DECAY ESTIMATES AND A VANISHING PHENOMENON FOR THE SOLUTIONS OF CRITICAL ANISOTROPIC EQUATIONS

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Abstract. We investigate the asymptotic behavior of solutions of anisotropic equations of the form $-\sum_{i=1}^{n} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i-2}\,\partial_{x_i} u\right) = f(x,u)$ in $\mathbb{R}^n$, where $p_i > 1$ for all $i = 1,\dots,n$ and f is a Caratheodory function with critical Sobolev growth. This problem arises in particular from the study of extremal functions for a class of anisotropic Sobolev inequalities. We establish decay estimates for the solutions and their derivatives, and we bring to light a vanishing phenomenon which occurs when the maximum value of the p_i exceeds a critical value.

1. Introduction and main results

We let $n \geq 2$ and $\overrightarrow{p} = (p_1,\dots,p_n)$ be such that $p_i > 1$ for all $i = 1,\dots,n$ and $\sum_{i=1}^{n} 1/p_i > 1$. In this paper, we are interested in the solutions of problems of the form

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = f(x,u) & \text{in } \mathbb{R}^n, \\ u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $\Delta_{\overrightarrow{p}} u := \sum_{i=1}^{n} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i-2}\,\partial_{x_i} u\right)$ is the anisotropic Laplace operator, $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{D^{1,\overrightarrow{p}}(\mathbb{R}^n)} := \sum_{i=1}^{n}\left(\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i}\,dx\right)^{1/p_i}$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that

$$|f(x,s)| \leq \Lambda\,|s|^{p^*-1} \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

for some real number $\Lambda > 0$. Here, p^* denotes the critical Sobolev exponent and is defined as

$$p^* := \frac{n}{\sum_{i=1}^{n}\frac{1}{p_i} - 1} = \frac{np}{n-p} \qquad \text{with} \quad \frac{1}{p} := \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}\,.$$

The problem (1.1) with $f(x,u) = |u|^{p^*-2}\,u$ appears in the study of extremal functions for a class of anisotropic Sobolev inequalities. Early references on anisotropic Sobolev inequalities are Nikol′skiĭ [24], Troisi [33], and Trudinger [34]. We also refer to Cianchi [5] for a more recent work on the topic. Here we are interested in an inequality which appeared first in Troisi [33]. Among different equivalent versions (see Theorem 2.1 below), this inequality can be stated as

$$\int_{\mathbb{R}^n} |u|^{p^*}\,dx \leq C\left(\sum_{i=1}^{n}\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i}\,dx\right)^{p^*/p} \tag{1.3}$$

for some constant $C = C(n,\overrightarrow{p})$ and for all functions $u \in C_c^\infty(\mathbb{R}^n)$. The inequality (1.3) enjoys an anisotropic scaling law (see (2.3) below). As a corollary of the work of El Hamidi–Rakotoson [14], we obtain in Theorem 2.2 below that there exist extremal functions for the inequality (1.3) provided that $p_i < p^*$ for all $i = 1,\dots,n$.

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In the presence of anisotropy, namely when the p_i are not all equal, there is no explicit formula for the extremal functions of (1.3). This motivates to find a priori estimates for these functions, and more generally for the solutions of equations of type (1.1). The main difficulties in this work come from the non-homogeneity of the problem and the lack of radial symmetry.

As a more general motivation, the solutions of problems of type (1.1) with $f(x,u) = |u|^{p^*-2}u$ turn out to play a central role in the blow-up theories of critical equations in general domains. Possible references in book form on this subject and its applications in the isotropic regime are Druet–Hebey–Robert [13], Ghoussoub [17], and Struwe [31]. A first step in the direction of a blow-up theory in the anisotropic regime was taken in El Hamidi–Vétois [15] where we extended the bubble tree decompositions of Struwe [30]. Now, if one wants to go further and investigate a pointwise blow-up theory, then it is essential to know the asymptotic behavior of the solutions of (1.1) with $f(x,u) = |u|^{p^*-2}u$. The results in this paper can be seen as a crucial step in this direction.

Anisotropic equations of type (1.1) have received much attention in recent years. In addition to the above cited references [14, 15] and without pretending to be exhaustive, we mention for instance the works by Cianchi [6] on symmetrization properties, Cîrstea–Vétois [7] on the fundamental solutions, Cupini–Marcellini–Mascolo [10] on the local boundedness of solutions, Fragalà–Gazzola–Kawohl [16] on the existence and non-existence of solutions in bounded domains, Lieberman [22] on gradient estimates, Namlyeyeva–Shishkov–Skrypnik [23] on singular solutions, and Vétois [37] on vanishing properties of solutions. More references can be found for instance in [37].

Throughout this paper, we denote

$$p_+ := \max\left(\{p_i \in \overrightarrow{p}\}\right) \quad \text{and} \quad p_* := \frac{n-1}{\sum_{i=1}^n \frac{1}{p_i} - 1} = \frac{p(n-1)}{n-p}. \tag{1.4}$$

The exponent p_* is known to play a critical role in several results on the asymptotic behavior of solutions of second order elliptic equations (see the historic paper of Serrin [28], see also for instance the more recent paper of Serrin–Zou [29] and the references therein).

Our first result is as follows.

Theorem 1.1. *Assume that $p_+ < p_*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true and u be a solution of* (1.1). *Then there exists a constant $C_0 = C_0(n, \overrightarrow{p}, \Lambda, u)$ such that*

$$|u(x)|^{p_*} + \sum_{i=1}^n |\partial_{x_i} u(x)|^{p_i} \le C_0 \left(1 + \sum_{i=1}^n |x_i|^{\frac{p_* p_i}{p_* - p_i}}\right)^{-1} \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.5}$$

where p_* is as in (1.4).

