STRONG MAXIMUM PRINCIPLES FOR ANISOTROPIC ELLIPTIC AND PARABOLIC EQUATIONS

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Abstract. We investigate vanishing properties of nonnegative solutions of anisotropic elliptic and parabolic equations. We describe the optimal vanishing sets, and we establish strong maximum principles.

1. Introduction and results

In dimension $n \geq 2$, given $\overrightarrow{p} = (p_1, \dots, p_n)$ with $p_i > 1$ for $i = 1, \dots, n$, the anisotropic Laplace operator $\Delta_{\overrightarrow{p}}$ is defined by

$$\Delta_{\overrightarrow{p}} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \nabla_{x_i}^{p_i} u \,, \tag{1.1}$$

where $\nabla_{x_i}^{p_i} u = |\partial u/\partial x_i|^{p_i - 2}\, \partial u/\partial x_i$. We are concerned with equations of the type

$$\Delta_{\overrightarrow{p}} u = f\left(x, u, \nabla u\right) \quad \text{in } \Omega \tag{1.2}$$

and

$$-\frac{\partial u}{\partial t} + \Delta_{\overrightarrow{p}} u = f\left(x, t, u, \nabla u\right) \quad \text{in } \Omega \times (0, T) \,, \tag{1.3}$$

where Ω is a domain in $\mathbb{R}^n$, T is a positive real number, f is a continuous function, and $\Delta_{\overrightarrow{p}}$ is as in (1.1). Anisotropic equations like (1.2) and (1.3) have strong physical background. They emerge, for instance, from the mathematical description of the dynamics of fluids with different conductivities in different directions. We refer to the extensive books by Antontsev–Díaz–Shmarev [3] and Bear [9] for discussions in this direction. They also appear in biology, see Bendahmane–Karlsen [10] and Bendahmane–Langlais–Saad [12], as a model describing the spread of an epidemic disease in heterogeneous environments.

In this paper, we investigate strong maximum principles for anisotropic equations of the type (1.2) and (1.3). Given a subset K of Ω, we say that equations (1.2) and (1.3) satisfy a strong maximum principle in K if any nonnegative solution which vanishes at some point in K is in fact identically zero on the whole set K. As is well known (see, for instance, Protter–Weinberger [41]), in case of the standard harmonic and heat equations, namely in case $f = 0$ and $p_i = 2$ for all $i = 1, \dots, n$, equations (1.2) and (1.3) satisfy a strong maximum principle in the whole domain Ω.

We show in this paper that in presence of anisotropy, the zeros of solutions may not spread over the whole domain Ω, but they spread along directions where the anisotropic configuration is minimal. We illustrate this fact with a first example. In the anisotropic configuration

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$p_1 = \dots = p_{n-1} = p_-$, $p_n = p_+$, $p_- < p_+$ one can check that a nonnegative stationary solution of equations (1.2) and (1.3) with $f = 0$ on $\Omega = (0, +\infty)^{n-1} \times \mathbb{R}$ is given by

$$\mathcal{U}_{\overrightarrow{p}}(x_1, \dots, x_n) = \frac{C\,|x_n|^{p_+/(p_+-p_-)}}{\left(\sum_{i=1}^{n-1} x_i^{p_-/(p_--1)}\right)^{(p_--1)/(p_+-p_-)}}\,, \tag{1.4}$$

for some constant $C = C(n, \overrightarrow{p}) > 0$ under the assumptions that $p_+ > p_-(n-2)/(n-1-p_-)$ and $p_- < n-1$. The function $\mathcal{U}_{\overrightarrow{p}}$ vanishes on the set $(0, +\infty)^{n-1} \times \{0\}$ without vanishing elsewhere in the domain. Functions of the form (1.4) were introduced, in a different context, by Giaquinta [26] and Marcellini [31]. This example can be generalized by observing that for any $C > 0$ and $\varepsilon > 0$, the function $\mathcal{U}_{\overrightarrow{p}}$ satisfies the inequality $\Delta_{\overrightarrow{p}} u \le \lambda u^{p_--1}$ on $\Omega = (\varepsilon, +\infty)^{n-1} \times \mathbb{R}$ for $\lambda > 0$ large.

In Theorem 1.1 below, we establish a strong maximum principle for elliptic inequalities of the type

$$\Delta_{\overrightarrow{p}} u \le f(u) \quad \text{in } \Omega\,. \tag{1.5}$$

In presence of anisotropy, the vanishing sets are of the form

$$\Omega_0 = \{x \in \mathbb{R}^n;\quad [x, \xi_0] \subset \Omega \quad \text{and} \quad x_i = \xi_{0,i} \quad \forall i \in \mathcal{I}_+\}\,, \tag{1.6}$$

for some point $\xi_0 = (\xi_{0,1}, \dots, \xi_{0,n})$ in $\mathbb{R}^n$, where $\mathcal{I}_+ = \{i \in \{1, \dots, n\};\quad p_i > p_-\}$ and $p_- = \min(p_1, \dots, p_n)$ is the minimum value in the anisotropic configuration. We prove our result under the assumptions that the function f in the right hand sides of (1.5) is continuous, nondecreasing, and such that

$$f(u) = \mathrm{O}\left(u^{p_--1}\right) \quad \text{as } u \to 0\,. \tag{1.7}$$

We let $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ be the Sobolev space defined by

$$W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) = \left\{u \in L^{p_+}_{\mathrm{loc}}(\Omega);\quad \frac{\partial u}{\partial x_i} \in L^{p_i}_{\mathrm{loc}}(\Omega) \quad \forall i = 1, \dots, n\right\},$$

where $p_+ = \max(p_1, \dots, p_n)$ and where, for any real number $p \ge 1$, $L^p_{\mathrm{loc}}(\Omega)$ is the space of all measurable functions on Ω which belong to $L^p(\Omega')$ for all compact subsets Ω' of Ω. We say that a function u in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ is a (weak) solution of the inequality (1.5) if for any nonnegative smooth function φ with compact support in Ω, there holds

$$-\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx \le \int_\Omega f(u)\,\varphi dx\,.$$

An historic reference on strong maximum principles for elliptic equations is Hopf [27]. We refer to Protter–Weinberger [41] for a reference in book form on this topic. Our first result states as follows.

