpha} r_{j,\alpha} = o\left(\sqrt{\mu_{j,\alpha}}\right) \text{ .} \end{aligned} \tag{H_i}$$

If $i = k$ we do not assume anything. Assuming (H_i) we aim to prove that (2.1) holds true for i. Let $i \in \{1, \ldots, k\}$, $i < k$, be arbitrary. Assuming that (H_i) holds true we get that for any $j \in \mathcal{D}_i$,

$$s_{j,i,\alpha}^{-2} = O\left(\ln \frac{1}{\mu_{j,\alpha}}\right) \tag{2.3}$$

where $\mathcal{D}_i$ is as in (1.20)–(1.21). Indeed, if $j \in \mathcal{D}_i$, then $j > i$. Moreover, for any $j \in \mathcal{D}_i$, we have that $i \in \mathcal{A}_j$ so that $s_{j,i,\alpha} \geq r_{j,\alpha}$, and we clearly have that $s_{j,i,\alpha}^2 = o\left(\mu_{j,\alpha}\mu_{i,\alpha}^{-1}\right) = o\left(\mu_{j,\alpha}\mu_\alpha^{-1}\right)$. The first equality is obvious. For the second one we remark that if $i > 1$, then for $j \in \mathcal{D}_i$,

$$d_g(x_{i,\alpha}, x_{j,\alpha})^2 = o\left(r_{i,\alpha}^2\right) = o\left(\mu_{i,\alpha}\mu_\alpha^{-1}\right)$$

since $1 \in \mathcal{A}_i$. In particular, $\sqrt{\mu_\alpha} r_{j,\alpha} = o\left(\sqrt{\mu_{j,\alpha}}\right)$, and (2.3) is a direct consequence of (H_i) since $\mathcal{H}_i(0) \neq 0$. In what follows we introduce the subsets $\Omega_{i,\alpha}$ of M given by

$$\Omega_{i,\alpha} = B_{x_{i,\alpha}}\left(\delta_i r_{i,\alpha}\right) \setminus \bigcup_{j \in \mathcal{D}_i} \Omega_{i,j,\alpha} , \tag{2.4}$$

where

$$\Omega_{i,j,\alpha} = B_{x_{j,\alpha}}\left(R_{i,j} s_{j,i,\alpha}\right) \tag{2.5}$$

for all $j \in \mathcal{D}_i$, δ_i is as in (1.18), and the $\mathcal{D}_i$'s and $R_{i,j}$'s are given by (1.20)–(1.21). It follows from (1.20) that the $\Omega_{i,j,\alpha}$'s are disjoint for α sufficiently large, and it is easily checked that $s_{j,i,\alpha} = o(r_{i,\alpha})$ for all $j \in \mathcal{D}_i$. We also have that $\mathcal{D}_i = \emptyset$ if $i = k$ so that no $\Omega_{i,j,\alpha}$'s have to be considered in that case. We define X_α to be the smooth 1-form given by

$$X_\alpha(x) = \left(1 - \frac{1}{6(n-1)} Rc_g^\sharp(x)\left(\nabla f_\alpha(x), \nabla f_\alpha(x)\right)\right) \nabla f_\alpha(x) , \tag{2.6}$$

where $f_\alpha(x) = \frac{1}{2}d_g(x_{i,\alpha}, x)^2$, and $Rc_g^\sharp$ is the $(0,2)$-tensor field we get from the $(2,0)$-Ricci tensor Rc_g thanks to the musical isomorphism. We let $X_\alpha(\nabla u_\alpha)$ be given by $X_\alpha(\nabla u_\alpha) = (X_\alpha, \nabla u_\alpha)$, where $(\cdot,\cdot)$ is the pointwise scalar product for 1-forms. Applying the Pohozaev identity in Druet and Hebey [4] to the u_α's in $\Omega_{i,\alpha}$ with the 1-forms X_α, we get that

$$\begin{aligned}&\int_{\Omega_{i,\alpha}} h_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g + \frac{1}{8}\int_{\Omega_{i,\alpha}} \left(\Delta_g\left(\mathrm{div}_g X_\alpha\right)\right) u_\alpha^2 dv_g \\ &+ \frac{1}{4}\int_{\Omega_{i,\alpha}} \left(\mathrm{div}_g X_\alpha\right) h_\alpha u_\alpha^2 dv_g \\ &= Q_\alpha - \sum_{j\in\mathcal{D}_i} Q_\alpha^j + R_{1,\alpha} + R_{2,\alpha} - \sum_{j\in\mathcal{D}_i} R_{2,\alpha}^j ,\end{aligned} \tag{2.7}$$

where, if $\nu = \nu_\alpha$ stands for the unit outer normal to $\partial\Omega_{i,\alpha}$, the Q_α's are given by

$$\begin{aligned}Q_\alpha =& \frac{1}{4}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} \left(\mathrm{div}_g X_\alpha\right)(\partial_\nu u_\alpha) u_\alpha d\sigma_g \\ &- \int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} \left(\frac{1}{2} X_\alpha(\nu)\left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha)(\partial_\nu u_\alpha)\right) d\sigma_g ,\end{aligned} \tag{2.8}$$

the Q_α^j's are given by

$$\begin{aligned}Q_\alpha^j =& \frac{1}{4}\int_{\partial\Omega_{i,j,\alpha}} \left(\mathrm{div}_g X_\alpha\right)(\partial_\nu u_\alpha) u_\alpha d\sigma_g \\ &- \int_{\partial\Omega_{i,j,\alpha}} \left(\frac{1}{2} X_\alpha(\nu)\left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha)(\partial_\nu u_\alpha)\right) d\sigma_g ,\end{aligned} \tag{2.9}$$

where $\Omega_{i,j,\alpha}$ is as in (2.5), the $R_{1,\alpha}$'s are given by

$$R_{1,\alpha} = -\int_{\Omega_{i,\alpha}} \left(\nabla X_\alpha - \frac{1}{4}\left(\mathrm{div}_g X_\alpha\right) g\right)^\sharp (\nabla u_\alpha, \nabla u_\alpha) dv_g , \tag{2.10}$$

the $R_{2,\alpha}$'s are given by

$$\begin{aligned}R_{2,\alpha} =& \frac{1}{4}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} X_\alpha(\nu) u_\alpha^{2^\star} d\sigma_g \\ &- \frac{1}{8}\int_{\partial B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} \left(\partial_\nu\left(\mathrm{div}_g X_\alpha\right)\right) u_\alpha^2 d\sigma_g ,\end{aligned} \tag{2.11}$$

and the $R_{2,\alpha}^j$'s are given by

$$R_{2,\alpha}^j = \frac{1}{4}\int_{\partial\Omega_{i,j,\alpha}} X_\alpha(\nu) u_\alpha^{2^\star} d\sigma_g - \frac{1}{8}\int_{\partial\Omega_{i,j,\alpha}} \left(\partial_\nu\left(\mathrm{div}_g X_\alpha\right)\right) u_\alpha^2 d\sigma_g . \tag{2.12}$$

We split the proof of Proposition 2.1 in several lemmas. All what follows is up to a subsequence. The first lemma is contained in Druet and Hebey [4]. We note that for X_α as in (2.6),

$$\begin{aligned}&\left|X_\alpha(x)\right| = O\left(d_g(x_{i,\alpha}, x)\right) ,\ \mathrm{div}_g X_\alpha(x) - 4 = O\left(d_g(x_{i,\alpha}, x)^2\right) , \\ &\left|\nabla\left(\mathrm{div}_g X_\alpha\right)(x)\right| = O\left(d_g(x_{i,\alpha}, x)\right) ,\ \text{and} \\ &\Delta_g\left(\mathrm{div}_g X_\alpha\right)(x) = \frac{4}{3} S_g(x_{i,\alpha}) + O\left(d_g(x_{i,\alpha}, x)\right) .\end{aligned} \tag{2.13}$$

Then we compute the right hand side in (2.7).

Lemma 2.1. *Let $i \in \{1, \dots, k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. Let*

$$\begin{aligned} I_\alpha^i = & \int_{\Omega_{i,\alpha}} h_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g + \frac{1}{8} \int_{\Omega_{i,\alpha}} (\Delta_g(div_g X_\alpha))\, u_\alpha^2 dv_g \\ & + \frac{1}{4} \int_{\Omega_{i,\alpha}} (div_g X_\alpha)\, h_\alpha u_\alpha^2 dv_g \,, \end{aligned} \tag{2.14}$$

where $\Omega_{i,\alpha}$ is as in (2.4) and X_α is as in (2.6). Then

$$I_\alpha^i = \Big(-128\omega_3 \mathcal{H}_i(0) + o(1)\Big)\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \tag{2.15}$$

if $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, and $I_\alpha^i = O\left(\mu_{i,\alpha}\mu_\alpha\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right)$ otherwise, where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11), $\mathcal{H}_i$ is as in (1.29), and μ_α is as in (1.22).

The next lemma in the proof of Proposition 2.1 is as follows.

Lemma 2.2. *Let $i \in \{1, \dots, k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. Let I_α^i be as in (2.14). There holds that*

$$\begin{aligned} I_\alpha^i = & -64\omega_3 \left(m_0^2 - \omega^2 - \frac{1}{6} S_g(x_i) + o(1)\right) \mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} \\ & + o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \\ & + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha^2 dv_g\right) + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g\right) \,, \end{aligned} \tag{2.16}$$

where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11), μ_α is as in (1.22), and x_i is the limit of the $x_{i,\alpha}$'s.

Proof of Lemma 2.2. Let h_α be as in (1.4). Then,

$$h_\alpha = m_0^2 - \omega_\alpha^2 + O\left(v_\alpha\right) \tag{2.17}$$

and we get by (2.13) and (2.17) that

$$\begin{aligned} I_\alpha^i = & \left(m_0^2 - \omega^2 + o(1)\right) \int_{\Omega_{i,\alpha}} u_\alpha X_\alpha(\nabla u_\alpha) dv_g \\ & + \frac{1}{8} \int_{\Omega_{i,\alpha}} (\Delta_g(\mathrm{div}_g X_\alpha))\, u_\alpha^2 dv_g \\ & + \frac{1}{4} \left(m_0^2 - \omega^2 + o(1)\right) \int_{\Omega_{i,\alpha}} (\mathrm{div}_g X_\alpha)\, u_\alpha^2 dv_g \\ & + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha^2 dv_g\right) + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g\right) \,. \end{aligned} \tag{2.18}$$

Using (1.23) it is easily checked that

$$\int_{\Omega_{i,\alpha}} d_g\left(x_{i,\alpha}, x\right) u_\alpha^2 dv_g = o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \,. \tag{2.19}$$

Integrating by parts, thanks to (1.26)–(1.27) and (2.13), there also holds that

$$\int_{\Omega_{i,\alpha}} u_\alpha X_\alpha(\nabla u_\alpha)dv_g = -2\int_{\Omega_{i,\alpha}} u_\alpha^2 dv_g + o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) + o\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right). \quad (2.20)$$

Then, by (2.18)–(2.20), we get that

$$\begin{aligned} I_\alpha^i = &-\left(m_0^2 - \omega^2 - \frac{1}{6}S_g(x_i) + o(1)\right)\int_{\Omega_{i,\alpha}} u_\alpha^2 dv_g \\ &+ o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) + o\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right) \\ &+ O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha^2 dv_g\right) + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g\right). \end{aligned} \quad (2.21)$$

We have that

$$\int_{\Omega_{i,j,\alpha}} (B_\alpha^i)^2 dv_g = o\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right) \quad (2.22)$$

for all $j \in \mathcal{D}_i$. Thanks to (2.21) and (2.22), and thanks to (1.23), we then get that

$$\begin{aligned} I_\alpha^i = &-\left(m_0^2 - \omega^2 - \frac{1}{6}S_g(x_i) + o(1)\right)\int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (B_\alpha^i)^2 dv_g \\ &+ o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) + o\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right) \\ &+ O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha^2 dv_g\right) + O\left(\int_{\Omega_{i,\alpha}} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g\right). \end{aligned} \quad (2.23)$$

We have that

$$\int_{B_{x_{i,\alpha}}(\delta_i r_{i,\alpha})} (B_\alpha^i)^2 dv_g = 64\omega_3 \mu_{i,\alpha}^2 \ln\frac{r_{i,\alpha}}{\mu_{i,\alpha}} + o\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right). \quad (2.24)$$

Combining (2.23) and (2.24), this ends the proof of Lemma 2.2. □

At this point it remains to handle the two terms involving the gauge potentials v_α in (2.16). We let

$$A_\alpha = \int_{\Omega_{i,\alpha}} v_\alpha u_\alpha^2 dv_g \text{ and } B_\alpha = \int_{\Omega_{i,\alpha}} v_\alpha u_\alpha |X_\alpha(\nabla u_\alpha)| dv_g \quad (2.25)$$

for all α. The following lemma reduces the problem of controlling these two terms to the problem of controlling the first term.