We point out that the decay rate in (1.5) is the same as the one obtained in Cîrstea–Vétois [7] for the fundamental solutions in $\mathbb{R}^n$, namely the solutions of the equation $-\Delta_{\overrightarrow{p}} u = \delta_0$ in $\mathbb{R}^n$, where δ_0 is the Dirac mass at the point 0.

In case all p_i are equal to p, as part of a more general result, Alvino–Ferone–Trombetti–Lions [1] proved that the best constant in the inequality (1.3) is attained by the functions

$$u_{a,b}(x) := \left(a + b \sum_{i=1}^n |x_i|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}} \tag{1.6}$$

for all $a, b > 0$. Moreover, Cordero-Erausquin–Nazaret–Villani [8] proved that the functions (1.6) are the only extremal functions of (1.3). In case where the norm of the gradient in (1.3) is replaced bv the Euclidean norm, the existence of radially symmetric extremal functions was found by Aubin [2], Rodemich [26], and Talenti [32]. Since $p/\left(p_*-p\right)=(n-p)/(p-1)$, the decay rate in (1.6) coincides with the one in (1.5).

In case of the Laplace operator ($p_i=2$), Caffarelli–Gidas–Spruck [3] (see also Chen–Li [4]) proved that every positive solution of (1.1) with $f(x,u)=u^{p^*-1}$ is of the form (1.6). This result can be extended to the case where all p_i are equal to $p\in(1,n)$ for positive solutions satisfying the one-dimensional symmetry $u(x)=u\big(\sum_{i=1}^n|x_i|^{\frac{p}{p-1}}\big)$ for all $x\in\mathbb{R}^n$. Indeed, this result has been proved by Guedda–Véron [19] in case of positive, radially symmetric solutions for the p–Laplace equation $-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)=u^{p^*-1}$ in $\mathbb{R}^n$, and it can easily be seen that both cases lead to the same ordinary differential equation. We also mention that radial symmetry results have been established for positive solutions in $D^{1,p}(\mathbb{R}^n)$ in the case of p–Laplace equations (see Damascelli–Merchán–Montoro–Sciunzi [12], Damascelli–Ramaswamy [11], Sciunzi [27], and Vétois [38]).

Theorem 1.1 has been proved in Vétois [38] in case of the p–Laplace operator. We also refer in case of the Laplace operator ($p_i=2$) to Jannelli–Solimini [20], where the decay estimate (1.5) has been proved to hold true for solutions of (1.1) with right-hand side $f(x,u)=\sum_{i=1}^N a_i(x)|u|^{q_i^*-2}u$, where $q_i^*:=2^*(1-1/q_i)$, $q_i\in(n/2,\infty]$, $|a_i(x)|=\mathrm{O}\big(|x|^{-n/q_i}\big)$ for large $|x|$, and a_i belongs to the Marcinkiewicz space $M^{q_i}(\mathbb{R}^n)$ for all $i=1,\dots,N$.

The next results concern the case $p_+\ge p_*$, namely $p_i\ge p_*$ for some index i. In particular, we are now exclusively in the case where the exponents p_i are not all equal.

In the limit case $p_+=p_*$, we prove the following result.

Theorem 1.2. *Assume that $p_+=p_*$. Let $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true and u be a solution of* (1.1). *Then for any $q>p_*$, there exists a constant $C_q=C(n,\overrightarrow{p},\Lambda,u,q)$ such that*

$$|u(x)|^q+\sum_{i=1}^n|\partial_{x_i}u(x)|^{p_i}\le C_q\Bigg(1+\sum_{i=1}^n|x_i|^{\frac{qp_i}{q-p_i}}\Bigg)^{-1}\quad\text{for a.e. } x\in\mathbb{R}^n. \tag{1.7}$$

Beyond this limit case, namely when $p_*<p_+<p^*$, we find the following result.

Theorem 1.3. *Assume that $p_*<p_+<p^*$. Let $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true and u be a solution of* (1.1). *Then there exist a real number $q_0=q_0(n,\overrightarrow{p})<p_+$ such that the two following assertions hold true.*

(i) *There exists a constant $R_0=R_0(n,\overrightarrow{p},\Lambda,u)$ such that*

$$u(x)=0\quad\text{for all } x\in\mathbb{R}^n \text{ such that } \sum_{i\in\mathcal{I}_0}|x_i|\ge R_0\,, \tag{1.8}$$

where $\mathcal{I}_0$ is the set of all indices i such that $p_i>q_0$. Moreover, $\mathcal{I}_0\ne\emptyset$ due to $q_0<p_+$.

(ii) *For any $q>q_0$, there exists a constant $C_q=C(n,\overrightarrow{p},\Lambda,u,q)$ such that*

$$|u(x)|^q+\sum_{i=1}^n|\partial_{x_i}u(x)|^{p_i}\le C_q\Bigg(1+\sum_{i\in\mathcal{I}_0^c}|x_i|^{\frac{qp_i}{q-p_i}}\Bigg)^{-1}\quad\text{for a.e. } x\in\mathbb{R}^n, \tag{1.9}$$

where $\mathcal{I}_0^c:=\{1,\dots,n\}\backslash\mathcal{I}_0$.

We are able, moreover, to give an explicit definition in terms of n and $\overrightarrow{p}$ of a real number q_0 satisfying the above result (see Section 7).

The dependence on u of the constants C_0, C_q, and R_0 in the above results will be made more precise in Remarks 6.3 and 7.3.

As a remark about the support of solutions, by a result in Vétois [37], we have that for any nonnegative solution u of (1.1) with $f(x,u)$ as in (1.2) (see [37] for the general assumptions), if $u(x) = 0$ for some $x \in \mathbb{R}^n$, then we have $u \equiv 0$ on the affine subspace $\{y \in \mathbb{R}^n : y_i = x_i \quad \forall i = \{1,\dots,n\} \setminus \mathcal{I}_-\}$, where $\mathcal{I}_-$ is the set of all indices i such that $p_i = \min(\{p_j \in \overrightarrow{p}\})$. In case all p_i are equal to p, we obtain that either $u > 0$ or $u \equiv 0$, thus recovering the same result as Vazquez [35] found for the p–Laplace operator. In the presence of anisotropy, as shows for instance Theorem 1.3, this result does not hold true in general on the whole $\mathbb{R}^n$.