Theorem 1.1. *Let Ω be a nonempty domain in $\mathbb{R}^n$ and f be a continuous nondecreasing function on $\mathbb{R}_+$ satisfying* (1.7)*. Let u be a nonnegative solution in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ of inequality* (1.5)*. If there holds $u(\xi_0) = 0$ for some point ξ_0 in Ω, then the function u is identically zero on the set Ω_0, where Ω_0 is as in* (1.6)*.*

The vanishing sets Ω_0 are the maximal sets on which the strong maximum principle holds true, see (1.4).

Condition (1.7) is optimal among pure nonlinearities of the type $f(u) = u^{p-1}$. Indeed, for any real number p in $[1, p_-)$, letting i be such that $p_i = p_-$, one can check that a nonnegative solution of the equation $\Delta_{\overrightarrow{p}} u = u^{p-1}$ in $\mathbb{R}^n$ is given by the function $\mathcal{U}_{p,p_-}(x) =$

$\left|1 - C_{p,p_-} x_i\right|^{p_-/(p_- - p)}$, where $C_{p,p_-} = (p_- - p)/((p_-^{p_- -1} p(p_- - 1)^2)^{1/p_-})$. Clearly, the functions $\mathcal{U}_{p,p_-}$ do not satisfy strong maximum principles on sets of the form (1.6).

In Theorem 1.2 below, we establish strong maximum principles for parabolic inequalities of the type

$$-\frac{\partial u}{\partial t} + \Delta_{\overrightarrow{p}} u \le f(u) \quad \text{in } \Omega \times (0,T) \,. \tag{1.8}$$

We let $L^{\overrightarrow{p}}_{\mathrm{loc}}(0,T;W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega))$ be the function space defined by

$$L^{\overrightarrow{p}}_{\mathrm{loc}}(0,T;W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)) = \left\{ u \in L^{p_+}_{\mathrm{loc}}(0,T;L^{p_+}_{\mathrm{loc}}(\Omega)); \quad \frac{\partial u}{\partial x_i} \in L^{p_i}_{\mathrm{loc}}(0,T;L^{p_i}_{\mathrm{loc}}(\Omega)) \quad \forall i = 1,\dots,n \right\},$$

where $p_+ = \max(p_1,\dots,p_n)$ and where, for any real number $p \ge 1$, $L^p_{\mathrm{loc}}(0,T;L^p_{\mathrm{loc}}(\Omega))$ is the space of all measurable functions u on $\Omega \times (0,T)$ such that $\int_{t_1}^{t_2}\int_{\Omega'} |u|^p \, dxdt < \infty$ for all real numbers $0 < t_1 < t_2 < T$ and all compact subsets Ω' of Ω. We say that a function u in $L^{\overrightarrow{p}}_{\mathrm{loc}}(0,T;W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)) \cap C^0(\Omega \times (0,T))$ is a (weak) solution of the inequality (1.8) if for any nonnegative smooth function φ with compact support in $\Omega \times (0,T)$, there holds

$$\int_0^T\int_\Omega u \frac{\partial \varphi}{\partial t} dxdt - \int_0^T\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dxdt \le \int_0^T\int_\Omega f(u)\, \varphi dxdt \,.$$

A strong maximum principle for parabolic equations involving the standard Laplace operator was obtained by Nirenberg [38]. We refer, once again, to the extensive book by Protter–Weinberger [41] on this topic. Our result states as follows.

Theorem 1.2. *Let Ω be a nonempty domain in $\mathbb{R}^n$, T be a positive real number, and f be a continuous nondecreasing function on $\mathbb{R}_+$ satisfying* (1.7). *Let u be a nonnegative solution in $L^{\overrightarrow{p}}_{\mathrm{loc}}(0,T;W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)) \cap C^0(\Omega \times (0,T))$ of inequality* (1.8). *Assume that there holds $u(\xi_0,t_0) = 0$ for some point ξ_0 in Ω and some real number t_0 in $(0,T)$. Let Ω_0 be as in* (1.6). *Then we get the following assertions.*

(i) *If $p_- < 2$, then the function u is identically zero on the set $\Omega_0 \times \{t_0\}$.*
(ii) *If $p_- = 2$, then the function u is identically zero on the set $\Omega_0 \times (0,t_0]$.*
(iii) *If $p_- > 2$, then the function u is identically zero on the set $\{\xi_0\} \times (0,t_0]$.*

The vanishing sets in Theorem 1.2 are optimal in the sense that in case $p_- < 2$, we get existence of solutions which extinct in finite time (see Antontsev–Shmarev [5–8]), and in case $p_- > 2$, we get existence of solutions which vanish only on a time segment. As an example in case $p_- > 2$, letting i be such that $p_i = p_-$, one can consider the function $\mathcal{U}_{p_-}(x,t) = \left(\left|1 - C_{p_-} x_i\right|^{p_-} / (1 - (p_- - 2)\, t)\right)^{1/(p_- -2)}$, where $C_{p_-} = (p_- - 2)/(2p_-^{p_- -1}(p_- - 1)^2)^{1/p_-}$. As is easily checked, the function $\mathcal{U}_{p_-}$ is a nonnegative solution of the equation $\partial u/\partial t = \Delta_{\overrightarrow{p}} u$ in $\mathbb{R}^n \times (0, 1/(p_- - 2))$, and we get $\mathcal{U}_{p_-}(x,t) = 0$ if and only if $x_i = 1/C_{p_-}$.