Lemma 2.3. *Let $i \in \{1, \dots, k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. There exists $C > 0$ such that*

$$B_\alpha^2 \le C\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}} + o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right)\right) A_\alpha \quad (2.26)$$

for all α, where A_α and B_α are as in (2.25), and $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11).

Proof of Lemma 2.3. By the Cauchy-Schwarz inequality, since the v_α's are bounded in L^∞, and by (2.13), there exists $C > 0$ such that

$$B_\alpha^2 \le C A_\alpha \int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha}, \cdot)^2 |\nabla u_\alpha|^2 dv_g \quad (2.27)$$

for all α. There also exists $C > 0$ such that

$$\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha},\cdot)^2|\nabla u_\alpha|^2dv_g \le C\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha},\cdot)^2|\nabla(u_\alpha - B^i_\alpha)|^2dv_g + C\int_{\Omega_{i,\alpha}} d_g(x_{i,\alpha},\cdot)^2|\nabla B^i_\alpha|^2dv_g \tag{2.28}$$

for all α, and we can conclude by noting that

$$\int_{\Omega_{i,\alpha}} d_g\left(x_{i,\alpha},x\right)^2\left|\nabla B^i_\alpha\right|^2 dv_g = O\left(\mu_{i,\alpha}^2 \ln\frac{1}{\mu_{i,\alpha}}\right)\ , \tag{2.29}$$

and that by (1.23), using Hölder's inequalities,

$$\int_{\Omega_{i,\alpha}} d_g\left(x_{i,\alpha},x\right)^2\left|\nabla\left(u_\alpha - B^i_\alpha\right)\right|^2 dv_g = o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) + o\left(\mu_{i,\alpha}^2\ln\frac{1}{\mu_{i,\alpha}}\right)\ . \tag{2.30}$$

This ends the proof of Lemma 2.3. □

Now we search for estimates on the A_α's in (2.25). We split v_α into a quasi-harmonic and a quasi-Poisson part. We define

$$\hat\Omega_{i,\alpha} = B_{x_{i,\alpha}}\left(\frac{3}{2}\delta_i r_{i,\alpha}\right)\setminus\bigcup_{j\in\mathcal{D}_i} B_{x_{j,\alpha}}\left(\frac{2}{3}R_{i,j}s_{j,i,\alpha}\right)\ , \tag{2.31}$$

where δ_i is as in (1.18), and the $\mathcal{D}_i$'s and $R_{i,j}$'s are given by (1.20)–(1.21). The $B_{x_{j,\alpha}}\left(\frac{2}{3}R_{i,j}s_{j,i,\alpha}\right)$'s are disjoint for $\alpha \gg 1$ and we have that $\Omega_{i,\alpha}\subset\hat\Omega_{i,\alpha}$ for all i and all α. Then we let

$$v_\alpha = w_{1,\alpha} + w_{2,\alpha} \tag{2.32}$$

for all α, where $w_{1,\alpha}$ is given by

$$\begin{cases}\Delta_g w_{1,\alpha} + m_1^2 w_{1,\alpha} = 0 \text{ in } \hat\Omega_{i,\alpha}\\ w_{1,\alpha} = v_\alpha \text{ on } \partial\hat\Omega_{i,\alpha}\end{cases} \tag{2.33}$$

for all α, $w_{2,\alpha}$ is given by

$$\begin{cases}\Delta_g w_{2,\alpha} + m_1^2 w_{2,\alpha} = W_\alpha \text{ in } \hat\Omega_{i,\alpha}\\ w_{2,\alpha} = 0 \text{ on } \partial\hat\Omega_{i,\alpha}\end{cases} \tag{2.34}$$

for all α, and

$$W_\alpha = \Delta_g v_\alpha + m_1^2 v_\alpha = q\left(1 - qv_\alpha\right)u_\alpha^2\ , \tag{2.35}$$

for all α. As shown in Hebey and Truong [10], working at the macroscopic level, we easily get that there exists $\beta\in(0,1]$ and $C > 0$ such that

$$v_\alpha \le C u_\alpha^\beta \text{ in } M \tag{2.36}$$

for all α. Now the next lemma establishes a first set of L^∞-estimates for the quasi-harmonic part of the decomposition of the v_α's.

Lemma 2.4. *Let $i\in\{1,\dots,k\}$ be arbitrary, and in case $i<k$, assume that (H_i) holds true. Suppose $r_{i,\alpha}\ne o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in* (1.11)*. Then*

$$\|w_{1,\alpha}\|_{L^\infty(\hat\Omega_{i,\alpha})}\to 0$$

as $\alpha\to+\infty$, where $\hat\Omega_{i,\alpha}$ is as in (2.31)*, and $w_{1,\alpha}$ is as in* (2.32)–(2.33)*.*

Proof of Lemma 2.4. Assuming that $r_{i,\alpha} \neq o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, we get that

$$\mu_{i,\alpha} = O\left(r_{i,\alpha}^2 \mu_\alpha\right) = o\left(r_{i,\alpha}^2\right) . \tag{2.37}$$

By (1.26)–(1.27) and (2.37) we then get that

$$\max_{\partial B_{x_{i,\alpha}}(\frac{3}{2}\delta_i r_{i,\alpha})} u_\alpha = o(1) . \tag{2.38}$$

Similarly, by (2.3), which follows from (H_i), and by (1.26)–(1.27), there also holds that

$$\max_{\partial B_{x_{j,\alpha}}(\frac{2}{3}R_{i,j}s_{j,i,\alpha})} u_\alpha = o(1) . \tag{2.39}$$

By the maximum principle,

$$0 \le w_{1,\alpha} \le \max_{\partial\hat{\Omega}_{i,\alpha}} w_{1,\alpha} \tag{2.40}$$

in $\hat{\Omega}_{i,\alpha}$, and since $w_{2,\alpha} \ge 0$, we get from (2.36) and (2.40) that

$$0 \le w_{1,\alpha} \le \max_{\partial\hat{\Omega}_{i,\alpha}} v_\alpha \le \max_{\partial\hat{\Omega}_{i,\alpha}} u_\alpha^\beta \tag{2.41}$$

in $\hat{\Omega}_{i,\alpha}$. Combining (2.38), (2.39), and (2.41), this ends the proof of Lemma 2.4. □

The next lemma provides a pointwise estimate for the $w_{2,\alpha}$'s.

Lemma 2.5. *There exists $C > 0$ such that*

$$w_{2,\alpha}(x) \le C \frac{\mu_{i,\alpha}^2 \ln\left(2 + \frac{d_g(x_{i,\alpha},x)^2}{\mu_{i,\alpha}^2}\right)}{\mu_{i,\alpha}^2 + d_g(x_{i,\alpha},x)^2} \tag{2.42}$$

for all α and all $x \in \hat{\Omega}_{i,\alpha}$, where $\hat{\Omega}_{i,\alpha}$ is as in (2.31)*, and $w_{2,\alpha}$ is as in* (2.32)*–*(2.34)*.*

Proof of Lemma 2.5. Let G_α be the Green's function of $\Delta_g + m_1^2$ in $\hat{\Omega}_{i,\alpha}$ with zero Dirichlet boundary condition on $\partial\hat{\Omega}_{i,\alpha}$, and G be the Green's function of the same operator in $B_{x_i}(\frac{2}{3}i_g)$ with zero Dirichlet boundary condition on $\partial B_{x_i}(\frac{2}{3}i_g)$, where x_i is the limit of the $x_{i,\alpha}$'s and i_g is the injectivity radius of (M,g). Let $x \in \hat{\Omega}_{i,\alpha}$, and $G_{\alpha,x}$, G_x be the functions defined for $y \neq x$ by $G_{\alpha,x}(y) = G_\alpha(x,y)$ and $G_x(y) = G(x,y)$. Since $G_{\alpha,x} \le G_x$ on $\partial\hat{\Omega}_{i,\alpha}$ we get that $G_{\alpha,x} \le G_x$ in $\hat{\Omega}_{i,\alpha}\backslash\{x\}$. In particular, see Robert [14], we get that there exists $C > 0$ such that

$$G_\alpha(x,y) \le \frac{C}{d_g(x,y)^2} \tag{2.43}$$

for all $x, y \in \hat{\Omega}_{i,\alpha}$, $x \neq y$. There holds that

$$\begin{aligned} w_{2,\alpha}(x) &= \int_{\hat{\Omega}_{i,\alpha}} G_{\alpha,x}\left(\Delta_g w_{2,\alpha} + m_1^2 w_{2,\alpha}\right) dv_g \\ &= \int_{\hat{\Omega}_{i,\alpha}} G_{\alpha,x} W_\alpha dv_g , \end{aligned} \tag{2.44}$$

where W_α is as in (2.35), and since $W_\alpha \le C u_\alpha^2$, it follows from (2.43) and (2.44) that there exists $C > 0$ such that

$$w_{2,\alpha}(x) \le C \int_{\hat{\Omega}_{i,\alpha}} \frac{u_\alpha^2 dv_g}{d_g(x,\cdot)^2} \tag{2.45}$$

for all α and all $x \in \hat{\Omega}_{i,\alpha}$. By (1.23) we then get that from (2.45) that

$$\begin{aligned} w_{2,\alpha}(x) &\le C \int_{\hat{\Omega}_{i,\alpha}} \frac{B_\alpha^i(y)^2 dv_g(y)}{d_g(x,y)^2} \\ &\le C \int_{B_{x_{i,\alpha}}(\frac{3}{2}\delta_i r_{i,\alpha})} \frac{B_\alpha^i(y)^2 dv_g(y)}{d_g(x,y)^2} \end{aligned} \tag{2.46}$$

and (2.42) easily follows from (2.46) by noting that there exists $C > 0$ such that

$$\int_{B_{x_{i,\alpha}}(\frac{3}{2}\delta_i r_{i,\alpha})} \frac{B_\alpha^i(y)^2 dv_g(y)}{d_g(x,y)^2} \le C \frac{\mu_{i,\alpha}^2 \ln\left(2 + \frac{d_g(x_{i,\alpha},x)^2}{\mu_{i,\alpha}^2}\right)}{\mu_{i,\alpha}^2 + d_g(x_{i,\alpha},x)^2}$$

for all x and all α. This ends the proof of Lemma 2.5. □

Thanks to Lemmas 2.4 and 2.5 we can prove the following.