We also point out that in the limit case $p_+ = p^*$, we are able to construct quasi-explicit examples of solutions of (1.1) with $f(x,u) = |u|^{p^*-2} u$ for anisotropic configurations of type $\overrightarrow{p} = (p_-,\dots,p_-,p_+,\dots,p_+)$ by using the method of separation of variables (see Vétois [36]). These solutions turn out to vanish in the i-th directions corresponding to $p_i = p_+$, exactly like what we prove to be true in Theorem 1.3 in case $p_* < p_+ < p^*$.

The paper is organized as follows. In Section 2, we present different equivalent versions of the anisotropic Sobolev inequality, and we study the existence and scaling properties of extremal functions for these inequalities.

Section 3 is concerned with preliminary properties satisfied by the solutions of (1.1), namely global boundedness results and a weak decay estimate.

In Sections 4 and 5, we perform a Moser-type iteration scheme inspired from the one developed in Cîrstea–Vétois [7] for the fundamental solutions. In order to treat a large part of the proofs in a unified way, we consider a general family of domains defined as

$$\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda) := \left\{x \in \mathbb{R}^n : \sum_{i\in\mathcal{I}_1} |x_i|^{q_i} < (1+\lambda) R_1 \text{ and } \left|\sum_{i\in\mathcal{I}_2} |x_i|^{q_i} - R_2\right| < \lambda R_2\right\}, \quad (1.10)$$

where $\lambda \in (0,1)$, $R_1, R_2 > 0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are two disjoint subsets of $\{1,\dots,n\}$, $\mathcal{I}_2 \neq \emptyset$, and $\overrightarrow{q} = (q_i)_{i\in\mathcal{I}_1\cup\mathcal{I}_2}$ is such that $q_i > 1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$. On these domains, we prove that the solutions of (1.1) satisfy reverse Hölder-type inequalities of the form

$$\|u\|^{\gamma}_{L^{\gamma}\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda)\right)} \le C \max_{i\in\mathcal{I}_1\cup\mathcal{I}_2} \left((\lambda'-\lambda)^{-p_i} R_{\delta_i}^{-\frac{p_i}{q_i}} \|u\|^{\gamma_i}_{L^{\gamma_i}\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')\right)} \right)^{\frac{n}{n-p}} \quad (1.11)$$

for all $\gamma > p_* - 1$ and $\lambda < \lambda' \in (0,1/2]$, where $\delta_i := 1$ if $i \in \mathcal{I}_1$, $\delta_i := 2$ if $i \in \mathcal{I}_2$, $\gamma_i := \frac{n-p}{n}\gamma + p_i - p$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$, and $C = C(n, \overrightarrow{p}, \overrightarrow{q}, u, \gamma)$ (see Lemma 4.1 for more details on the dependence of the constant with respect to u and γ). Since the right-hand side of (1.11) involves different exponents γ_i in the anisotropic case, the number of exponents in the estimates may grow exponentially when iterating this inequality. We overcome this issue in Section 5 by controlling the values of the exponents with respect to the number of iterations.

In Section 6, we prove a vanishing result which will give Point (i) in Theorem 1.3. We prove this result by applying our iteration scheme with $R_1 = R_2^{1/\varepsilon}$ for small real numbers $\varepsilon > 0$ and $\mathcal{I}_1$, $\mathcal{I}_2$ being the sets of all indices i such that $p_i < p_0$, $p_i = p_0$, respectively, for some large enough real number $p_0 \in \overrightarrow{p}$ (see (6.3) for the exact condition on p_0). Passing to the limit into our iteration scheme, we obtain a pointwise estimate of the form

$$\|u\|_{L^\infty(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R^{1/\varepsilon},\mathcal{I}_2,R,1/4))} \le \left(CR^{-\frac{1}{p_\varepsilon}}\right)^{\frac{1}{\varepsilon}} \tag{1.12}$$

for some constant $C = C\left(n, \overrightarrow{p}, u\right)$ (see Lemma 6.2). When R is large enough, the right-hand side of (1.12) converges to 0 as $\varepsilon \to 0$, and we thus obtain our vanishing result.

In Section 7, we prove Theorems 1.1, 1.2, and we complete the proof of Theorem 1.3 by proving the decay estimates (1.9). The proofs of these results rely again on our iteration scheme, this time applied with $\mathcal{I}_1 = \emptyset$ and $\mathcal{I}_2$ being the set of all indices i such that $p_i \le \overline{p}_0$ for some real number $\overline{p}_0$ (see (6.1)).

Finally, in Appendix A, we prove a weak version of Kato's inequality which is used in Sections 3 and 4.

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2. Application to the extremal functions of a class of anisotropic Sobolev inequalities

As mentioned in the introduction, one of our main motivation in this paper is to apply our results to the extremal functions of a class of anisotropic Sobolev inequalities which originates from Troisi [33]. In this section, we first present in Theorem 2.1 below different equivalent versions of these inequalities, and we then prove in Theorem 2.2 that all these inequalities have extremal functions, and that with a suitable change of scale, these extremal functions are solutions of (1.1) with $f(x,u) = |u|^{p^*-2}u$.

We state the equivalent versions of the anisotropic Sobolev inequalities as follows.