We refer to Antontsev–Shmarev [5–8] for several results on the existence of solutions with finite waiting time or finite extinction time and on the localization of solutions of parabolic equations like (1.3). Other possible references on anisotropic parabolic equations are Antontsev–Chipot [2], Bendahmane–Karlsen [10, 11], Bendahmane–Langlais–Saad [12], and Lieberman [29]. Elliptic equations like (1.2) also received much attention in recent years. Possible references on elliptic equations like (1.2) are Alves–El Hamidi [1], Antontsev–Shmarev [4], Cianchi [13], D'Ambrosio [14], Di Castro [16], Di Castro–Montefusco [17],El Hamidi–Rakotoson [19, 20], El Hamidi–Vétois [21], Fragalà–Gazzola–Kawohl [23], Fragalà–Gazzola–Lieberman [24], García-Melián–Rossi–Sabina de Lis [25], Li [28], Lieberman [29, 30], Marcellini [32], Mihăilescu–Pucci–Rădulescu [34], Mihăilescu–Rădulescu–Tersian [35], Namlyeyeva–Shishkov–Skrypnik [36], Skrypnik [42], Tersenov–Tersenov [43], and Vétois [45–48]. We refer to Mercaldo–Rossi–Segura de León–Trombetti [33] for a description of the asymptotic behavior of solutions of equations like (1.2) as $p_- \to 1$, and we refer to Di Castro–Pérez-Llanos–Urbano [18] and Pérez-Llanos–Rossi [40] for the case $p_- \to \infty$, where $p_- = \min(p_1, \dots, p_n)$ and $p_+ = \max(p_1, \dots, p_n)$ are the minimum and maximum values in the anisotropic configuration.

In the isotropic configuration where $p_i = p$ for all $i = 1, \dots, n$, the operator (1.1) is comparable, though slightly different, to the p-Laplace operator $\Delta_p = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$. We refer to Vázquez [44] where the strong maximum principle was established for elliptic equations involving the p-Laplace operator. As for parabolic equations involving the p-Laplace operator, the question of the strong maximum principle was addressed in Nazaret [37]. For more material on p-Laplace equations, we refer to the lecture notes by Peral [39].

We also mention the work by Fortini–Mugnai–Pucci [22] where maximum principles are established for a general class of anisotropic inequalities in divergence form, in particular in the case of variable exponents (see also Zhang [49] concerning this case).

The proofs of Theorems 1.1 and 1.2 rely on the comparison of solutions with a family of anisotropic test functions (see (2.5) and (3.3)). We prove Theorem 1.1 in Section 2, and we prove Theorem 1.2 in Section 3.

2. Anisotropic elliptic equations

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Renumbering, if necessary, the coordinates, we may assume that there exists an index n_- such that $p_1 = \dots = p_{n_-} = p_-$ and $p_- < p_i$ for all $i > n_-$. We let $\xi_0 = (\xi_{0,1}, \dots, \xi_{0,n})$ be a point in Ω such that $u(\xi_0) = 0$. We proceed by contradiction and assume that the function u is not identically zero on Ω_0, where Ω_0 is as in (1.6). We let P be the set of points x in Ω such that $u(x) > 0$. Since Ω_0 is arcwise connected and since both the sets $P \cap \Omega_0$ and $\Omega_0 \backslash P$ are nonempty, we get that $\partial P \cap \Omega_0$ is nonempty. We choose a point $\xi_1 = (\xi_{1,1}, \dots, \xi_{1,n})$ in $P \cap \Omega_0$ such that

$$\inf_{x \in \Omega_0 \backslash P} \sum_{i=1}^{n_-} |x_i - \xi_{1,i}|^{\frac{p_-}{p_- - 1}} < \inf_{x \in \partial\Omega} \sum_{i=1}^{n_-} |x_i - \xi_{1,i}|^{\frac{p_-}{p_- - 1}}, \tag{2.1}$$

where, by convention, $\inf \emptyset = +\infty$. Since P is open, it follows from (2.1) that there exist a positive real number r_0 and a point $\zeta_0 = (\zeta_{0,1}, \dots, \zeta_{0,n})$ in $\Omega_0 \backslash P$ such that $\zeta_0 \in \partial B_{\xi_1}^{p_-}(r_0)$ and $\overline{B_{\xi_1}^{p_-}(r_0)} \backslash \{\zeta_0\} \subset P$, where

$$B_{\xi_1}^{p_-}(r_0) = \left\{ x \in \mathbb{R}^n ; \quad \sum_{i=1}^{n_-} |x_i - \xi_{1,i}|^{\frac{p_-}{p_- - 1}} < r_0 \quad \text{and} \quad x_i = \xi_{0,i} \quad \forall i > n_- \right\} \tag{2.2}$$

and

$$\partial B_{\xi_1}^{p_-}(r_0) = \left\{ x \in \mathbb{R}^n ; \quad \sum_{i=1}^{n_-} |x_i - \xi_{1,i}|^{\frac{p_-}{p_- - 1}} = r_0 \quad \text{and} \quad x_i = \xi_{0,i} \quad \forall i > n_- \right\}. \tag{2.3}$$

For any positive real numbers λ, δ, and ε, we let $A_{\xi_{1-\delta}}^{\overrightarrow{p}}(r_0, \lambda, \varepsilon)$ be the annular set defined by

$$A_{\xi_{1-\delta}}^{\overrightarrow{p}}(r_0, \lambda, \varepsilon) = \left\{ x \in \mathbb{R}^n ; \quad r_0 - \varepsilon < \sum_{i=1}^{n} \lambda^{\frac{p_- - p_i}{p_i - 1}} |x_i - \delta\zeta_{0,j} - (1-\delta)\xi_{1,j}|^{\frac{p_i}{p_i - 1}} < r_0 \right\}. \tag{2.4}$$

Since $\overline{B^{p_-}_{\xi_1}(r_0)}\backslash\{\zeta_0\}\subset P$, we get that for δ, λ small and any ε, $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda,\varepsilon)$ is included in Ω. Moreover, we get that for ε fixed and δ, λ small, the point ζ_0 belongs to $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda,\varepsilon)$. For any positive real numbers λ and δ, we define our test function $v_{\lambda,\delta}$ on $\mathbb{R}^n$ by

$$v_{\lambda,\delta}(x)=\lambda\delta\Big(e^{\lambda\big(r_0-\sum_{i=1}^n(\lambda^2\delta)^{\frac{p_--p_i}{p_i-1}}|x_i-\delta\zeta_{0,i}-(1-\delta)\xi_{1,i}|^{\frac{p_i}{p_i-1}}\big)}-1\Big). \tag{2.5}$$