Lemma 2.6. *Let $i \in \{1,\dots,k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. Suppose $r_{i,\alpha} \ne o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in (1.11). Then*

$$\int_{\hat{\Omega}_{i,\alpha}} v_\alpha u_\alpha^2 dv_g = o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right), \tag{2.47}$$

where $\hat{\Omega}_{i,\alpha}$ is as in (2.31).

Proof of Lemma 2.6. By Lemma 2.5, for any $R > 0$ there exists $\varepsilon_R > 0$ such that

$$\|w_{2,\alpha}\|_{L^\infty\left(\hat{\Omega}_{i,\alpha}\backslash B_{x_{i,\alpha}}(R\mu_{i,\alpha})\right)} \le \varepsilon_R \tag{2.48}$$

for all α, and such that $\varepsilon_R \to 0$ as $R \to +\infty$. By (1.23), Lemma 2.4, and (2.48), we then get that

$$\begin{aligned} \int_{\hat{\Omega}_{i,\alpha}} v_\alpha u_\alpha^2 dv_g &\le \int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha u_\alpha^2 dv_g + \varepsilon_R \int_{\hat{\Omega}_{i,\alpha}} u_\alpha^2 dv_g \\ &\le \int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha u_\alpha^2 dv_g + \varepsilon_R \mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} \end{aligned} \tag{2.49}$$

for $\alpha \gg 1$. By Hölder's inequality,

$$\begin{aligned} &\int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha u_\alpha^2 dv_g \\ &\le \left(\int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha^4 dv_g\right)^{1/4} \left(\int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} u_\alpha^{8/3} dv_g\right)^{3/4} \end{aligned} \tag{2.50}$$

and by (1.23), there exists $C > 0$ such that

$$\int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} u_\alpha^{8/3} dv_g \le C\mu_{i,\alpha}^{4/3} \tag{2.51}$$

for all α. Also there holds that

$$\int_{B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha^4 dv_g \le C\mathrm{Vol}_g\left(B_{x_{i,\alpha}}(R\mu_{i,\alpha})\right) \le C\mu_{i,\alpha}^4 \tag{2.52}$$

for all α. By (2.50)–(2.52) we then get that

$$\int_{\hat{\Omega}_{i,\alpha}\cap B_{x_{i,\alpha}}(R\mu_{i,\alpha})} v_\alpha u_\alpha^2 dv_g = O\left(\mu_{i,\alpha}^2\right) \tag{2.53}$$

and since $\varepsilon_R \to 0$ as $R \to +\infty$, we get (2.47) by combining (2.49) and (2.53). This ends the proof of Lemma 2.6. □

Now we handle the cases where $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$. The first lemma in this direction is as follows.

Lemma 2.7. *Let $i \in \{1,\dots,k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. Suppose $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in* (1.11)*. Then*

$$r_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} \geq \varepsilon_0 \tag{2.54}$$

for all α, where $\varepsilon_0 > 0$ is independent of α.

Proof of Lemma 2.7. We proceed by contradiction and assume that $r_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} \to 0$ as $\alpha \to +\infty$. Since there also holds that $r_{i,\alpha} \to 0$ when $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, we get that

$$r_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} \to 0 \tag{2.55}$$

as $\alpha \to +\infty$. By (1.23), since the v_α's are bounded, there exists $C > 0$ such that

$$\int_{\hat{\Omega}_{i,\alpha}} v_\alpha u_\alpha^2 dv_g \leq C\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} \tag{2.56}$$

for all α. By Lemmas 2.1 to 2.3, and (2.56), there holds that

$$\Big(-128\omega_3\mathcal{H}_i(0) + o(1)\Big)\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) . \tag{2.57}$$

Combining (2.55) and (2.57) it follows that $\mathcal{H}_i(0) = 0$, a contradiction with the fact that $\mathcal{H}_i(0) \neq 0$. This ends the proof of Lemma 2.7. □

From Lemma 2.7 we get that the following key estimate holds true.

Lemma 2.8. *Let $i \in \{1,\dots,k\}$ be arbitrary, and in case $i < k$, assume that (H_i) holds true. Suppose $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$, where $r_{i,\alpha}$ is the range of influence of $x_{i,\alpha}$ as in* (1.11)*. Then*

$$\int_{\hat{\Omega}_{i,\alpha}} v_\alpha u_\alpha^2 dv_g = o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) , \tag{2.58}$$

where $\hat{\Omega}_{i,\alpha}$ is as in (2.31)*.*

Proof of Lemma 2.8. By Lemma 2.7, $r_{i,\alpha}^{-2} \leq C \ln \frac{1}{\mu_{i,\alpha}}$, and by (1.26)–(1.27) and (2.36) we then get that there exists $C > 0$ such that

$$v_\alpha \leq C\left(\mu_{i,\alpha} \ln \frac{1}{\mu_{i,\alpha}}\right)^\beta \tag{2.59}$$

on $\partial B_{x_{i,\alpha}}\left(\frac{3}{2}\delta_i r_{i,\alpha}\right)$, for all α. Still by (1.26)–(1.27) and (2.36), it follows from (2.3), and thus from (H_i), that there exists $C > 0$ such that

$$v_\alpha \le C\left(\mu_{j,\alpha} s_{j,i,\alpha}^{-2}\right)^\beta \le C\left(\mu_{j,\alpha} \ln \frac{1}{\mu_{j,\alpha}}\right)^\beta \tag{2.60}$$

on $\partial B_{x_{j,\alpha}}\left(\frac{2}{3}R_{i,j}s_{j,i,\alpha}\right)$ for all $j \in \mathcal{D}_i$ and all α. By the maximum principle, the quasi-harmonic part $w_{1,\alpha}$ of the decomposition of the v_α's satisfies that

$$0 \le w_{1,\alpha} \le \max_{\partial\hat{\Omega}_{i,\alpha}} w_{1,\alpha}$$

and since $w_{1,\alpha} \le v_\alpha$ for all α, it follows from (2.59) and (2.60) that

$$\|w_{1,\alpha}\|_{L^\infty(\hat{\Omega}_{i,\alpha})} \to 0 \tag{2.61}$$

as $\alpha \to +\infty$. Now we handle the quasi-Poisson part $w_{2,\alpha}$ of the decomposition of the v_α's. By (1.23), since the v_α's are bounded, there exists $C_0 > 0$ such that $W_\alpha \le C_0(B_\alpha^i)^2$ in $\hat{\Omega}_{i,\alpha}$ for all α, where W_α is as in (2.35). Define w_α by

$$\begin{cases} \Delta_g w_\alpha + m_1^2 w_\alpha = C_0(B_\alpha^i)^2 \ \text{ in } B_{x_{i,\alpha}}\left(\frac{3}{2}\delta_i r_{i,\alpha}\right) \\ w_\alpha = 0 \ \text{ on } \partial B_{x_{i,\alpha}}\left(\frac{3}{2}\delta_i r_{i,\alpha}\right) \ . \end{cases} \tag{2.62}$$

By the maximum principle, $0 \le w_{2,\alpha} \le w_\alpha$ in $\hat{\Omega}_{i,\alpha}$ for all α, and by (1.23) and Hölder's inequality we then get that

$$\begin{aligned} \int_{\hat{\Omega}_{i,\alpha}} w_{2,\alpha} u_\alpha^2 dv_g &\le \left(\int_{\hat{\Omega}_{i,\alpha}} w_{2,\alpha}^4 dv_g\right)^{1/4} \left(\int_{\hat{\Omega}_{i,\alpha}} u_\alpha^{8/3} dv_g\right)^{3/4} \\ &\le C\left(\int_{B_i(\alpha)} w_\alpha^4 dv_g\right)^{1/4} \left(\int_{B_i(\alpha)} (B_\alpha^i)^{8/3} dv_g\right)^{3/4} \end{aligned} \tag{2.63}$$

for all α, where $B_i(\alpha) = B_{x_{i,\alpha}}\left(\frac{3}{2}\delta_i r_{i,\alpha}\right)$. There holds that

$$\int_{B_i(\alpha)} (B_\alpha^i)^{8/3} dv_g = O\left(\mu_{i,\alpha}^{4/3}\right) \ , \tag{2.64}$$

and we can write using (2.64) that there exists $C > 0$ such that

$$\begin{aligned} \|w_\alpha\|_{H^1(B_i(\alpha))}^2 &\le C\int_{B_i(\alpha)} \left(\Delta_g w_\alpha + m_1^2 w_\alpha\right) w_\alpha dv_g \\ &\le C\int_{B_i(\alpha)} (B_\alpha^i)^2 w_\alpha dv_g \\ &\le C\left(\int_{B_i(\alpha)} w_\alpha^4 dv_g\right)^{1/4} \left(\int_{B_i(\alpha)} (B_\alpha^i)^{8/3} dv_g\right)^{3/4} \\ &\le C\mu_{i,\alpha}\left(\int_{B_i(\alpha)} w_\alpha^4 dv_g\right)^{1/4} \end{aligned} \tag{2.65}$$

for all α. By the Euclidean Sobolev inequality, $\|w_\alpha\|_{L^4(B_i(\alpha))} \le K\|\nabla w_\alpha\|_{L^2(B_i(\alpha))}$ for all α and some $K > 0$ independent of α, and by (2.65) it follows that

$$\left(\int_{B_i(\alpha)} w_\alpha^4 dv_g\right)^{1/4} = O\left(\mu_{i,\alpha}\right) \ . \tag{2.66}$$

Combining (2.63), (2.64), and (2.66), we then get that

$$\int_{\hat{\Omega}_{i,\alpha}} w_{2,\alpha} u_\alpha^2 dv_g = O\left(\mu_{i,\alpha}^2\right) = o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) . \tag{2.67}$$

We easily get (2.58) from (2.61) and (2.67) since $v_\alpha = w_{1,\alpha} + w_{2,\alpha}$ and since, by (1.23),

$$\|u_\alpha\|^2_{L^2(\hat{\Omega}_{i,\alpha})} = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) .$$

This ends the proof of Lemma 2.8. □

At this point we can prove Proposition 2.1. This is the subject of what follows.