Theorem 2.1. *The following inequalities hold true.*

(i) *There exists a constant $C = C\left(n, \overrightarrow{p}\right)$ such that*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{n/p^*} \le C \prod_{i=1}^n \left(\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx\right)^{1/p_i} \quad \forall u \in C_c^\infty(\mathbb{R}^n). \tag{2.1}$$

(ii) *For any $\overrightarrow{\theta} = (\theta_1, \dots, \theta_n)$ such that $\theta_i > 0$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n 1/\theta_i = n/p$, there exists a constant $C_{\overrightarrow{\theta}} = C\left(n, \overrightarrow{p}, \overrightarrow{\theta}\right)$ such that*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{p/p^*} \le C_{\overrightarrow{\theta}} \sum_{i=1}^n \left(\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx\right)^{\theta_i/p_i} \quad \forall u \in C_c^\infty(\mathbb{R}^n). \tag{2.2}$$

In particular, we get (1.3) in case $\theta_i = p_i$ for all $i = 1, \dots, n$.

As a remark, the inequalities (2.1) and (2.2) enjoy an anisotropic scaling law. Indeed, it can easily be seen that every integral in these inequalities are invariant with respect to the change of scale $u \mapsto u_\lambda$, where

$$u_\lambda(x) = \lambda u\left(\lambda^{(p^*-p_1)/p_1} x_1, \dots, \lambda^{(p^*-p_n)/p_n} x_n\right) \tag{2.3}$$

for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

Proof of Theorem 2.1. We refer to Troisi [33, Theorem 1.2] for the proof of the inequality (2.1). Then the inequality (2.2) follows from (2.1) by applying an inequality of weighted arithmetic and geometric means. As a remark, we can also obtain (2.1) from (2.2) by applying the change of scale (2.5) below. □

Regarding the extremal functions of (2.1) and (2.2), we prove the following result. The existence part in this result will be obtained as a corollary of the work of El Hamidi–Rakotoson [14] and Proposition 2.3 below.

Theorem 2.2. *If $p_+ < p^*$, then there exist extremal functions $u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, $u \neq 0$, of (2.1) and (2.2). Moreover, for any extremal function u of (2.1) or (2.2), there exist $\mu_1, \dots, \mu_n > 0$ such that the function $x \in \mathbb{R}^n \mapsto u(\mu_1 x_1, \dots, \mu_n x_n)$ is a constant-sign solution of (1.1) with $f(x,u) = |u|^{p^*-2} u$. In particular, every extremal function of (2.1) or (2.2) satisfies the a priori estimates in Theorems 1.1, 1.2, and 1.3.*

As a remark, due to the scaling law (2.3), every extremal function of (2.1) or (2.2) generates in fact an infinite family of extremal functions.

Preliminary to the proof of Theorem 2.2, we prove the following result.

Proposition 2.3. *Let $\overrightarrow{\theta} = (\theta_1, \dots, \theta_n)$ be such that $\theta_i > 0$ for all $i = 1, \dots, n$ and $\sum_{i=1}^n 1/\theta_i = n/p$. Then the following assertions hold true.*

(i) *For any extremal function u of (2.2), $u \circ \tau_{\overrightarrow{\theta}}$ is an extremal function of (2.1), where*

$$\tau_{\overrightarrow{\theta}}(x) := \left(\lambda_{\overrightarrow{\theta},1} x_1, \dots, \lambda_{\overrightarrow{\theta},n} x_n\right), \quad \lambda_{\overrightarrow{\theta},i} := \theta_i^{1/\theta_i} \prod_{j=1}^n \theta_j^{-p/(n\theta_i\theta_j)}, \tag{2.4}$$

for all $x \in \mathbb{R}^n$ and $i = 1, \dots, n$.

(ii) *For any extremal function u of (2.1), $u \circ \sigma_{\overrightarrow{\theta},u} \circ \tau_{\overrightarrow{\theta}}^{-1}$ is an extremal function of (2.2), where $\tau_{\overrightarrow{\theta}}$ is as in (2.4) and*

$$\sigma_{\overrightarrow{\theta},u}(x) := \left(\mu_{\overrightarrow{\theta},1}(u) x_1, \dots, \mu_{\overrightarrow{\theta},n}(u) x_n\right), \quad \mu_{\overrightarrow{\theta},i}(u) := \frac{\prod_{j=1}^n \left(\int_{\mathbb{R}^n} |\partial_{x_j} u|^{p_j} dx\right)^{p/(n\theta_i p_j)}}{\left(\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx\right)^{1/p_i}}, \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and $i = 1, \dots, n$.

Proof of Proposition 2.3. We begin with proving Point (i). We fix an extremal function u_0 of (2.2). Since $\sum_{i=1}^n 1/\theta_i = n/p$, we obtain

$$\prod_{i=1}^n \left(\int_{\mathbb{R}^n} \left|\partial_{x_i}\left(u_0 \circ \tau_{\overrightarrow{\theta}}\right)\right|^{p_i} dx\right)^{p/(np_i)} \le \frac{p}{n} \sum_{i=1}^n \frac{1}{\theta_i} \left(\int_{\mathbb{R}^n} \left|\partial_{x_i}\left(u_0 \circ \tau_{\overrightarrow{\theta}}\right)\right|^{p_i} dx\right)^{\theta_i/p_i}. \tag{2.6}$$

For any function $u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, simple calculations give

$$\sum_{i=1}^n \frac{1}{\theta_i} \left(\int_{\mathbb{R}^n} \left|\partial_{x_i}\left(u \circ \tau_{\overrightarrow{\theta}}\right)\right|^{p_i} dx\right)^{\theta_i/p_i} = \left(\prod_{j=1}^n \theta_j^{-p/(n\theta_j)}\right) \sum_{i=1}^n \left(\int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx\right)^{\theta_i/p_i} \tag{2.7}$$

and

$$\int_{\mathbb{R}^n} \left|u \circ \tau_{\overrightarrow{\theta}}\right|^{p^*} dx = \int_{\mathbb{R}^n} |u|^{p^*} dx\,. \tag{2.8}$$