Letting $\Delta_{\overrightarrow{p}}$ be as in (1.1), we find

$$\begin{aligned}\Delta_{\overrightarrow{p}}v_{\lambda,\delta}(x)=\left(\lambda^2\delta\right)^{p_--1}\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1}&e^{(p_i-1)\lambda\big(r_0-\sum_{j=1}^n(\lambda^2\delta)^{\frac{p_--p_j}{p_j-1}}|x_j-\delta\zeta_{0,j}-(1-\delta)\xi_{1,j}|^{\frac{p_j}{p_j-1}}\big)}\\&\times\left(p_i\lambda^{\frac{2p_--p_i-1}{p_i-1}}\delta^{\frac{p_--p_i}{p_i-1}}\left|x_i-\delta\zeta_{0,i}-(1-\delta)\xi_{1,i}\right|^{\frac{p_i}{p_i-1}}-1\right).\end{aligned} \tag{2.6}$$

For any point x in $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$, by (1.7), (2.5), and (2.6), we get

$$\begin{aligned}&-\Delta_{\overrightarrow{p}}v_{\lambda,\delta}(x)+f(v_{\lambda,\delta}(x))\\&\le(\lambda\delta)^{p_--1}\left(\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1}\lambda^{p_--1}e^{(p_i-1)\lambda\varepsilon}-\frac{p_-^{p_-}\lambda^{p_-}(r_0-\varepsilon)}{(p_--1)^{p_--1}}+C\left(e^{\lambda\varepsilon}-1\right)^{p_--1}\right)\end{aligned} \tag{2.7}$$

when $\lambda\delta$ and $\lambda\varepsilon$ are small, for some positive constant C independent of λ, δ, ε, and x. Choosing λ large enough so that

$$\lambda>\frac{(p_--1)^{p_--1}}{p_-^{p_-}r_0}\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1},$$

and then, choosing δ and ε small, it follows from (2.7) that $v_{\lambda,\delta}$ is a C^1-solution of the inequality

$$-\Delta_{\overrightarrow{p}}v_{\lambda,\delta}+f(v_{\lambda,\delta})<0\quad\text{in }A^{\overrightarrow{p}}_{\xi_{1-\delta}}\left(r_0,\lambda^2\delta,\varepsilon\right), \tag{2.8}$$

where $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ is as in (2.4). We let $\partial_1A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ and $\partial_2A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ stand for the respective interior and exterior boundaries of the annular set $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$. Since the function u is positive on $B^{p_-}_{\xi_1}(r_0)$, by continuity, we get the existence of a positive constant C_ε such that $u>C_\varepsilon$ on $B^{p_-}_{\xi_1}(r_0-\varepsilon)$, where $B^{p_-}_{\xi_1}(r_0-\varepsilon)$ is as in (2.2). Still by continuity of u, it follows that $u\ge C_\varepsilon$ on $\partial_1A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ for δ small. Since $v_{\lambda,\delta}=\lambda\delta\left(e^{\lambda\varepsilon}-1\right)$ on $\partial_1A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ and $v_{\lambda,\delta}=0$ on $\partial_2A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$, we then get $v_{\lambda,\delta}\le u$ on $\partial A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$ for δ small. In particular, there holds $(v_{\lambda,\delta}-u)_+=0$ on $\partial A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$, where $(v_{\lambda,\delta}-u)_+=\max(v_{\lambda,\delta}-u,0)$. Testing (1.5) and (2.8) against $(v_{\lambda,\delta}-u)_+$ and integrating by parts on $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$, we then get

$$\begin{aligned}\sum_{i=1}^n\int_{W_{\lambda,\delta,\varepsilon}}\left(\left|\frac{\partial v_{\lambda,\delta}}{\partial x_i}\right|^{p_i-2}\frac{\partial v_{\lambda,\delta}}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial v_{\lambda,\delta}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)dx&\\+\int_{W_{\lambda,\delta,\varepsilon}}\left(f(v_{\lambda,\delta})-f(u)\right)(v_{\lambda,\delta}-u)\,dx\le0\,,&\end{aligned} \tag{2.9}$$

where

$$W_{\lambda,\delta,\varepsilon}=\left\{x\in A^{\overrightarrow{p}}_{\xi_{1-\delta}}\left(r_0,\lambda^2\delta,\varepsilon\right);\quad v_{\lambda,\delta}(x)>u(x)\right\}.$$

Since the function f is nondecreasing, it follows from (2.9) that

$$\sum_{i=1}^{n}\int_{W_{\lambda,\delta,\varepsilon}}\left(\left|\frac{\partial v_{\lambda,\delta}}{\partial x_i}\right|^{p_i-2}\frac{\partial v_{\lambda,\delta}}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial v_{\lambda,\delta}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)dx=0\,,$$

and thus that $\nabla u=\nabla v_{\lambda,\delta}$ almost everywhere in $W_{\lambda,\delta,\varepsilon}$. Since $W_{\lambda,\delta,\varepsilon}$ is open, we then get that the function $v_{\lambda,\delta}-u$ is constant in $W_{\lambda,\delta,\varepsilon}$. By continuity of u and $v_{\lambda,\delta}$, it follows that $|W_{\lambda,\delta,\varepsilon}|=0$, i.e. $v_{\lambda,\delta}\le u$ in $A^{\overrightarrow{p}}_{\xi_{1-\delta}}(r_0,\lambda^2\delta,\varepsilon)$. In particular, we get $u(\zeta_0)\ge v_{\lambda,\delta}(\zeta_0)>0$. There is a contradiction. This ends the proof of Theorem 1.1. □

3. Anisotropic parabolic equations

This section is devoted to the proof of Theorem 1.2. Renumbering, if necessary, the coordinates, we may assume in what follows that there exists an index n_- such that $p_1=\dots=p_{n_-}=p_-$ and $p_-<p_i$ for all $i>n_-$. For any positive real numbers μ, r, and any point (ξ,t) in $\mathbb{R}^n\times\mathbb{R}_+$, we define the sets $B^{p_-}_{(\xi,t)}(\mu,r)$ and $\partial B^{p_-}_{(\xi,t)}(\mu,r)$ by

$$B^{p_-}_{(\xi,t)}(\mu,r)=\left\{(x,s)\in\mathbb{R}^n\times\mathbb{R}_+\,;\ \sum_{i=1}^{n_-}|x_i-\xi_i|^{\frac{p_-}{p_--1}}+\mu\,|s-t|^{\frac{p_-}{p_--1}}<r\text{ and }x_i=\xi_i\ \forall i>n_-\right\}\tag{3.1}$$

and

$$\partial B^{p_-}_{(\xi,t)}(\mu,r)=\left\{(x,s)\in\mathbb{R}^n\times\mathbb{R}_+\,;\ \sum_{i=1}^{n_-}|x_i-\xi_i|^{\frac{p_-}{p_--1}}+\mu\,|s-t|^{\frac{p_-}{p_--1}}=r\text{ and }x_i=\xi_i\ \forall i>n_-\right\}.\tag{3.2}$$

As a preliminary step in the proof of Theorem 1.2, we prove the following lemma.