Proof of Proposition 2.1. First we let $i \in \{1, \dots, k\}$ be arbitrary, and in case $i < k$, we assume that (H_i) holds true. By Lemma 2.3, Lemma 2.6, and Lemma 2.8, we always have that

$$\begin{aligned} A_\alpha &= o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) + o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) , \text{ and} \\ B_\alpha &= o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) + o\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) , \end{aligned} \tag{2.68}$$

where A_α and B_α are as in (2.25). As is easily checked, (2.1) and (2.2) follow from Lemma 2.1, Lemma 2.2, and (2.68). It remains to prove that $\nabla \mathcal{H}_i(0) \equiv 0$ if we assume that $r_{i,\alpha} = o\left(\sqrt{\frac{\mu_{i,\alpha}}{\mu_\alpha}}\right)$ and that $m_0^2 - \omega^2 \notin \frac{1}{6}\mathrm{Rg}(S_g)$. Let Y be an arbitrary 1-form in $\mathbb{R}^n$. We apply again the Pohozaev identity in Druet and Hebey [4] to u_α in $\Omega_{i,\alpha}$, but we choose here $X = X_\alpha$ to be given in the exponential chart at $x_{i,\alpha}$ by

$$(X_\alpha)_\kappa = Y_\kappa - \frac{2}{3} R_{\kappa jkl}(x_{i,\alpha}) x^j x^k Y^l ,$$

where $Y^l = Y_l$ for all l and the $R_{\kappa jkl}$ are the components of the Riemann tensor Rm_g at $x_{i,\alpha}$ in the exponential chart. As is easily checked, still in geodesic normal coordinates at $x_{i,\alpha}$, $(\nabla X_\alpha)_{\kappa j} = -\mathcal{R}_{\kappa jkl}(x_{i,\alpha}) x^k Y^l + O\left(|x|^2\right)$ so that we obtain $\mathrm{div}_g\left(X_\alpha\right) = O\left(|x|^2\right)$. Then, thanks to the symmetries of the Riemann tensor, we get with the Pohozaev identity that

$$\begin{aligned} &\int_{\partial\Omega_{i,\alpha}} \left(\frac{1}{2} X_\alpha(\nu) \left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha)\partial_\nu u_\alpha\right) d\sigma_g + \int_{\Omega_{i,\alpha}} h_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g \\ &= O\left(\int_{\Omega_{i,\alpha}} u_\alpha^2 dv_g\right) + O\left(\int_{\Omega_{i,\alpha}} d_g\left(x_{i,\alpha}, x\right)^2 \left|\nabla u_\alpha\right|^2 dv_g\right) \\ &+ O\left(\int_{\partial\Omega_{i,\alpha}} u_\alpha^{2^\star} d\sigma_g\right) + O\left(\int_{\partial\Omega_{i,\alpha}} u_\alpha^2 d\sigma_g\right) + O\left(\int_{\partial\Omega_{i,\alpha}} |\partial_\nu u_\alpha| u_\alpha d\sigma_g\right) , \end{aligned} \tag{2.69}$$

where $\nu = \nu_\alpha$ stands for the unit outer normal to $\partial\Omega_{i,\alpha}$. Estimating the right-hand side of (2.69) thanks to (2.3), and thanks to (1.23) and (1.26), we get that

$$\begin{aligned} &\int_{\partial\Omega_{i,\alpha}} \left(\frac{1}{2} X_\alpha(\nu) \left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha)\partial_\nu u_\alpha\right) d\sigma_g \\ &+ \int_{\Omega_{i,\alpha}} h_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) + O\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) . \end{aligned} \tag{2.70}$$

By (2.17),

$$\int_{\Omega_{i,\alpha}} h_\alpha u_\alpha X_\alpha(\nabla u_\alpha) dv_g = \left(m_0^2 - \omega_\alpha^2\right) \int_{\Omega_{i,\alpha}} u_\alpha X_\alpha(\nabla u_\alpha) dv_g + O\left(B_\alpha\right) \ , \tag{2.71}$$

and, integrating by parts, it is easily checked that

$$\int_{\Omega_{i,\alpha}} u_\alpha X_\alpha(\nabla u_\alpha) dv_g = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) + O\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) \ .$$

Coming back to (2.71), thanks to (2.68), it follows that

$$\begin{aligned} &\int_{\partial\Omega_{i,\alpha}} \left(\frac{1}{2} X_\alpha\left(\nu\right) \left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha\right) d\sigma_g \\ &= O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) + O\left(\mu_{i,\alpha}^2 r_{i,\alpha}^{-2}\right) \ . \end{aligned} \tag{2.72}$$

By (1.26)–(1.29), thanks also to (2.3), we can write that

$$\begin{aligned} &\int_{\partial\Omega_{i,\alpha}} \left(\frac{1}{2} X_\alpha\left(\nu\right) \left|\nabla u_\alpha\right|^2 - X_\alpha(\nabla u_\alpha) \partial_\nu u_\alpha\right) d\sigma_g \\ &= \left(128\omega_3 \left(Y\left(\nabla \mathcal{H}_i\right)\right)_0 + o(1)\right) \mu_{i,\alpha}^2 r_{i,\alpha}^{-3} + \hat{\Theta}_\alpha \ , \end{aligned} \tag{2.73}$$

where $\left(Y\left(\nabla\mathcal{H}_i\right)\right)_0 = Y^\kappa\left(\nabla_\kappa \mathcal{H}_i\right)(0)$, and

$$\hat{\Theta}_\alpha = o\left(\mu_{i,\alpha}^2 \left(\ln \frac{1}{\mu_{i,\alpha}}\right)^{3/2}\right) \ .$$

If we assume that $m_0^2 - \omega^2 \notin \frac{1}{6}\mathrm{Rg}(S_g)$, then we get as a consequence of (2.1) that

$$r_{i,\alpha}\left(\ln \frac{1}{\mu_{i,\alpha}}\right)^{1/2} = O\left(1\right) \ .$$

Coming back to (2.72) and (2.73), it follows that $\left(Y\left(\nabla\mathcal{H}_i\right)\right)_0 = 0$, and since Y is arbitrary, we get that $\nabla\mathcal{H}_i(0) \equiv 0$. This ends the proof of Proposition 2.1. □

3. Proof of Theorem 0.1

We prove Theorem 0.1 in this section. We let (M,g) be a smooth compact Riemannian 4-manifold, and $m_0, m_1, q > 0$ be positive real numbers. We let $\omega \in (-m_0, +m_0)$, $\left(u_\alpha e^{i\omega_\alpha t}\right)_\alpha$ be an arbitrary sequence of finite energy standing waves, $u_\alpha \geq 0$, and $\left(v_\alpha\right)_\alpha$ be an arbitrary sequence of gauge potentials, satisfying that they solve (1.2) for all α, and that $\omega_\alpha \to \omega$ in $\mathbb{R}$ as $\alpha \to +\infty$. We assume that

$$m_0^2 - \omega^2 \notin \frac{1}{6}\mathrm{Rg}(S_g) \tag{3.1}$$

and that (1.3) holds true, and we want to get a contradiction. As in the preceding section, we order the blow-up points such that (1.13) holds true. First we claim that u_∞ in (1.8) has to be zero. To prove this, we proceed by contradiction and assume that $u_\infty \not\equiv 0$. Then, by the definition of the range of influence, $r_{1,\alpha} = O(\sqrt{\mu_\alpha})$, where $\mu_\alpha = \mu_{1,\alpha}$ is as in (1.22). In particular $r_{1,\alpha} = o(1)$ and we can apply Proposition 2.1 with $i = 1$. As we know, $\mathcal{H}_1(0) \neq 0$ and, therefore, it follows from (2.1) in Proposition 2.1 that

$$r_{1,\alpha}^2 \ln \frac{1}{\mu_\alpha} \geq C \ , \tag{3.2}$$

where $C > 0$ is independent of α. Noting that (3.2) leads to a contradiction when combined with the estimate $r_{1,\alpha} = O(\sqrt{\mu_\alpha})$, this proves that $u_\infty \equiv 0$. Now we remark that by (3.1) and (2.2) in Proposition 2.1 there necessarily holds that $r_{1,\alpha} \to 0$ as $\alpha \to +\infty$. In particular, $\mathcal{A}_1 \neq \emptyset$, where $\mathcal{A}_1$ is as in (1.12). Moreover, by (2.1) in Proposition 2.1 and (3.1) we get that for any $i \in \mathcal{A}_1 \cup \{1\}$, there exists $C_i > 0$ such that

$$r_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}} \to C_i \tag{3.3}$$

as $\alpha \to +\infty$. By (3.3), for any $i \in \mathcal{A}_1 \cup \{1\}$,

$$\mu_{i,\alpha} = o\left(r_{i,\alpha}^2\right) . \tag{3.4}$$

It follows from (3.4) that for any $i \in \mathcal{A}_1 \cup \{1\}$, $\mathcal{A}_i \cap \mathcal{B}_i \neq \emptyset$, where the $\mathcal{B}_i$'s are as in (1.16). By the explicit formula (1.30),

$$\mathcal{H}_i(z) = \sum_{j \in \mathcal{A}_i \cap \mathcal{B}_i} \frac{\lambda_{i,j}}{|z - z_{i,j}|^2} , \tag{3.5}$$

where $\mathcal{H}_i$ is as in (1.29). Let $\mathcal{E}_1 = (\mathcal{A}_1 \cap \mathcal{B}_1) \cup \{1\}$. For any $i \in \mathcal{A}_1 \cap \mathcal{B}_1$, we have that $\mathcal{A}_i \cap \mathcal{B}_i = \mathcal{E}_1 \backslash \{i\}$. Let $i \in \mathcal{E}_1$ be such that

$$d_g\left(x_{1,\alpha}, x_{i,\alpha}\right) \geq d_g\left(x_{1,\alpha}, x_{j,\alpha}\right)$$

for all $j \in \mathcal{E}_1$ and all α. Then the $z_{i,j}$'s all lie in a ball whose boundary contains zero. In particular they all lie in a half space and we get that there exists $\nu_i \in \mathbb{R}^n$, $|\nu_i| = 1$, such that $\langle \nu_i, z_{i,j} \rangle > 0$ for all $j \in \mathcal{A}_i \cap \mathcal{B}_i$. By Proposition 2.1, $\nabla \mathcal{H}_i(0).(\nu_i) = 0$, and by (3.5) we have that

$$\nabla \mathcal{H}_i(0).(\nu_i) = 2 \sum_{j \in \mathcal{A}_i \cap \mathcal{B}_i} \frac{\lambda_{i,j}}{|z_{i,j}|^4} \langle \nu_i, z_{i,j} \rangle ,$$

a contradiction since $\lambda_{ij} > 0$. This ends the proof of Theorem 0.1.