By invertibility of $\tau_{\overrightarrow{\theta}}$ and since u_0 is an extremal function of (2.2), it follows from (2.7) and (2.8) that

$$\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}\left(u_0\circ\tau_{\overrightarrow{\theta}}\right)\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}\left|u_0\circ\tau_{\overrightarrow{\theta}}\right|^{p^*}dx\right)^{p/p^*}}=\inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\\ u\neq 0}}\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\,. \tag{2.9}$$

Now, we claim that

$$\inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\\ u\neq 0}}\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\le\frac{n}{p}\cdot\inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\\ u\neq 0}}\frac{\prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{p/(np_i)}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\,. \tag{2.10}$$

We prove this claim. For any function $u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, $u\neq 0$, by applying the change of scale (2.5), we obtain

$$\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}\left(u\circ\sigma_{\overrightarrow{\theta},u}\right)\right|^{p_i}dx\right)^{\theta_i/p_i}=\prod_{j=1}^{n}\left(\int_{\mathbb{R}^n}\left|\partial_{x_j}u\right|^{p_j}dx\right)^{p/(np_j)} \tag{2.11}$$

for all $i=1,\dots,n$, and

$$\int_{\mathbb{R}^n}\left|u\circ\sigma_{\overrightarrow{\theta},u}\right|^{p^*}dx=\int_{\mathbb{R}^n}|u|^{p^*}dx\,. \tag{2.12}$$

Since $\sum_{i=1}^{n}1/\theta_i=n/p$, it follows from (2.11) and (2.12) that

$$\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}\left(u\circ\sigma_{\overrightarrow{\theta},u}\right)\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}\left|u\circ\sigma_{\overrightarrow{\theta},u}\right|^{p^*}dx\right)^{p/p^*}}=\frac{n}{p}\cdot\frac{\prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{p/(np_i)}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\,, \tag{2.13}$$

and hence we obtain (2.10). It follows from (2.6), (2.9) and (2.10) that $u_0\circ\tau_{\overrightarrow{\theta}}$ is an extremal function of (2.1). This ends the proof of Point (i).

Now, we prove Point (ii). We fix an extremal function u_0 of (2.1). By (2.13) and since $\sum_{i=1}^{n}1/\theta_i=n/p$, we obtain

$$\begin{aligned}\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}\left(u_0\circ\sigma_{\overrightarrow{\theta},u_0}\right)\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}\left|u_0\circ\sigma_{\overrightarrow{\theta},u_0}\right|^{p^*}dx\right)^{p/p^*}}&=\frac{n}{p}\cdot\inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\\ u\neq 0}}\frac{\prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{p/(np_i)}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\\&\le\inf_{\substack{u\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\\ u\neq 0}}\frac{\sum_{i=1}^{n}\frac{1}{\theta_i}\left(\int_{\mathbb{R}^n}\left|\partial_{x_i}u\right|^{p_i}dx\right)^{\theta_i/p_i}}{\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{p/p^*}}\,.\end{aligned} \tag{2.14}$$

It follows from (2.7) and (2.14) that $u_0\circ\sigma_{\overrightarrow{\theta},u_0}\circ\tau_{\overrightarrow{\theta}}^{-1}$ is an extremal function of (2.2). This ends the proof of Point (ii). □

Now, we can prove Theorem 2.2 by using Proposition 2.3.

Proof of Theorem 2.2. We prove the results for the sole inequality (1.3). The results for (2.1) and (2.2) then follow from Proposition 2.3.

First, in case $p_+ < p^*$, the existence of extremal functions of (1.3) follows from the work of El Hamidi–Rakotoson [14]. Indeed, it has been proven in [14] that there exist minimizers for

$$\mathcal{I} := \inf_{\substack{u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n) \\ \int_{\mathbb{R}^n} |u|^{p^*} dx = 1}} \sum_{i=1}^{n} \frac{1}{p_i} \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} \, dx \,. \tag{2.15}$$

This infimum is connected with (1.3) by the change of scale $u \mapsto \mu_{\overrightarrow{p},u}^{-1} \cdot u \circ \rho_{\overrightarrow{p},u}$, where

$$\rho_{\overrightarrow{p},u}(x) := \mu_{\overrightarrow{p},u} \cdot \tau_{\overrightarrow{p}}(x) \,, \quad \mu_{\overrightarrow{p},u} := \left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/(np^*)} ,$$

and $\tau_{\overrightarrow{p}}(x)$ is as in (2.4) for all $x \in \mathbb{R}^n$ and $u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, $u \neq 0$. More precisely, simple calculations give

$$\sum_{i=1}^{n} \frac{1}{p_i} \int_{\mathbb{R}^n} \left| \partial_{x_i} \left(\mu_{\overrightarrow{p},u}^{-1} \cdot u \circ \rho_{\overrightarrow{p},u} \right) \right|^{p_i} dx = \left(\prod_{j=1}^{n} p_j^{-p/(np_j)} \right) \frac{\sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| \partial_{x_i} u \right|^{p_i} dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p^*}} \tag{2.16}$$

and

$$\int_{\mathbb{R}^n} \left| \mu_{\overrightarrow{p},u}^{-1} \cdot u \circ \rho_{\overrightarrow{p},u} \right|^{p^*} dx = 1 \,, \tag{2.17}$$

and hence

$$\mathcal{I} \le \left(\prod_{j=1}^{n} p_j^{-p/(np_j)} \right) \inf_{\substack{u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n) \\ u \neq 0}} \frac{\sum_{i=1}^{n} \int_{\mathbb{R}^n} \left| \partial_{x_i} u \right|^{p_i} dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p^*}} \,. \tag{2.18}$$

In particular, for any minimizer u of (2.15), since $\mu_{\overrightarrow{p},u} = 1$ and $\rho_{\overrightarrow{p},u} = \tau_{\overrightarrow{p}}$, it follows from (2.16)–(2.18) that $u \circ \tau_{\overrightarrow{p}}^{-1}$ is an extremal function of (1.3).