Lemma 3.1. *Let Ω, T, f, and u be as in Theorem 1.2. Let μ be a positive real number. Assume that there exist a positive real number r_0 and two points (ξ_0,t_0) and (ξ_1,t_1) in $\Omega\times(0,T)$ such that $u(\xi_0,t_0)=0$, $(\xi_0,t_0)\in\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$, $B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)\subset\Omega_0\times(0,T)$, and $u(x,t)>0$ for all points (x,t) in $B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)\setminus\{(\xi_0,t_0)\}$, where Ω_0 is as in (1.6), $B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$ is as in (3.1), and $\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$ is as in (3.2). Then we get the following assertions.*

(i) *If $p_-\le 2$, then $\xi_0=\xi_1$.*
(ii) *If $p_-=2$ and $\mu>\frac{1}{4r_0}\big(\sum_{i=1}^{n}\big(\frac{p_i}{p_i-1}\big)^{p_i-1}\big)^2$, then $t_0=t_1-\sqrt{r_0/\mu}$.*
(iii) *If $p_->2$, then $t_0\le t_1$.*

Proof of Lemma 3.1. We proceed by contradiction and assume that $\xi_0\neq\xi_1$ if $p_-<2$, either $\xi_0\neq\xi_1$ or $t_0>t_1$ if $p_-=2$, and $t_0>t_1$ if $p_->2$. Moreover, decreasing, if necessary, the real number r_0, we may assume that $u(x,t)>0$ for all points (x,t) on $\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)\setminus\{(\xi_0,t_0)\}$, where $\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$ is as in (3.2). For any positive real numbers λ, μ, and δ, we define our test function $v_{\lambda,\mu,\delta}$ on $\mathbb{R}^n\times\mathbb{R}_+$ by

$$v_{\lambda,\mu,\delta}(x)=\lambda\delta\Big(e^{\lambda\left(r_0-\sum_{i=1}^n\left(\lambda^2\delta\right)^{\frac{p_--p_i}{p_i-1}}|x_i-\delta\xi_{0,i}-(1-\delta)\xi_{1,i}|^{\frac{p_i}{p_i-1}}-\mu|t-\delta t_0-(1-\delta)t_1|^{\frac{p_-}{p_--1}}\right)}-1\Big),\tag{3.3}$$

where $\xi_0=(\xi_{0,1},\dots,\xi_{0,n})$ and $\xi_1=(\xi_{1,1},\dots,\xi_{1,n})$. We find

$$\frac{\partial v_{\lambda,\mu,\delta}}{\partial t}(x,t)=\frac{p_-}{p_--1}\mu\lambda^2\delta e^{\lambda\left(r_0-\sum_{i=1}^n\left(\lambda^2\delta\right)^{\frac{p_--p_i}{p_i-1}}|x_i-\delta\xi_{0,i}-(1-\delta)\xi_{1,i}|^{\frac{p_i}{p_i-1}}-\mu|t-\delta t_0-(1-\delta)t_1|^{\frac{p_-}{p_--1}}\right)}$$
$$\times\left|\delta t_0+(1-\delta)t_1-t\right|^{\frac{2-p_-}{p_--1}}\left(\delta t_0+(1-\delta)t_1-t\right).\tag{3.4}$$

Moreover, letting $\Delta_{\overrightarrow{p}}$ be as in (1.1), we find

$$\Delta_{\overrightarrow{p}} v_{\lambda,\mu,\delta}(x,t) = \left(\lambda^2\delta\right)^{p_- -1} \sum_{i=1}^{n} \left(\frac{p_i}{p_i - 1}\right)^{p_i - 1} \times e^{(p_i-1)\lambda\left(r_0 - \sum_{j=1}^{n}\left(\lambda^2\delta\right)^{\frac{p_- - p_j}{p_j - 1}} |x_j - \delta\xi_{0,j} - (1-\delta)\xi_{1,j}|^{\frac{p_j}{p_j-1}} - \mu |t - \delta t_0 - (1-\delta) t_1|^{\frac{p_-}{p_- -1}}\right)} \times \left(p_i \lambda^{\frac{2p_- - p_i - 1}{p_i - 1}} \delta^{\frac{p_- - p_i}{p_i - 1}} |x_i - \delta\xi_{0,i} - (1-\delta)\xi_{1,i}|^{\frac{p_i}{p_i - 1}} - 1\right). \quad (3.5)$$

As is easily seen, for δ small, for any $i = 1, \dots, n$ and any point (x,t) in $\mathbb{R}^n \times \mathbb{R}_+$, there holds

$$\left| |x_i - \delta\xi_{0,i} - (1-\delta)\xi_{1,i}|^{\frac{p_i}{p_i-1}} - |\xi_{0,i} - \xi_{1,i}|^{\frac{p_i}{p_i-1}} \right| \le C\left(|\xi_{0,i} - \xi_{1,i}|^{\frac{1}{p_i-1}} |x_i - \xi_{0,i}| + |x_i - \xi_{0,i}|^{\frac{p_i}{p_i-1}} + \delta |\xi_{0,i} - \xi_{1,i}|^{\frac{p_i}{p_i-1}} \right), \quad (3.6)$$

$$\left| |t - \delta t_0 - (1-\delta) t_1|^{\frac{p_-}{p_- -1}} - |t_0 - t_1|^{\frac{p_-}{p_- -1}} \right| \le C\left(|t_0 - t_1|^{\frac{1}{p_- -1}} |t - t_0| + |t - t_0|^{\frac{p_-}{p_- -1}} + \delta |t_0 - t_1|^{\frac{p_-}{p_- -1}} \right), \quad (3.7)$$