4. Proof of Theorem 0.2

We prove Theorem 0.2 in this section. The proof is based on a finite-dimensional reduction method, the so-called Lyapounov-Schmidt method (introduced originally by Floer and Weinstein [8] in the one-dimensional case). An early reference on the Lyapounov-Schmidt method for a Sobolev critical equation is Rey [12]. Different techniques based on the finite dimensional reduction method have been developed for such problems such as, among others, the localized energy method (see del Pino, Felmer, and Musso [3], Rey and Wei [13], and Wei [16]). Here, we use the Lyapunov–Schmidt method with respect to the H^1-norm, as it is used, for instance, in Esposito, Pistoia, and Vétois [7] for the Yamabe equation. In our case, on the sphere, we restrict ourselves to radial functions in H^1. We let $(\mathbb{S}^4, g)$ be the round 4-sphere, $m_0, m_1, q > 0$ be positive real numbers such that $m_0^2 \geq 2$, and $\omega \in (-m_0, +m_0)$. We let Φ be the map given by (1.1). Then, see Hebey and Truong [10], Φ is locally Lipschitz and differentiable, and its differential $D\Phi(u) = V_u$ at u is given by

$$\Delta_g V_u(\varphi) + (m_1^2 + q^2 u^2) V_u(\varphi) = 2qu\left(1 - q\Phi(u)\right)\varphi \tag{4.1}$$

for all $\varphi \in H^1(M)$. There also holds that $0 \le \Phi \le \frac{1}{q}$ in H^1. Letting $I_\omega : H^1 \to \mathbb{R}$ be given by

$$\begin{aligned} I_\omega(u) = &\frac{1}{2}\int_{\mathbb{S}^4} |\nabla u|_g^2 dv_g + \frac{1}{2}(m_0^2 - \omega^2)\int_{\mathbb{S}^4} u^2 dv_g \\ &- \frac{1}{4}\int_{\mathbb{S}^4} u_+^4 dv_g + \frac{q}{2}\omega^2 \int_{\mathbb{S}^4} \Phi(u) u^2 dv_g \end{aligned} \tag{4.2}$$

for all $u \in H^1$, where $u_+ = \max(u, 0)$, it follows that I_ω is C^1 in H^1 and that its critical points $u \in H^1$ are such that

$$\Delta_g u + m_0^2 u = u_+^3 + \omega^2 \left(q\Phi(u) - 1\right)^2 u \ . \tag{4.3}$$

By the maximum principle, since $m_0^2 > \omega^2 \left(q\Phi(u) - 1\right)^2$, we then get that $u \ge 0$. In particular, $(u, \Phi(u))$ solves (0.2). We define

$$\varepsilon_\omega = 2 + \omega^2 - m_0^2 \tag{4.4}$$

so that $\varepsilon_\omega \to 0$ if and only if $\omega^2 \to m_0^2 - 2$. We fix $x_0 \in \mathbb{S}^4$, and we define the radial Sobolev space

$$H_r^1(\mathbb{S}^4) = \left\{u \in H^1(\mathbb{S}^4) \text{ s.t. } u \circ \sigma = u \text{ for all } \sigma \in G_{x_0}\right\} \ ,$$

where G_{x_0} is the group of rotations about x_0. In particular, $H_r^1 \subset H^1$ is a closed subspace of H^1. Similarly, for any $p \ge 1$, we define the radial Lebesgue space

$$L_r^p(\mathbb{S}^4) = \left\{u \in L^p(\mathbb{S}^4) \text{ s.t. } u \circ \sigma = u \text{ for all } \sigma \in G_{x_0}\right\} \ .$$

We let $\mathcal{L}_g$ be the conformal Laplacian on $(\mathbb{S}^4, g)$ given by $\mathcal{L}_g = \Delta_g + 2$. As is easily checked by using standard minimization technics, for any $f \in L_r^{4/3}$, there exists a unique $u = \mathcal{L}_g^{-1}(f)$, $u \in H_r^1$, such that $\mathcal{L}_g u = f$. Given $u \in H_r^1$ we define

$$f_\omega(u) = u_+^3 + \varepsilon_\omega u - q\omega^2 \left(2 - q\Phi(u)\right)\Phi(u) u \ , \tag{4.5}$$

where Φ is given by (1.1), and ε_ω is as in (4.4). Now, given $\delta > 0$, we let W_δ be the fundamental solution of the conformal critical equation. Then

$$W_\delta = \frac{\sqrt{2}\delta}{\sqrt{1+\delta^2} - \cos d_g(x_0, x)} \tag{4.6}$$

and, as is well known, W_δ satisfies that

$$\mathcal{L}_g W_\delta = W_\delta^3 \tag{4.7}$$

in $\mathbb{S}^4$ for all $\delta > 0$. There also holds, see for instance Druet, Hebey and Robert [6], that

$$W_\delta = \frac{\mu_\delta}{\mu_\delta^2 + \frac{d_g(x_0,\cdot)^2}{8}} + R_\delta \tag{4.8}$$

for all $\delta > 0$, where $\mu_\delta = \frac{\delta}{\sqrt{2}(1+\sqrt{1+\delta^2})}$, and $R_\delta \to 0$ in H^1 as $\delta \to 0$. We aim to construct solutions of (4.3) of the form $u = W_\delta + \psi_\delta$. We define $Z_\delta \in H_r^1$ by

$$Z_\delta = \delta \frac{dW_\delta}{d\delta} \ . \tag{4.9}$$

By Bianchi-Egnell [2], Z_δ is, up to the product by a constant, the only solution in H_r^1 of the linearized equation associated to (4.7), and thus the only solution in H_r^1 of

$$\mathcal{L}_g Z_\delta = 3W_\delta^2 Z_\delta \ . \tag{4.10}$$

We let $\langle \cdot, \cdot \rangle_{\mathcal{L}_g}$ be the scalar product associated with $\mathcal{L}_g$ so that

$$\langle u, v \rangle_{\mathcal{L}_g} = \int_{\mathbb{S}^4} \left((\nabla u \nabla v) + 2uv \right) dv_g \tag{4.11}$$

for all $u, v \in H_r^1$, and let $\| \cdot \|_{\mathcal{L}_g}$ be the corresponding norm. Then we define

$$Z_\delta^\perp = \left\{ u \in H_r^1 \text{ s.t. } \langle u, Z_\delta \rangle_{\mathcal{L}_g} = 0 \right\} \ , \tag{4.12}$$

and let $\Pi_\delta^\perp : H_r^1 \to Z_\delta^\perp$ be the projection onto $Z_\delta^\perp$. It follows that

$$\Pi_\delta^\perp u = u - \frac{\langle u, Z_\delta \rangle_{\mathcal{L}_g}}{\| Z_\delta \|_{\mathcal{L}_g}^2} Z_\delta$$

for all u. By (4.7) and (4.10), $W_\delta \in Z_\delta^\perp$ for all $\delta > 0$. At last we define $L_{\delta,\omega}$, $N_{\delta,\omega}$, and $R_{\delta,\omega}$ by

$$\begin{aligned} L_{\delta,\omega}(u) &= \Pi_\delta^\perp \left(u - \mathcal{L}_g^{-1} \left(D f_\omega(W_\delta).u \right) \right) \ , \\ N_{\delta,\omega} &= \Pi_\delta^\perp \left(\mathcal{L}_g^{-1} \left(f_\omega(W_\delta + u) - f_\omega(W_\delta) - D f_\omega(W_\delta).u \right) \right) \ , \\ R_{\delta,\omega} &= \Pi_\delta^\perp \left(\mathcal{L}_g^{-1} \left(f_\omega(W_\delta) \right) - W_\delta \right) \end{aligned} \tag{4.13}$$

for all $u \in H_r^1$, where f_ω is given by (4.5). We split the proof of Theorem 4 in several lemmas. The first lemma consists in the following rewriting of equation (4.3).

Lemma 4.1. *A function $\psi \in Z_\delta^\perp$ is such that $u = W_\delta + \psi$ solves* (4.3) *in $Z_\delta^\perp$ if and only if*

$$L_{\delta,\omega}(\psi) = N_{\delta,\omega}(\psi) + R_{\delta,\omega} \ , \tag{4.14}$$

where $L_{\delta,\omega}$, $N_{\delta,\omega}$, and $R_{\delta,\omega}$ are as in (4.13).

Proof of Lemma 4.1. Let $u = W_\delta + \psi_\delta$. As we easily check, ψ satisfies (4.14) if and only if $\Pi_\delta^\perp \left(u - \mathcal{L}_g^{-1} \left(f_\omega(u) \right) \right) = 0$, and thus if and only if $u - \mathcal{L}_g^{-1} \left(f_\omega(u) \right) = \alpha Z_\delta$, where $\alpha \in \mathbb{R}$. In particular, ψ satisfies (4.14) if and only if

$$\Delta_g u + m_0^2 u = u_+^3 + \omega^2 \left(q \Phi(u) - 1 \right)^2 u + \alpha \mathcal{L}_g(Z_\delta) \ ,$$

and thus if and only if u solves (4.3) in $Z_\delta^\perp$. This ends the proof of Lemma 4.1. □

Now we prove that the following estimate holds true.

Lemma 4.2. *There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any $\varepsilon \in (0, \varepsilon_0)$, $L_{\delta,\omega} : Z_\delta^\perp \to Z_\delta^\perp$ is invertible. Moreover, there exists $C > 0$, independent of δ and ω, such that*

$$\| L_{\delta,\omega}(\psi) \|_{\mathcal{L}_g} \geq C \| \psi \|_{\mathcal{L}_g} \tag{4.15}$$

for all $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, and $\psi \in Z_\delta^\perp$.

Proof of Lemma 4.2. In order to apply the Fredholm alternative, we show that for any $\delta > 0$ and $\omega \in \mathbb{R}$ (fixed), the map $\widetilde{L}_{\delta,\omega} : \psi \mapsto \psi - L_{\delta,\omega}(\psi)$ is compact. We let $(\psi_\alpha)_\alpha$ be a bounded sequence in $Z_\delta^\perp$. For any $\alpha \in \mathbb{N}$, we define the function $\varphi_\alpha = \mathcal{L}_g^{-1} \left(D f_\omega(W_\delta).\psi_\alpha \right)$ so that $\widetilde{L}_{\delta,\omega}(\psi_\alpha) = \Pi_\delta^\perp \varphi_\alpha$. Since $(\psi_\alpha)_{\alpha \in \mathbb{N}}$ is bounded in H^1, we get from (4.1) that $\left(D f_\omega(W_\delta).\psi_\alpha \right)_\alpha$ is bounded in H^1. By elliptic regularity theory, it follows that $(\varphi_\alpha)_{\alpha \in \mathbb{N}}$ is compact in H^1, and thus that $(\widetilde{L}_{\delta,\omega}(\psi_\alpha))_\alpha$ is compact in $Z_\delta^\perp$. This proves that the map $\widetilde{L}_{\delta,\omega}$ is compact. Now, by the Fredholm alternative, in order to get the invertibility of $L_{\delta,\omega}$, it suffices to prove that the kernel of $L_{\delta,\omega}$ is reduced to $\{0\}$, which is a consequence of (4.15). We prove (4.15) by contradiction. We assume that there exist $(\delta_\alpha)_\alpha$, $(\omega_\alpha)_\alpha$, and $(\psi_\alpha)_\alpha$ such that $\delta_\alpha \to 0$ and $\varepsilon_{\omega_\alpha} \to 0$ as $\alpha \to +\infty$, such that $\psi_\alpha \in Z_{\delta_\alpha}^\perp$ and $\|\psi_\alpha\|_{\mathcal{L}_g} = 1$ for all α, and such that

$$\|L_{\delta_\alpha,\omega_\alpha}(\psi_\alpha)\|_{\mathcal{L}_g} \to 0 \tag{4.16}$$

as $\alpha \to +\infty$. We claim that

$$\left\|\psi_\alpha - \mathcal{L}_g^{-1}\left(Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right)\right\|_{\mathcal{L}_g} = o(1)\ . \tag{4.17}$$

By (4.16), in order to obtain (4.17) it suffices to prove that

$$\left\langle \psi_\alpha - \mathcal{L}_g^{-1}\left(Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right), Z_{\delta_\alpha}\right\rangle_{\mathcal{L}_g} = o(1) \tag{4.18}$$

By (4.10) we get that

$$\left\langle \psi_\alpha - \mathcal{L}_g^{-1}\left(Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right), Z_{\delta_\alpha}\right\rangle_{\mathcal{L}_g} = \int_{\mathbb{S}^4}\left(3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right)Z_{\delta_\alpha}dv_g\ ,$$

and by Hölder's inequality and Lemma 4.3 it follows that

$$\begin{aligned}
&\left|\left\langle \psi_\alpha - \mathcal{L}_g^{-1}\left(Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right), Z_{\delta_\alpha}\right\rangle_{\mathcal{L}_g}\right| \\
&= O\left(\|3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\|_{L^{4/3}}\|Z_{\delta_\alpha}\|_{L^4}\right) \\
&= O\left(\|3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\|_{L^{4/3}}\right)\ .
\end{aligned} \tag{4.19}$$