Next, we prove that the extremal functions of (1.3) do not change sign. We let C_0 be the best constant and u be an extremal function of (1.3). By writing $u = u_+ - u_-$, where $u_+ := \max(u, 0)$ and $u_- := \max(-u, 0)$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} \, dx &= \left(\frac{1}{C_0} \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p_*} = \left(\frac{1}{C_0} \int_{\mathbb{R}^n} u_-^{p^*} dx + \frac{1}{C_0} \int_{\mathbb{R}^n} u_+^{p^*} dx \right)^{p/p_*} \\ &\le \left(\left(\sum_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_{x_i} u_-|^{p_i} \, dx \right)^{p^*/p} + \left(\sum_{i=1}^{n} \int_{\mathbb{R}^n} |\partial_{x_i} u_+|^{p_i} \, dx \right)^{p^*/p} \right)^{p/p_*} . \end{aligned} \tag{2.19}$$

It follows from (2.19) that either $u_- = 0$ or $u_+ = 0$, and hence we obtain that the function u has constant sign.

Finally, from the Euler-Lagrange equation satisfied by u, namely

$$-\sum_{i=1}^{n} p_i \partial_{x_i} \left(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \right) = \lambda(u) \, |u|^{p^*-2} u, \quad \text{where} \quad \lambda(u) := \frac{\sum_{i=1}^{n} p_i \int_{\mathbb{R}^n} \left| \partial_{x_i} u \right|^{p_i} dx}{\int_{\mathbb{R}^n} |u|^{p^*} dx} \,,$$

we derive that the function $x \in \mathbb{R}^n \mapsto \mu u(\mu_1 x_1, \dots, \mu_n x_n)$ with $\mu_i := (\lambda(u)/p_i)^{1/p_i}$ for all $i = 1, \dots, n$ is a solution of (1.1) with $f(x, u) = |u|^{p^*-2} u$. This ends the proof of Theorem 2.2. □

3. Preliminary results

From now on, we are concerned with the general case of an arbitrary solution of (1.1).

For any $s \in (0, \infty)$ and any domain $\Omega \subset \mathbb{R}^n$, we define the weak Lebesgue space $L^{s,\infty}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{s,\infty}(\Omega)} := \sup_{h>0} \left(h \cdot \operatorname{meas}\left(\{|u| > h\}\right)^{1/s}\right) < \infty\,,$$

where $\operatorname{meas}(\{|u| > h\})$ is the measure of the set $\{x \in \Omega : |u(x)| > h\}$. The map $\|\cdot\|_{L^{s,\infty}(\Omega)}$ defines a quasi-norm on $L^{s,\infty}(\Omega)$. We refer, for instance, to the book of Grafakos [18] for the material on weak Lebesgue spaces.

The first result in this section is as follows.

Lemma 3.1. *Assume that $p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that (1.2) holds true. Then any solution of (1.1) belongs to $W^{1,\infty}(\mathbb{R}^n) \cap L^{p_*-1,\infty}(\mathbb{R}^n)$, and hence by interpolation, to $L^s(\mathbb{R}^n)$ for all $s \in (p_* - 1, \infty]$.*

Proof of Lemma 3.1. The L^∞–boundedness of the solutions follows from a straightforward adaptation of El Hamidi–Rakotoson [14, Propositions 1 and 2], the first proposition being in turn adapted from Fragalà–Gazzola–Kawohl [16, Theorem 2].

Once we have the L^∞–boundedness of the solutions, we obtain the L^∞–boundedness of the derivatives by applying Lieberman's gradient estimates [22].

The proof of the $L^{p_*-1,\infty}$–boundedness of the solutions follows exactly the same arguments as in Vétois [38, Lemma 2.2]. One only has to replace $|\nabla u|^p$ by $\sum_{i=1}^n |\partial_{x_i} u|^{p_i}$. ☐

For any solution u of (1.2), by Proposition A.1 in Appendix A, we obtain

$$-\Delta_{\overrightarrow{p}} |u| \le f(x, u) \cdot \operatorname{sgn}(u) \le \Lambda |u|^{p^*-1} \quad \text{in } \mathbb{R}^n, \tag{3.1}$$

where $\operatorname{sgn}(u)$ denotes the sign of u and the inequality is in the sense that for any nonnegative, smooth function φ with compact support in $\mathbb{R}^n$, we have

$$\sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_{x_i} |u||^{p_i-2} (\partial_{x_i} |u|)(\partial_{x_i}\varphi)\, dx \le \Lambda \int_{\mathbb{R}^n} |u|^{p^*-1} \varphi\, dx\,.$$

We prove the following result.

Lemma 3.2. *For any real number $\Lambda > 0$ and any nonnegative, nontrivial solution $v \in D^{1,p}(\mathbb{R}^n)$ of the inequality $-\Delta_{\overrightarrow{p}} v \le \Lambda v^{p^*-1}$ in $\mathbb{R}^n$, we have $\|v\|_{L^{p^*}(\mathbb{R}^n)} \ge \kappa_0$ for some constant $\kappa_0 = \kappa_0(n, p, \Lambda) > 0$.*

Proof. By testing the inequality $-\Delta_{\overrightarrow{p}} v \le \Lambda v^{p^*-1}$ with the function v, and applying the anisotropic Sobolev inequality, we obtain

$$\Lambda \int_{\mathbb{R}^n} v^{p^*} dx \ge \sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_{x_i} v|^{p_i}\, dx \ge K \left(\int_{\mathbb{R}^n} v^{p^*} dx\right)^{\frac{n-p}{n}} \tag{3.2}$$

for some constant $K = K(n, \overrightarrow{p})$. The result then follows from (3.2) with $\kappa_0 := (K/\Lambda)^{\frac{n-p}{p^2}}$. ☐

As a last result in this section, we prove the following decay estimate. This result is not sharp, but it turns out to be a crucial ingredient in what follows.