and

$$\left| |t - \delta t_0 - (1-\delta) t_1|^{\frac{2-p_-}{p_- -1}} (t - \delta t_0 - (1-\delta) t_1) - |t_0 - t_1|^{\frac{2-p_-}{p_- -1}} (t_0 - t_1) \right| \le \begin{cases} C\Big(|t_0 - t_1|^{\frac{2-p_-}{p_- -1}} |t - t_0| + |t - t_0|^{\frac{1}{p_- -1}} + \delta |t_0 - t_1|^{\frac{1}{p_- -1}} \Big) & \text{if } p_- \le 2 \\ C\left(|t - t_0|^{\frac{1}{p_- -1}} + \delta^{\frac{1}{p_- -1}} |t_0 - t_1|^{\frac{1}{p_- -1}} \right) & \text{if } p_- > 2 \end{cases} \quad (3.8)$$

for some positive constant C independent of δ, x, and t. For any positive real numbers μ, δ, and ε, we define the ellipsoidal ball $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\delta,\varepsilon)$ by

$$B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\delta,\varepsilon) = \left\{ (x,s) \in \mathbb{R}^n \times \mathbb{R}_+ ;\quad \sum_{i=1}^{n} \delta^{\frac{p_- - p_i}{p_i - 1}} |x_i - \xi_{0,i}|^{\frac{p_i}{p_i-1}} + \mu |t - t_0|^{\frac{p_-}{p_- -1}} < \varepsilon \right\}. \quad (3.9)$$

Clearly, for δ and ε small, $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\delta,\varepsilon)$ is included in $\Omega \times (0,T)$. For any positive real numbers λ, μ, δ, ε, and any point (x,t) in $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\lambda^2\delta,\varepsilon)$, by (1.7), (3.3)–(3.8), and since

$(\xi_0,t_0)\in\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$, we get

$$\frac{\partial v_{\lambda,\mu,\delta}}{\partial t}(x,t)\le\frac{p_-}{p_--1}\mu\lambda^2\delta \tag{3.10}$$
$$\times\begin{cases} e^{C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\left(|t_1-t_0|^{\frac{2-p_-}{p_--1}}(t_1-t_0)+C\mu^{-\frac{1}{p_-}}\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)\right) & \text{if } p_-\le2 \text{ and } t_0\le t_1\\ e^{-C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}|t_1-t_0|^{\frac{2-p_-}{p_--1}}(t_1-t_0) & \\ \qquad +C\mu^{-\frac{1}{p_-}}e^{C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right) & \text{if } p_-\le2 \text{ and } t_0>t_1\\ e^{-C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}|t_1-t_0|^{\frac{2-p_-}{p_--1}}(t_1-t_0) & \\ \qquad +C\mu^{-\frac{1}{p_-}}e^{C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\left(\varepsilon^{\frac{1}{p_-}}+\delta^{\frac{1}{p_--1}}\right) & \text{if } p_->2 \text{ and } t_0>t_1\end{cases}$$

and

$$\begin{aligned}-\Delta_{\overrightarrow{p}}v_{\lambda,\mu,\delta}(x,t)+f(v_{\lambda,\mu,\delta}(x,t))\le(\lambda\delta)^{p_--1}\Bigg(&\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1}\lambda^{p_--1}e^{(p_i-1)C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\\ &-\frac{p_-^{p_-}\lambda^{p_-}}{(p_--1)^{p_--1}}e^{-(p_+-1)C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\left(r_0-\mu|t_1-t_0|^{\frac{p_-}{p_--1}}\right)\\ &+Ce^{C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)+C\left(e^{C\lambda(\mu+1)\left(\varepsilon^{\frac{p_--1}{p_-}}+\delta\right)}-1\right)^{p_--1}\Bigg)\end{aligned} \tag{3.11}$$

when δ, ε, and $\lambda(\mu+1)\left(\varepsilon^{(p_--1)/p_-}+\delta\right)$ are small, for some positive constant C independent of λ, μ, δ, ε, x, and t. In case $p_-\le2$ and $\xi_0\neq\xi_1$, since $(\xi_0,t_0)\in\partial B^{p_-}_{(\xi_1,t_1)}(\mu,r_0)$, we get $\mu|t_1-t_0|^{p_-/(p_--1)}<r_0$. We choose λ large enough so that

$$\begin{cases}\lambda>\dfrac{(p_--1)^{p_--1}}{p_-^{p_-}\left(r_0-\mu|t_1-t_0|^{\frac{p_-}{p_--1}}\right)}\displaystyle\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1} & \text{if } p_-<2 \text{ and } \xi_0\neq\xi_1\\ \lambda>\dfrac{1}{4\left(r_0-\mu(t_1-t_0)^2\right)}\left(2\mu(t_1-t_0)+\displaystyle\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1}\right) & \text{if } p_-=2 \text{ and } \xi_0\neq\xi_1\,.\end{cases} \tag{3.12}$$

It follows from (3.10), (3.11), and (3.12) that in case $p_-\le2$ and $\xi_0\neq\xi_1$, for δ and ε small, the function $v_{\lambda,\mu,\delta}$ is a C^1-solution of the inequality

$$\frac{\partial v_{\lambda,\mu,\delta}}{\partial t}-\Delta_{\overrightarrow{p}}v_{\lambda,\mu,\delta}+f(v_{\lambda,\mu,\delta})<0\quad\text{in } B^{\overrightarrow{p}}_{(\xi_0,t_0)}\left(\mu,\lambda^2\delta,\varepsilon\right), \tag{3.13}$$