By the definition (4.5) of f_{ω_α} there holds that

$$\begin{aligned}
&3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha \\
&\quad = -\varepsilon_{\omega_\alpha}\psi_\alpha + q\omega_\alpha^2\left(2 - q\Phi(W_{\delta_\alpha})\right)\Phi(W_{\delta_\alpha})\psi_\alpha \\
&\qquad + 2q\omega_\alpha^2\left(1 - q\Phi(W_{\delta_\alpha})\right)W_{\delta_\alpha}D\Phi(W_{\delta_\alpha}).\psi_\alpha
\end{aligned}$$

and since $0 \le \Phi(W_{\delta_\alpha}) \le \frac{1}{q}$, $\varepsilon_{\omega_\alpha} \to 0$ as $\alpha \to +\infty$, $\|\psi_\alpha\|_{\mathcal{L}_g} = 1$ for all α, and $\|W_\delta\|_{L^2} \to 0$ as $\delta \to 0$, it follows that

$$\begin{aligned}
&\left\|3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right\|_{L^{4/3}} \\
&= o\left(\|\psi_\alpha\|_{L^{4/3}}\right) + O\left(\|\Phi(W_{\delta_\alpha})\|_{H^1}\|\psi_\alpha\|_{H^1}\right) \\
&\quad + O\left(\|W_{\delta_\alpha}D\Phi(W_{\delta_\alpha}).\psi_\alpha\|_{L^{4/3}}\right) \\
&= o\left(\|\psi_\alpha\|_{H^1}\right) + O\left(\|\Phi(W_{\delta_\alpha})\|_{H^1}\|\psi_\alpha\|_{H^1}\right) \\
&\quad + O\left(\|W_{\delta_\alpha}\|_{L^2}\|D\Phi(W_{\delta_\alpha}).\psi_\alpha\|_{H^1}\right) \\
&= o(1) + O\left(\|\Phi(W_{\delta_\alpha})\|_{H^1}\right) + o\left(\|D\Phi(W_{\delta_\alpha}).\psi_\alpha\|_{H^1}\right)\ .
\end{aligned} \tag{4.20}$$

By (1.1) and (4.1),

$$\|\Phi(W_{\delta_\alpha})\|_{H^1} = O\left(\|W_{\delta_\alpha}\|_{L^{8/3}}^2\right) = o(1) \tag{4.21}$$

and

$$\|D\Phi(W_{\delta_\alpha}).\psi_\alpha\|_{H^1} = O\left(\|W_{\delta_\alpha}\|_{L^2}\|\psi_\alpha\|_{H^1}\right) = o\left(\|\psi_\alpha\|_{H^1}\right) = o(1)\ . \tag{4.22}$$

By (4.21) and (4.22), coming back to (4.20), we obtain that

$$\left\|3W_{\delta_\alpha}^2\psi_\alpha - Df_{\omega_\alpha}(W_{\delta_\alpha}).\psi_\alpha\right\|_{L^{4/3}} = o(1)\ . \tag{4.23}$$

Then, by (4.19) and (4.23), we deduce (4.18), and thus also (4.17). Then, by (4.17) and (4.23), for any sequence $(\varphi_\alpha)_\alpha$ in H^1,

$$\langle \psi_\alpha, \varphi_\alpha \rangle_{\mathcal{L}_g} = 3 \int_{\mathbb{S}^4} W_{\delta_\alpha}^2 \psi_\alpha \varphi_\alpha dv_g + o\left(\|\varphi_\alpha\|_{H^1}\right) . \tag{4.24}$$

Given $\varphi \in C_c^\infty(\mathbb{R}^4)$, we apply (4.24) with φ_α given by

$$\varphi_\alpha(x) = \frac{1}{\delta_\alpha} \varphi\left(\frac{1}{\delta_\alpha} \exp_{x_0}^{-1} x\right)$$

for all $x \in \mathbb{S}^4$. In particular, $\|\varphi_\alpha\|_{H^1} = O(1)$, and by using (4.8), we obtain from (4.24) that

$$\int_{\mathbb{R}^4} (\nabla \tilde{\psi}_\alpha \nabla \varphi) dx = 24 \int_{\mathbb{R}^4} \frac{\tilde{\psi}_\alpha \varphi}{(1+|x|^2)^2} dx + o(1) , \tag{4.25}$$

where $\tilde{\psi}_\alpha$ is given by

$$\tilde{\psi}_\alpha(x) = \delta_\alpha \psi_\alpha\left(\exp_{x_0}(\delta_\alpha x)\right) .$$

Since $\|\psi_\alpha\|_{H^1} = O(1)$, there holds that $\|\nabla \tilde{\psi}_\alpha\|_{L^2} = O(1)$, and we deduce that, up to a subsequence, $\tilde{\psi}_\alpha \rightharpoonup \tilde{\psi}_\infty$ in $\dot{H}^1$ as $\alpha \to +\infty$. Passing into the limit in (4.25), and since (4.25) holds for all $\varphi \in C_c^\infty(\mathbb{R}^4)$, we obtain that $\tilde{\psi}_\infty$ solves the equation

$$\Delta \tilde{\psi}_\infty = \frac{24}{(1+|x|^2)^2} \tilde{\psi}_\infty \tag{4.26}$$

in $\mathbb{R}^4$. We have that $\psi_\alpha \in H_r^1$ by the definition of $Z_{\delta_\alpha}^\perp$ in (4.12), and thus $\tilde{\psi}_\infty$ is radially symmetric. By Bianchi and Egnell, the sole radially symmetric solutions of (4.26) are of the form

$$\tilde{\psi}_\infty(x) = \frac{\lambda(|x|^2-1)}{(1+|x|^2)^2} \tag{4.27}$$

for some $\lambda \in \mathbb{R}$. At this point we claim that $\lambda = 0$. Since $\psi_\alpha \in Z_{\delta_\alpha}^\perp$, and Z_{δ_α} satisfies (4.10), we have that

$$\int_{\mathbb{S}^4} W_{\delta_\alpha}^2 Z_{\delta_\alpha} \psi_\alpha dv_g = 0 \tag{4.28}$$

for all α. Rescaling (4.28), splitting the integral over $B_0(R)$ and $\mathbb{R}^4 \backslash B_0(R)$, using Lemma 4.3 for the noncompact part of the integral, we obtain that

$$\lim_{R\to+\infty} \int_{B_0(R)} \frac{(|x|^2-1)\tilde{\psi}_\infty(x)}{(1+|x|^2)^4} dx = 0 . \tag{4.29}$$

Then, combining (4.27) and (4.29), it follows that $\lambda = 0$ and then that $\tilde{\psi}_\infty \equiv 0$. Finally we apply (4.24) with $\varphi_\alpha = \psi_\alpha$. We obtain that

$$\|\psi_\alpha\|_{\mathcal{L}_g}^2 = \int_{\mathbb{S}^4} W_{\delta_\alpha}^2 \psi_\alpha^2 dv_g + o(1) .$$

We may assume that $\tilde{\psi}_\alpha \to \tilde{\psi}_\infty$ in L^2 of any compact subset of $\mathbb{R}^4$. Here again, by rescaling and splitting the integral over $B_0(R)$ and $\mathbb{R}^4 \backslash B_0(R)$, and since we just got that $\tilde{\psi}_\infty \equiv 0$, we obtain that $\|\psi_\alpha\|_{\mathcal{L}_g}^2 \to 0$ as $\alpha \to +\infty$, in obvious contradiction with the fact that $\|\psi_\alpha\|_{\mathcal{L}_g} = 1$ for all α. This proves (4.15) and Lemma 4.2. □

The following estimate on the $\mathcal{L}_g$-norm of Z_δ was used in the proof of Lemma 4.2. It will be used again in the concluding argument in the proof of Theorem 0.2. We state it as a lemma.

Lemma 4.3. *There holds that*

$$\|Z_\delta\|_{\mathcal{L}_g} = \frac{16\omega_3}{5}(1+o(1))$$

as $\delta \to 0$*, where* Z_δ *is given by* (4.9)*, and* ω_3 *is the volume of the round* 3*-sphere.*

Proof of Lemma 4.3. Let $\beta = \sqrt{1+\delta^2}$. We compute

$$Z_\delta = \frac{\sqrt{2}\delta(1-\beta\cos r)}{\beta(\beta-\cos r)^2} ,$$

where $r = d_g(x_0,\cdot)$. Independently, by (4.10), we obtain that

$$\|Z_\delta\|^2_{\mathcal{L}_g} = 3\int_{\mathcal{S}^4} W_\delta^2 Z_\delta^2 dv_g .$$

The result follows from direct computations noting that in geodesic normal coordinates $r^3 dv_g = (\sin r)^3 dx$. This ends the proof of Lemma 4.3. □

Now we define the map $T_{\delta,\omega} : Z_\delta^\perp \to Z_\delta^\perp$ given by

$$T_{\delta,\omega}(\psi) = L^{-1}_{\delta,\omega}\left(N_{\delta,\omega}(\psi) + R_{\delta,\omega}\right) \tag{4.30}$$

for all $\psi \in Z_\delta^\perp$. By Lemma 4.1, solving (4.3) in $Z_\delta^\perp$ amounts to finding a fixed point of $T_{\delta,\omega}$.

Lemma 4.4. *There exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\delta \in (0,\delta_0)$ *and any* $\varepsilon \in (0,\varepsilon_0)$*, the map* $T_{\delta,\omega}$ *in* (4.30) *has a fixed point* $\psi_{\delta,\omega} \in Z_\delta^\perp$*. Moreover, there exists* $C > 0$ *such that*

$$\|\psi_{\delta,\omega}\|_{H^1} \le C\left(\varepsilon_\omega\delta + \delta^2\sqrt{|\ln\delta|}\right) \tag{4.31}$$

for all $\delta \in (0,\delta_0)$ *and all* $\varepsilon \in (0,\varepsilon_0)$*. In addition, the map* $(\delta,\omega) \to \psi_{\delta,\omega}$*, with values in* H^1*, is* C^1 *with respect to* δ *and* ω*.*

Proof of Lemma 4.4. First we claim that there exists $C > 0$ such that

$$\|R_{\delta,\omega}\|_{H^1} \le C\left(\varepsilon_\omega\delta + \delta^2\sqrt{|\ln\delta|}\right) \tag{4.32}$$

for all $\delta > 0$ and all $\varepsilon > 0$. It is easily checked that there exists $C > 0$ such that

$$\|\varphi\|_{H^1} \le C\|\mathcal{L}_g\varphi\|_{L^{4/3}}$$

for all $\varphi \in H^1$. Since W_δ solves (4.7), $\|\Pi_\delta^\perp\| \le 1$, and $0 \le \Phi(W_\delta) \le \frac{1}{q}$,we then get that

$$\begin{aligned}\|R_{\delta,\omega}\|_{H^1} &\le C\,\|f_\omega(W_\delta) - \mathcal{L}_g W_\delta\|_{L^{4/3}}\\ &\le C\left(\varepsilon_\omega\|W_\delta\|_{L^{4/3}} + \|\Phi(W_\delta)W_\delta\|_{L^{4/3}}\right)\end{aligned} \tag{4.33}$$

for all $\delta > 0$ and $\varepsilon_\omega > 0$, where $C > 0$ is independent of δ and ε_ω. There holds that

$$\|\Phi(W_\delta)W_\delta\|_{L^{4/3}} = O\left(\|\Phi(W_\delta)\|_{H^1}\|W_\delta\|_{L^2}\right) = O\left(\|W_\delta\|^2_{L^{8/3}}\|W_\delta\|_{L^2}\right) \tag{4.34}$$

by (4.21). Direct computations give that

$$\|W_\delta\|_{L^{4/3}} = O(\delta)\ ,\ \|W_\delta\|_{L^2} = O\left(\delta\sqrt{|\ln\delta|}\right)\ ,\ \|W_\delta\|_{L^{8/3}} = O\left(\sqrt{\delta}\right) . \tag{4.35}$$