Lemma 3.3. *Assume that $p_+ < p^*$. Let κ_0 be as in Lemma 3.2, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that (1.2) holds true, and u be a solution of (1.1). For any $\kappa > 0$, we define*

$$r_\kappa(u) := \inf\left(\left\{r > 0 \,:\, \|u\|_{L^{p^*}\left(\mathbb{R}^n\backslash B_{\overrightarrow{p}}(0,r)\right)} < \kappa\right\}\right), \tag{3.3}$$

where $B_{\overrightarrow{p}}(0,r)$ is the open ball of center 0 and radius r with respect to the distance function $d_{\overrightarrow{p}}$ defined as

$$d_{\overrightarrow{p}}(x,y) := \sum_{i=1}^n |x_i - y_i|^{\frac{\delta p_i}{p^* - p_i}} \quad \text{with} \quad \delta := \frac{p^* - p_+}{p_+}, \tag{3.4}$$

for all $x, y \in \mathbb{R}^n$. Then for any $\kappa \in (0, \kappa_0)$ and $r > r_\kappa(u)$, there exists a constant $K_0 = K_0\big(n, \overrightarrow{p}, \Lambda, \kappa, r, r_\kappa(u), \|u\|_{L^{p^*}(\mathbb{R}^n)}\big)$ such that

$$|u(x)| \le K_0 \left(\sum_{i=1}^n |x_i|^{\frac{p_i}{p^* - p_i}}\right)^{-1} \quad \text{for all } x \in \mathbb{R}^n \backslash B_{\overrightarrow{p}}(0,r). \tag{3.5}$$

Proof of Lemma 3.3. This proof is adapted from Vétois [38, Lemma 3.1] We fix $\Lambda > 0$, $\kappa \in (0, \kappa_0)$, $\kappa' > \kappa_0$, $r > 0$, and $r' \in (0, r)$. We claim that in order to obtain Lemma 3.3, it is sufficient to prove that there exists a constant $K_1 = K_1(n, \overrightarrow{p}, \kappa, \kappa', r, r')$ such that for any solution u of (1.1) such that $r_\kappa(u) \le r'$ and $\|u\|_{L^{p^*}(\mathbb{R}^n)} \le \kappa'$, we have

$$d_{\overrightarrow{p}}\left(x, B_{\overrightarrow{p}}(0, r'')\right) |u(x)|^\delta \le K_1 \quad \text{for all } x \in \mathbb{R}^n \backslash B_{\overrightarrow{p}}(0,r), \tag{3.6}$$

where $r'' := (r + r')/2$. Indeed, for any $x \in \mathbb{R}^n \backslash B_{\overrightarrow{p}}(0,r)$, we can write

$$d_{\overrightarrow{p}}(x,0) \le d_{\overrightarrow{p}}\left(x, B_{\overrightarrow{p}}(0,r'')\right) + r'' \le d_{\overrightarrow{p}}\left(x, B_{\overrightarrow{p}}(0,r'')\right) + \frac{r''}{r} d_{\overrightarrow{p}}(x,0), \tag{3.7}$$

and hence by putting together (3.6) and (3.7), we obtain

$$d_{\overrightarrow{p}}(x,0) |u(x)|^\delta \le \frac{r}{r - r''} \cdot K_1 = \frac{2r}{r - r'} \cdot K_1. \tag{3.8}$$

By definition of $d_{\overrightarrow{p}}$, (3.5) then follows from (3.8). This proves our claim.

We prove (3.6) by contradiction. Suppose that for any $\alpha \in \mathbb{N}$, there exists a Caratheodory function $f_\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that (1.2) holds true, a solution u_α of (1.1) with $f = f_\alpha$ such that $r_\kappa(u_\alpha) \le r'$ and $\|u_\alpha\|_{L^{p^*}(\mathbb{R}^n)} \le \kappa'$, and a point $x_\alpha \in \mathbb{R}^n \backslash B(0,r)$ such that

$$d_{\overrightarrow{p}}\left(x_\alpha, B_{\overrightarrow{p}}(0,r'')\right) |u_\alpha(x_\alpha)|^\delta > 2\alpha. \tag{3.9}$$

It follows from (3.9) and Poláčik–Quittner–Souplet [25, Lemma 5.1] that there exists $y_\alpha \in \mathbb{R}^n \backslash B_{\overrightarrow{p}}(0, r'')$ such that

$$d_{\overrightarrow{p}}\left(y_\alpha, B_{\overrightarrow{p}}(0,r'')\right) |u_\alpha(y_\alpha)|^\delta > 2\alpha, \quad |u_\alpha(x_\alpha)| \le |u_\alpha(y_\alpha)|, \tag{3.10}$$

and

$$|u_\alpha(y)| \le 2^{1/\delta} |u_\alpha(y_\alpha)| \quad \text{for all } y \in B_{\overrightarrow{p}}\big(y_\alpha, \alpha\, |u_\alpha(y_\alpha)|^{-\delta}\big). \tag{3.11}$$

For any α and $y \in \mathbb{R}^n$, we define

$$\widetilde{u}_\alpha(y) := |u_\alpha(y_\alpha)|^{-1} \cdot u_\alpha(\tau_\alpha(y)), \tag{3.12}$$

where

$$\tau_\alpha(y) := y_\alpha + \Big(|u_\alpha(y_\alpha)|^{\frac{p_1 - p^*}{p_1}} y_1, \dots, |u_\alpha(y_\alpha)|^{\frac{p_n - p^*}{p_n}} y_n\Big).$$