where $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\lambda^2\delta,\varepsilon)$ is as in (3.9). In case $p_-=2$ and $t_0=t_1+\sqrt{r_0/\mu}$, we assume that $\mu>\frac{1}{4r_0}\left(\sum_{i=1}^n\left(\frac{p_i}{p_i-1}\right)^{p_i-1}\right)^2$, we let λ be an arbitrary positive real number, and we also find (3.13) for δ and ε small. In case $p_->2$ and $t_0>t_1$, without assumption on λ and μ, we still get (3.13) for δ and ε small. Now, we claim that there exists a positive constant C_ε such that $u\ge C_\varepsilon$ on $B^{\overrightarrow{p}}_{(\xi_{1-\delta},t_{1-\delta})}(\mu,\lambda^2\delta,r_0)\cap\partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\lambda^2\delta,\varepsilon)$ for δ small, where $(\xi_{1-\delta},t_{1-\delta}):=\delta(\xi_0,t_0)+(1-\delta)(\xi_1,t_1)$, and $B^{\overrightarrow{p}}_{(\xi_{1-\delta},t_{1-\delta})}(\mu,\lambda^2\delta,r_0)$, $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu,\lambda^2\delta,\varepsilon)$ are as in (3.9). In order to prove this claim, we proceed by contradiction and assume that there exist a sequence of positive real numbers $(\delta_\alpha)_\alpha$ and a sequence of points $(\xi_\alpha, t_\alpha)_\alpha$ such that $\delta_\alpha \to 0$, $u(\xi_\alpha, t_\alpha) \to 0$ as $\alpha \to +\infty$, and $(\xi_\alpha, t_\alpha) \in B^{\overrightarrow{p}}_{(\xi_{1-\delta_\alpha}, t_{1-\delta_\alpha})}(\mu, \lambda^2\delta_\alpha, r_0) \cap \partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta_\alpha, \varepsilon)$ for all α. Up to a subsequence, we get that (ξ_α, t_α) converges to a point (ξ_∞, t_∞) in $\overline{B^{p_-}_{(\xi_1,t_1)}(\mu, r_0)} \cap \partial B^{p_-}_{(\xi_0,t_0)}(\mu, \varepsilon)$, where $B^{p_-}_{(\xi_1,t_1)}(\mu, r_0)$ and $\partial B^{p_-}_{(\xi_0,t_0)}(\mu, \varepsilon)$ are as in (3.1) and (3.2). By continuity of the function u, we get $u(\xi_\infty, t_\infty) = 0$, and thus $(\xi_\infty, t_\infty) = (\xi_0, t_0)$. Since $(\xi_\alpha, t_\alpha) \in B^{\overrightarrow{p}}_{(\xi_{1-\delta_\alpha}, t_{1-\delta_\alpha})}(\mu, \lambda^2\delta_\alpha, r_0)$ for all α and since $\xi_{1,i} = \xi_{0,i}$ for all $i > n_-$, it follows that

$$\sum_{i=n_-+1}^{n} \left(\lambda^2\delta_\alpha\right)^{\frac{p_- - p_i}{p_i - 1}} |\xi_{\alpha,i} - \xi_{0,i}|^{\frac{p_i}{p_i-1}} < r_0 - \mu\,|t_\alpha - t_{1-\delta_\alpha}|^{\frac{p_-}{p_--1}} - \sum_{i=1}^{n_-} |\xi_{\alpha,i} - \xi_{1-\delta_\alpha,i}|^{\frac{p_-}{p_--1}} = \mathrm{o}(1) \tag{3.14}$$

as $\alpha \to +\infty$. On the other hand, since $(\xi_\alpha, t_\alpha) \in \partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta_\alpha, \varepsilon)$ for all α, we get

$$\sum_{i=n_-+1}^{n} \left(\lambda^2\delta_\alpha\right)^{\frac{p_- - p_i}{p_i - 1}} |\xi_{\alpha,i} - \xi_{0,i}|^{\frac{p_i}{p_i-1}} = \varepsilon - \mu\,|t_\alpha - t_0|^{\frac{p_-}{p_--1}} - \sum_{i=1}^{n_-} |\xi_{\alpha,i} - \xi_{0,i}|^{\frac{p_-}{p_--1}} = \varepsilon + \mathrm{o}(1) \tag{3.15}$$

as $\alpha \to +\infty$. There is a contradiction between (3.14) and (3.15). This ends the proof of our claim, namely that $u \geq C_\varepsilon$ on $B^{\overrightarrow{p}}_{(\xi_{1-\delta}, t_{1-\delta})}(\mu, \lambda^2\delta, r_0) \cap \partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta, \varepsilon)$ for δ small, for some positive constant C_ε. Since $v_{\lambda,\mu,\delta} \leq \lambda\delta\left(e^{\lambda r_0} - 1\right)$ in $B^{\overrightarrow{p}}_{(\xi_{1-\delta}, t_{1-\delta})}(\mu, \lambda^2\delta, r_0)$ and $v_{\lambda,\mu,\delta} \leq 0$ in $(\mathbb{R}^n \times \mathbb{R}_+) \backslash B^{\overrightarrow{p}}_{(\xi_{1-\delta}, t_{1-\delta})}(\mu, \lambda^2\delta, r_0)$, we then get $v_{\lambda,\mu,\delta} \leq u$ on $\partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta, \varepsilon)$ for δ small. In particular, there holds $(v_{\lambda,\mu,\delta} - u)_+ = 0$ on $\partial B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta, \varepsilon)$, where $(v_{\lambda,\mu,\delta} - u)_+ = \max(v_{\lambda,\mu,\delta} - u, 0)$. Testing (1.8) and (3.13) against $(v_{\lambda,\mu,\delta} - u)_+$ on $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta, \varepsilon)$ (up to an approximation in terms of Steklov averages, see, for instance, DiBenedetto [15]), we get

$$\begin{aligned} &\frac{1}{2}\int_{W_{\lambda,\mu,\delta,\varepsilon,t}} |v_{\lambda,\mu,\delta} - u|^2\,dx \\ &\quad + \sum_{i=1}^{n} \int_0^t \int_{W_{\lambda,\mu,\delta,\varepsilon,s}} \left(\left|\frac{\partial v_{\lambda,\mu,\delta}}{\partial x_i}\right|^{p_i-2} \frac{\partial v_{\lambda,\mu,\delta}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right)\left(\frac{\partial v_{\lambda,\mu,\delta}}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dxds \\ &\qquad + \int_0^t \int_{W_{\lambda,\mu,\delta,\varepsilon,s}} \left(f(v_{\lambda,\mu,\delta}) - f(u)\right)(v_{\lambda,\mu,\delta} - u)\,dxds \leq 0 \end{aligned} \tag{3.16}$$

for all real numbers t in $(0, T)$, where

$$W_{\lambda,\mu,\delta,\varepsilon,t} = \left\{x \in \mathbb{R}^n\,;\quad (x,t) \in B^{\overrightarrow{p}}_{(\xi_0,t_0)}\left(\mu, \lambda^2\delta, \varepsilon\right) \quad \text{and} \quad v_{\lambda,\mu,\delta}(x,t) > u(x,t)\right\}.$$