Then (4.32) follows from (4.33)–(4.35). Now, we aim to apply the fixed point theorem to the map $T_{\delta,\omega}$ in the set

$$B_{\delta,\omega}(\Lambda) = \left\{\psi \in Z_\delta^\perp \text{ s.t. } \|\psi\|_{H^1} \le \Lambda\left(\varepsilon_\omega\delta + \delta^2\sqrt{|\ln\delta|}\right)\right\}$$

for some $\Lambda > 0$. By Lemma 4.2, there exist $\delta_0 > 0$, $\varepsilon_0 > 0$, and $C > 0$ such that

$$\|T_{\delta,\omega}(\psi)\|_{H^1} \le C\left(\|N_{\delta,\omega}(\psi)\|_{H^1} + \|R_{\delta,\omega}\|_{H^1}\right) \tag{4.36}$$

for all $\delta \in (0, \delta_0)$, $\varepsilon_\omega \in (0, \varepsilon_0)$, and all $\psi \in Z_\delta^\perp$, and such that

$$\|T_{\delta,\omega}(\psi_1) - T_{\delta,\omega}(\psi_2)\|_{H^1} \le C\|N_{\delta,\omega}(\psi_1) - N_{\delta,\omega}(\psi_2)\|_{H^1} \tag{4.37}$$

for all $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$, and all $\psi_1, \psi_2 \in Z_\delta^\perp$. Direct computations, using Hölder's inequalities, and estimates like in (4.21) and (4.22), then give that there exists $\varepsilon_\delta > 0$, $\varepsilon_\delta = \varepsilon_\delta(\Lambda)$, such that $\varepsilon_\delta \to 0$ as $\delta \to 0$, and such that

$$\|N_{\delta,\omega}(\psi_1) - N_{\delta,\omega}(\psi_2)\|_{H^1} \le \varepsilon_\delta \|\psi_1 - \psi_2\|_{H^1} \tag{4.38}$$

for all $\delta \in (0, \delta_0)$, $\varepsilon_\omega \in (0, \varepsilon_0)$, and $\psi_1, \psi_2 \in B_{\delta,\omega}(\Lambda)$. By (4.32), (4.36), and (4.38), taking $\psi_2 \equiv 0$ and $\psi_1 = \psi$ in (4.38), we get there exists $\delta_0 > 0$ small, $\varepsilon_0 > 0$ small, and $\Lambda > 0$ sufficiently large, such that

$$T_{\delta,\omega} : B_{\delta,\omega}(\Lambda) \to B_{\delta,\omega}(\Lambda)$$

for all $\delta \in (0, \delta_0)$ and $\varepsilon_\omega \in (0, \varepsilon_0)$. In addition, (4.37) and (4.38) give that $T_{\delta,\omega}$ is a contraction map in $B_{\delta,\omega}(\Lambda)$. By the fixed point theorem it follows that $T_{\delta,\omega}$ admits a fixed point $\psi_{\delta,\omega} \in B_{\delta,\omega}(\Lambda)$. This proves both the existence of the fixed point and (4.31). It remains to prove that $\psi_{\delta,\omega}$ is C^1 with respect to δ and ω. To be more precise, we prove the regularity of $\psi_{\delta,\omega}$ with respect to δ and ε_ω by applying the implicit function theorem to the map $\mathcal{G} : (0, +\infty)^2 \times H^1 \to H^1$ given by

$$\mathcal{G}(\delta, \varepsilon_\omega, \psi) = \psi - \Pi_\delta^\perp \psi - \Pi_\delta^\perp\left(\mathcal{L}_g^{-1}\left(f_\omega(\Pi_\delta W_\delta + \Pi_\delta^\perp \psi)\right)\right)$$

for all $\psi \in H^1$. It is easily checked that $\mathcal{G}$ is C^1 with respect to its three variables, while we just proved that $\mathcal{G}(\delta, \varepsilon_\omega, W_\delta + \psi_{\delta,\omega}) = 0$ when $\delta > 0$ and $\varepsilon_\omega > 0$ are sufficiently small. There holds that

$$D_\psi \mathcal{G}(\delta, \varepsilon_\omega, W_\delta + \psi_{\delta,\omega}).(v) = v - \Pi_\delta^\perp\left(\mathcal{L}_g^{-1}\left(Df_\omega(W_\delta + \psi_{\delta,\omega}).\Pi_\delta^\perp v\right)\right)$$

for all $v \in H^1$. Using the Fredholm alternative, we obtain the invertibility of the differential $D_\psi \mathcal{G}(\delta, \varepsilon_\omega, W_\delta + \psi_{\delta,\omega})$ if we prove that its kernel is reduced to $\{0\}$. Direct computations, similar to the ones needed to get (4.38), give that there exists $C > 0$ such that

$$\|D_\psi \mathcal{G}(\delta, \varepsilon_\omega, W_\delta + \psi_{\delta,\omega}).(v)\|_{H^1} \ge C\|v\|_{H^1}$$

for all $v \in H^1$. In particular, $\mathrm{Ker} D_\psi \mathcal{G}(\delta, \varepsilon_\omega, W_\delta + \psi_{\delta,\omega}) = \{0\}$, and by the implicit function theorem, this ends the proof Lemma 4.4. ☐

Now we define the reduced functional $\mathcal{I}_\omega$ by

$$\mathcal{I}_\omega(\delta) = I_\omega(W_\delta + \psi_{\delta,\omega}) , \tag{4.39}$$

where I_ω is as in (4.2) and $\psi_{\delta,\omega}$ is given by Lemma 4.4. We prove that the following holds true.

Lemma 4.5. *There holds that*

$$\mathcal{I}_\omega(\delta) = I_\omega(W_\delta) + O\left(\varepsilon_\omega^2 \delta^2 + \delta^4 |\ln \delta|\right) \tag{4.40}$$

for all $\delta > 0$ small and $\varepsilon_\omega > 0$ small.

Proof of Lemma 4.5. Using that W_δ is a solution of (4.7), we get that

$$\begin{aligned}\mathcal{I}_\omega(\delta) - I_\omega(W_\delta) &= \frac{1}{2}\int_{\mathbb{S}^4} |\nabla\psi_{\delta,\omega}|^2 dv_g + \frac{1}{2}(2-\varepsilon_\omega)\int_{\mathbb{S}^4} \psi_{\delta,\omega}^2 dv_g \\ &- \varepsilon_\omega \int_{\mathbb{S}^4} W_\delta \psi_{\delta,\omega} dv_g \\ &- \frac{1}{4}\int_{\mathbb{S}^4} \left((W_\delta+\psi_{\delta,\omega})_+^4 - W_\delta^4 - 4W_\delta^3\psi_{\delta,\omega}\right) dv_g \\ &+ \frac{q}{2}\omega^2 \int_{\mathbb{S}^4} \left(\Phi(W_\delta+\psi_{\delta,\omega}) - \Phi(W_\delta)\right)(W_\delta+\psi_{\delta,\omega})^2 dv_g \\ &+ \frac{q}{2}\omega^2 \int_{\mathbb{S}^4} \Phi(W_\delta)\psi_{\delta,\omega}^2 dv_g + q\omega^2 \int_{\mathbb{S}^4} \Phi(W_\delta) W_\delta \psi_{\delta,\omega} dv_g \;.\end{aligned} \tag{4.41}$$

For any $a > 0$ and b real numbers, $|(a+b)_+^4 - a^4 - 4a^3 b| \le C(a^2+b^2)b^2$ for some $C > 0$ independent of a and b. By Hölder's and Sobolev inequalities, it follows that

$$\begin{aligned}&\int_{\mathbb{S}^4} \left|(W_\delta+\psi_{\delta,\omega})_+^4 - W_\delta^4 - 4W_\delta^3\psi_{\delta,\omega}\right| dv_g \\ &= O\left((\|W_\delta\|_{H^1}^2 + \|\psi_{\delta,\varepsilon}\|_{H^1}^2)\|\psi_{\delta,\omega}\|_{H^1}^2\right) \\ &= O\left(\|\psi_{\delta,\omega}\|_{H^1}^2\right) \;.\end{aligned} \tag{4.42}$$

We also have that

$$\frac{1}{2}\int_{\mathbb{S}^4} |\nabla\psi_{\delta,\omega}|^2 dv_g + \frac{1}{2}(2-\varepsilon_\omega)\int_{\mathbb{S}^4} \psi_{\delta,\omega}^2 dv_g = O\left(\|\psi_{\delta,\omega}\|_{H^1}^2\right) \;, \tag{4.43}$$

and by Hölder's and Sobolev inequalities, we get that

$$\int_{\mathbb{S}^4} |W_\delta\psi_{\delta,\omega}|\, dv_g = O\left(\|W_\delta\|_{L^{4/3}}\|\psi_{\delta,\omega}\|_{H^1}\right) \;, \tag{4.44}$$

that

$$\begin{aligned}&\int_{\mathbb{S}^4} \left|\left(\Phi(W_\delta+\psi_{\delta,\omega}) - \Phi(W_\delta)\right)\right| (W_\delta+\psi_{\delta,\omega})^2 dv_g \\ &= O\left(\|\Phi(W_\delta+\psi_{\delta,\omega}) - \Phi(W_\delta)\|_{H^1}\left(\|W_\delta\|_{L^{8/3}}^2 + \|\psi_{\delta,\omega}\|_{H^1}^2\right)\right) \;,\end{aligned} \tag{4.45}$$

and that

$$\begin{aligned}&\int_{\mathbb{S}^4} \left|\Phi(W_\delta)\psi_{\delta,\omega}^2 + q\omega^2\Phi(W_\delta)W_\delta\psi_{\delta,\omega}\right| dv_g \\ &= O\left(\|\Phi(W_\delta)\|_{H^1}\|\psi_{\delta,\omega}\|_{H^1}\left(\|\psi_{\delta,\omega}\|_{L^2} + \|W_\delta\|_{L^2}\right)\right) \;.\end{aligned} \tag{4.46}$$

By (1.1),

$$\|\Phi(W_\delta+\psi_{\delta,\omega}) - \Phi(W_\delta)\|_{H^1} = O\left((\|W_\delta\|_{L^2} + \|\psi_{\delta,\omega}\|_{L^2})\|\psi_{\delta,\omega}\|_{H^1}\right) \;, \tag{4.47}$$

and it follows from (4.21), (4.35), and (4.41)–(4.47) that

$$\mathcal{I}_\omega(\delta) - I_\omega(W_\delta) = O\left(\|\psi_{\delta,\omega}\|_{H^1}^2 + \left(\varepsilon_\omega\delta + \delta^2\sqrt{|\ln\delta|}\right)\|\psi_{\delta,\omega}\|_{H^1}\right) \;. \tag{4.48}$$

Then (4.40) follows from (4.31) and (4.48). This ends the proof of Lemma 4.5. □

Now we compute $I_\omega(W_\delta)$.

Lemma 4.6. *There holds that*

$$I_\omega(W_\delta) = \frac{8}{3}\pi^2 - 8\pi^2\varepsilon_\omega\delta^2|\ln\delta| + \frac{q}{2}\omega^2\int_{\mathbb{S}^4}\Phi(W_\delta)W_\delta^2dv_g + o\left(\varepsilon_\omega\delta^2|\ln\delta|\right) \tag{4.49}$$

as $\delta \to 0$ uniformly with respect to ω, and there exists $C > 1$ such that

$$\frac{1}{C}\delta^2 \le \int_{\mathbb{S}^4}\Phi(W_\delta)W_\delta^2dv_g \le C\delta^2 \tag{4.50}$$

for all $\delta > 0$ small.

Proof of Lemma 4.6. By (4.7),

$$I_\omega(W_\delta) = \frac{1}{4}\int_{\mathbb{S}^4}W_\delta^4dv_g - \frac{1}{2}\varepsilon_\omega\int_{\mathbb{S}^4}W_\delta^2dv_g + \frac{q}{2}\omega^2\int_{\mathbb{S}^4}\Phi(W_\delta)W_\delta^2dv_g\,. \tag{4.51}$$

Direct computations give that

$$\int_{\mathbb{S}^4}W_\delta^4dv_g = \frac{32}{3}\pi^2 \quad\text{and}\quad \int_{\mathbb{S}^4}W_\delta^2dv_g = 16\pi^2\delta^2|\ln\delta| + o(\delta^2|\ln\delta|)\,. \tag{4.52}$$

Then (4.49) follows from (4.51) and (4.52). Independently, testing equation (1.1) with $u = W_\delta$ against $\Phi(W_\delta)$, we get by Hölder's and Sobolev inequalities that

$$\|\Phi(W_\delta)\|_{H^1}^2 = O\left(\int_{\mathbb{S}^4}\Phi(W_\delta)W_\delta^2dv_g\right) = O\left(\|W_\delta\|_{L^{8/3}}^2\|\Phi(W_\delta)\|_{H^1}\right) \tag{4.53}$$

and testing (1.1) with $u = W_\delta$ against W_δ, we get that

$$\begin{aligned}\int_{\mathbb{S}^4}W_\delta^3dv_g &= O\left(\|W_\delta\|_{H^1}\|\Phi(W_\delta)\|_{H^1} + \|W_\delta\|_{L^4}^3\|\Phi(W_\delta)\|_{L^4}\right)\\ &= O\left(\|\Phi(W_\delta)\|_{H^1}\right)\,.\end{aligned} \tag{4.54}$$

Noting that there exists $C > 0$ such that $\int_{\mathbb{S}^4}W_\delta^3dv_g \ge C\delta$ for all $\delta > 0$ small, it follows from (4.35), (4.53), and (4.54) that (4.50) holds true. This ends the proof of Lemma 4.6. ☐

Now, we prove the following lemma establishing the relation between being a critical point of the reduced functional $\mathcal{I}_\omega$ and getting a solution $u_{\delta,\omega} = W_\delta + \psi_{\delta,\omega}$ of (4.3).

Lemma 4.7. *If $\delta > 0$ and $\varepsilon_\omega > 0$ are small and $\mathcal{I}'(\delta) = 0$, then $u_{\delta,\omega} = W_\delta + \psi_{\delta,\omega}$ is a solution of* (4.3)*.*

Proof of Lemma 4.7. We let $(\delta_\alpha)_\alpha$ and $(\omega_\alpha)_\alpha$ be such that $\mathcal{I}'_{\omega_\alpha}(\delta_\alpha) = 0$ and $\varepsilon_{\omega_\alpha} > 0$ for all α, and such that $\delta_\alpha \to 0$ and $\varepsilon_{\omega_\alpha} \to 0$ as $\alpha \to +\infty$. By Lemmas 4.1 and 4.4, if we let

$$u_\alpha = W_{\delta_\alpha} + \psi_{\delta_\alpha,\omega_\alpha}\,,$$

then

$$u_\alpha - \mathcal{L}_g^{-1}\left(f_{\omega_\alpha}(u_\alpha)\right) = \lambda_\alpha Z_{\delta_\alpha}$$

for some $\lambda_\alpha \in \mathbb{R}$, and u_α solves (4.3) if and only if $\lambda_\alpha = 0$ (the inequality $u_\alpha \ge 0$ follows from the maximum principle when $\lambda_\alpha = 0$). There holds that

$$\mathcal{I}'_{\omega_\alpha}(\delta_\alpha) = \lambda_\alpha\Big\langle Z_{\delta_\alpha}, \frac{du_{\delta,\omega_\alpha}}{d\delta}|_{\delta=\delta_\alpha}\Big\rangle_{\mathcal{L}_g}\,. \tag{4.55}$$

In particular, by (4.55), we get that either u_α solves (4.3), or

$$\Big\langle Z_{\delta_\alpha}, \frac{du_{\delta,\omega_\alpha}}{d\delta}|_{\delta=\delta_\alpha}\Big\rangle_{\mathcal{L}_g} = 0\,. \tag{4.56}$$

We assume (4.56) by contradiction. Since $\psi_{\delta,\omega_\alpha} \in Z_\delta^\perp$ for all $\delta > 0$, we get by (4.31) that

$$\begin{aligned}\left\langle Z_{\delta_\alpha}, \frac{d\psi_{\delta,\omega_\alpha}}{d\delta}|_{\delta=\delta_\alpha}\right\rangle_{\mathcal{L}_g} &= -\left\langle \frac{dZ_\delta}{d\delta}|_{\delta=\delta_\alpha}, \psi_{\delta_\alpha,\omega_\alpha}\right\rangle_{\mathcal{L}_g} \\ &= O\left(\|\frac{dZ_\delta}{d\delta}|_{\delta=\delta_\alpha}\|_{H^1}\|\psi_{\delta_\alpha,\omega_\alpha}\|_{H^1}\right) .\end{aligned} \tag{4.57}$$

Direct computations give that

$$\left\|\frac{dZ_\delta}{d\delta}|_{\delta=\delta_\alpha}\right\|_{H^1} = O\left(\frac{1}{\delta_\alpha^2}\right) . \tag{4.58}$$

Noting that

$$\left\langle Z_{\delta_\alpha}, \frac{du_{\delta,\omega_\alpha}}{d\delta}|_{\delta=\delta_\alpha}\right\rangle_{\mathcal{L}_g} = \frac{1}{\delta_\alpha}\|Z_{\delta_\alpha}\|_{H^1}^2 + \left\langle Z_{\delta_\alpha}, \frac{d\psi_{\delta,\omega_\alpha}}{d\delta}|_{\delta=\delta_\alpha}\right\rangle_{\mathcal{L}_g} ,$$

we then get by (4.31), and by (4.56)–(4.58), that

$$\|Z_{\delta_\alpha}\|_{H^1}^2 = O\left(\frac{1}{\delta_\alpha}\|\psi_{\delta_\alpha,\omega_\alpha}\|_{H^1}\right) = o(1) . \tag{4.59}$$

The contradiction follows from Lemma 4.3 and (4.59). This ends the proof of Lemma 4.7. □

At this point we are ready to prove Theorem 0.2. This is the subject of what follows.

Proof of Theorem 0.2. We may assume that $m_0^2 > 2$. In case $m_0^2 = 2$, then $\omega = 0$, and letting $\omega_\alpha = 0$ we get the desired sequence of solutions with $u_\alpha = W_{\delta_\alpha}$ for any sequence $(\delta_\alpha)_\alpha$ of positive real numbers converging to zero. Now, for any $t > 0$ and $\omega \in \mathbb{R}$ such that $\varepsilon_\omega > 0$, where ε_ω is as in (4.4), we let

$$\delta_\omega(t) = e^{\frac{-1}{\varepsilon_\omega}t} . \tag{4.60}$$

In particular, $\delta_\omega(t) \to 0$ uniformly in compact subsets of $(0, +\infty)$ as $\varepsilon_\omega \to 0$. By (4.40) and (4.49) in Lemmas 4.5 and 4.6 there holds that

$$\mathcal{I}_\omega\left(\delta_\omega(t)\right) = \frac{8}{3}\pi^2 + E_\omega(t) + o\left(e^{\frac{-2t}{\varepsilon_\omega}}\right) \tag{4.61}$$

uniformly in compact subsets of $(0, +\infty)$ as $\varepsilon_\omega \to 0$, where

$$E_\omega(t) = -8\pi^2 t e^{\frac{-2t}{\varepsilon_\omega}} + \frac{q}{2}\omega^2\int_{\mathbb{S}^4}\Phi(W_{\delta(t)})W_{\delta(t)}^2 dv_g . \tag{4.62}$$

By continuity, for any $K > 1$, there exists $t_{K,\omega} \in [\frac{1}{K}, K]$ such that

$$\mathcal{I}_\omega\left(\delta_\omega(t_{K,\omega})\right) = \min_{t\in[\frac{1}{K},K]}\mathcal{I}_\omega\left(\delta_\omega(t)\right) , \tag{4.63}$$

and, by the definition of ε_ω, we have that $\varepsilon_\omega \to 0$ if and only if $\omega^2 \to m_0^2 - 2$. By (4.50) in Lemma 4.6,

$$\left(-8\pi^2 t + \frac{q\omega^2}{2C}\right)e^{\frac{-2t}{\varepsilon_\omega}} \le E_\omega(t) \le \left(-8\pi^2 t + \frac{qC\omega^2}{2}\right)e^{\frac{-2t}{\varepsilon_\omega}} . \tag{4.64}$$

In particular, we get by (4.64) that there exists $0 < t_1 < t_2 < t_3$, depending only on C and m_0, such that

$$\begin{aligned} &\mathcal{I}_\omega\left(\delta_\omega(t)\right) > \frac{8}{3}\pi^2 \text{ for all } t \le t_1 , \\ &\mathcal{I}_\omega\left(\delta_\omega(t_2)\right) < \frac{8}{3}\pi^2 \text{ , and} \\ &\mathcal{I}_\omega\left(\delta_\omega(t)\right) > \mathcal{I}_\omega\left(\delta_\omega(t_2)\right) \text{ for all } t \ge t_3 , \end{aligned} \tag{4.65}$$

uniformly with respect to ω for $0 < \varepsilon_\omega \ll 1$ sufficiently small. Letting $K > 1$ be such that $[t_1, t_3] \subset \left(\frac{1}{K}, K\right)$, we get with (4.65) that $t_\omega = t_{K,\omega}$ satisfies that $\frac{1}{K} < t_\omega < K$ for all ω such that $0 < \varepsilon_\omega \ll 1$. Then, by Lemma 4.7,

$$u_{\delta_\omega(t_\omega),\omega} = W_{\delta_\omega(t_\omega)} + \psi_{\delta_\omega(t_\omega),\omega} \tag{4.66}$$

is a solution of (4.3). We know from (4.31) in Lemma 4.4 that $\|\psi_{\delta_\omega(t_\omega),\omega}\|_{H^1} \to 0$ as $\varepsilon_\omega \to 0$, and by combining (4.8) and (4.66), we get that $u_{\delta_\omega(t_\omega),\omega}$ has one bubble in its H^1-decomposition as $\varepsilon_\omega \to 0$. By elliptic theory, this automatically implies that $\|u_{\delta_\omega(t_\omega),\omega}\|_{L^\infty} \to +\infty$ as $\varepsilon_\omega \to 0$. Theorem 0.2 is proved. □

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