Since the function f is nondecreasing, it follows from (3.16) that for any real number t in $(0, T)$, there holds

$$\int_{W_{\lambda,\mu,\delta,\varepsilon,t}} |v_{\lambda,\mu,\delta} - u|^2\,dx = 0\,.$$

We then get $|W_{\lambda,\mu,\delta,\varepsilon,t}| = 0$, i.e. $v_{\lambda,\mu,\delta} \leq u$ in $B^{\overrightarrow{p}}_{(\xi_0,t_0)}(\mu, \lambda^2\delta, \varepsilon)$. In particular, we get $u(\xi_0, t_0) \geq v_{\lambda,\mu,\delta}(\xi_0, t_0) > 0$. There is a contradiction. This ends the proof of Lemma 3.1. ☐

Now, we can prove Theorem 1.2 by using Lemma 3.1.

Proof of Theorem 1.2. To begin with, we assume that $p_- \leq 2$ and prove that the function u is identically zero on the set $\Omega_0 \times \{t_0\}$, where Ω_0 is as in (1.6). We let P be the set of points (x, t) in $\Omega \times (0, T)$ such that $u(x, t) > 0$. We proceed by contradiction and assume that $P \cap (\Omega_0 \times \{t_0\})$ is not empty. In a similar way as in the proof of Theorem 1.1, we can choose a positive real number r_0 and two points $\zeta_0 = (\zeta_{0,1}, \dots, \zeta_{0,n})$ and $\xi_1 = (\xi_{1,1}, \dots, \xi_{1,n})$ in Ω_0 such that $u(\zeta_0, t_0) = 0$, $\zeta_0 \in \partial B^{p_-}_{\xi_1}(r_0)$, and $B^{p_-}_{\xi_1}(r_0) \times \{t_0\} \subset P$, where $B^{p_-}_{\xi_1}(r_0)$ and $\partial B^{p_-}_{\xi_1}(r_0)$ are as in (2.2) and (2.3). We let $h : [0,1] \to \mathbb{R}_+$ be defined by

$$h(\delta) = \inf_{(x,t)\in(\Omega_0\times(0,T))\backslash P} \left(\sum_{i=1}^{n_-} |x_i - \delta\xi_{1,i} - (1-\delta)\zeta_{0,i}|^{\frac{p_-}{p_- - 1}} + |t - t_0|^{\frac{p_-}{p_- - 1}} \right). \qquad (3.17)$$

Since $u(\zeta_0, t_0) = 0$, we get $h(\delta) \to 0$ as $\delta \to 0$. In particular, we get $\partial B^{p_-}_{(\delta\xi_1+(1-\delta)\zeta_0,t_0)}(1, h(\delta)) \subset \Omega_0 \times (0,T)$ and $B^{p_-}_{(\delta\xi_1+(1-\delta)\zeta_0,t_0)}(1, h(\delta)) \subset P$ for δ small, where $B^{p_-}_{(\delta\xi_1+(1-\delta)\zeta_0,t_0)}(1, h(\delta))$ and $\partial B^{p_-}_{(\delta\xi_1+(1-\delta)\zeta_0,t_0)}(1, h(\delta))$ are as in (3.1) and (3.2). Since P is open, it follows that for δ small, the infimum in (3.17) is achieved, i.e. there exists a point (ζ_δ, t_δ) on $\partial B^{p_-}_{(\delta\xi_1+(1-\delta)\zeta_0,t_0)}(1, h(\delta))$ such that $u(\zeta_\delta, t_\delta) = 0$. By Lemma 3.1, we get $\zeta_\delta = \delta\xi_1 + (1-\delta)\zeta_0$, and thus $h(\delta) = |t_\delta - t_0|^{p_-/(p_- - 1)}$ for δ small. It follows from (3.17) that for δ_1 and δ_2 small, there holds

$$h(\delta_1) \le \sum_{i=1}^{n_-} |\xi_{1,i} - \zeta_{0,i}|^{\frac{p_-}{p_- - 1}} |\delta_2 - \delta_1|^{\frac{p_-}{p_- - 1}} + h(\delta_2).$$

In particular, for δ small, the function h is differentiable and $h' = 0$ on $[0, \delta]$. It follows that the function h is constant on $[0, \delta]$. Since $h(0) = 0$, we then get $h = 0$ on $[0, \delta]$, i.e. $u = 0$ on $[\delta\xi_1 + (1-\delta)\zeta_0, \zeta_0] \times \{t_0\}$. There is a contradiction. This ends the proof of the first part of Theorem 1.2. Now, we assume that $p_- \ge 2$, and we prove that the function u is identically zero on the set $\{\xi_0\} \times (0, t_0]$. We proceed by contradiction and assume that there exists a real number t_1 in $(0, t_0)$ such that $u(\xi_0, t_1) > 0$. Since P is open, we get $B^{p_-}_{(\xi_0,t_1)}(\mu, \varepsilon) \subset P$ for μ large and ε small, where $B^{p_-}_{(\xi_0,t_1)}(\mu, \varepsilon)$ is as in (3.1). We may assume that the real number μ is large enough so that $\mu > \frac{1}{2\varepsilon} \sum_{i=1}^n \left(\frac{p_i}{p_i - 1}\right)^{p_i - 1}$. Increasing, if necessary, the real number t_1, since P is open, we may assume that

$$t_1 = \sup \left\{ t \in (0, t_0)\,; \quad B^{p_-}_{(\xi_0,t_1)}(\mu, \varepsilon) \subset P \right\}.$$

It follows that there exists a point (ξ_2, t_2) on $\partial B^{p_-}_{(\xi_0,t_1)}(\mu, \varepsilon)$ such that $t_2 > t_1$ and $u(\xi_2, t_2) = 0$. We get a contradiction with Lemma 3.1. This ends the proof of Theorem 1.2. $\square$

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