It follows from (3.11) and (3.12) that

$$|\widetilde{u}_\alpha(0)| = 1 \quad \text{and} \quad |\widetilde{u}_\alpha(y)| \le 2^{1/\delta} \quad \text{for all } y \in B_{\overrightarrow{p}}(0, \alpha). \tag{3.13}$$

Moreover, by (1.1), we obtain

$$-\Delta_{\overrightarrow{p}}\widetilde{u}_\alpha = |u_\alpha(y_\alpha)|^{1-p^*}\cdot f_\alpha\left(\tau_\alpha(y), |u_\alpha(y_\alpha)|\cdot\widetilde{u}_\alpha\right) \quad \text{in } \mathbb{R}^n, \tag{3.14}$$

and (1.2) gives

$$|u_\alpha(y_\alpha)|^{1-p^*}\cdot\left|f_\alpha\left(\tau_\alpha(y), |u_\alpha(y_\alpha)|\cdot\widetilde{u}_\alpha\right)\right| \le \Lambda\,|\widetilde{u}_\alpha|^{p^*-1}. \tag{3.15}$$

By Lieberman's gradient estimates [22], it follows from (3.13) and (3.15) that there exists a constant $C>0$ such that for any $R>0$, we have

$$\|\nabla\widetilde{u}_\alpha\|_{L^\infty(B_{\overrightarrow{p}}(0,R))} \le C \tag{3.16}$$

for large α. By Arzela–Ascoli Theorem and a diagonal argument, it follows from (3.13) and (3.16) that $(\widetilde{u}_\alpha)_\alpha$ converges up to a subsequence in $C^0_{\text{loc}}(\mathbb{R}^n)$ to some Lipschitz continuous function $\widetilde{u}_\infty$ such that $|\widetilde{u}_\infty(0)|=1$. Moreover, by testing (3.14)–(3.15) with $\widetilde{u}_\alpha$, we obtain

$$\sum_{i=1}^n\int_{\mathbb{R}^n}|\partial_{x_i}\widetilde{u}_\alpha|^{p_i}\,dx \le \Lambda\int_{\mathbb{R}^n}|\widetilde{u}_\alpha|^{p^*}\,dx = \Lambda\int_{\mathbb{R}^n}|u_\alpha|^{p^*}\,dx \le \Lambda\,(\kappa')^{p^*}. \tag{3.17}$$

Since $|\partial_{x_i}|\widetilde{u}_\alpha|| = |\partial_{x_i}\widetilde{u}_\alpha|$ a.e. in $\mathbb{R}^n$, it follows from (3.17) that $(|\widetilde{u}_\alpha|)_\alpha$ converges weakly up to a subsequence to $|\widetilde{u}_\infty|$ in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. Passing to the limit into (3.14)–(3.15), we then obtain that $|\widetilde{u}_\infty|$ is a weak solution of the inequality

$$-\Delta_{\overrightarrow{p}}|\widetilde{u}_\infty| \le \Lambda\,|\widetilde{u}_\infty|^{p^*-1} \quad \text{in } \mathbb{R}^n. \tag{3.18}$$

In particular, since $|\widetilde{u}_\infty(0)|=1$, it follows from Lemma 3.2 that $\|u_\infty\|_{L^{p^*}(\mathbb{R}^n)} \ge \kappa_0$, and hence there exists a real number $R>0$ such that

$$\|u_\infty\|_{L^{p^*}(B(0,R))} > \kappa\,. \tag{3.19}$$

On the other hand, we have

$$\|\widetilde{u}_\alpha\|_{L^{p^*}(B_{\overrightarrow{p}}(0,R))} = \|u_\alpha\|_{L^{p^*}\left(B_{\overrightarrow{p}}\left(y_\alpha, R\cdot|u_\alpha(y_\alpha)|^{-\delta}\right)\right)}. \tag{3.20}$$

By (3.10) and since $r_\kappa(u_\alpha) < r''$, we obtain

$$B_{\overrightarrow{p}}\left(y_\alpha, R\cdot|u_\alpha(y_\alpha)|^{-\delta}\right)\cap B_{\overrightarrow{p}}\left(0, r_\kappa(u_\alpha)\right) = \emptyset \tag{3.21}$$

for large α. By definition of $r_\kappa(u_\alpha)$, it follows from (3.20) and (3.21) that

$$\|\widetilde{u}_\alpha\|_{L^{p^*}(B_{\overrightarrow{p}}(0,R))} \le \kappa \tag{3.22}$$

for large α, which is in contradiction with (3.19). This ends the proof of Lemma 3.3. $\square$

4. The reverse Hölder-type inequalities

The following result is a key step in the Moser-type iteration scheme that we develop in the next section.

Lemma 4.1. *Assume that $p_+ < p^*$. Let $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function such that (1.2) holds true, u be a solution of (1.1), and κ, r, and K_0 be as in Lemma 3.3. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two disjoint subsets of $\{1,\dots,n\}$, $\mathcal{I}_2\ne\emptyset$, and $\overrightarrow{q}=(q_i)_{i\in\mathcal{I}_1\cup\mathcal{I}_2}$ be such that $q_i>1$ for all $i\in\mathcal{I}_1\cup\mathcal{I}_2$. Then there exists a constant $c_0=c_0(n,\overrightarrow{p},\Lambda,K_0)>1$ such that for any $R_1,R_2>0$, $\lambda<\lambda'\in(0,1/2]$, and $\gamma>p_*-1$ such that $\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')\cap B_{\overrightarrow{p}}(0,\max(r,1))=\emptyset$ and*

$\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')\cap\operatorname{supp}(u)$ is bounded, where $\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda)$ is as in (1.10), we have

$$\|u\|^{\gamma}_{L^{\gamma}\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda)\right)}\le c_0\gamma^{p^*}\max_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\Big(\min(1,\gamma-p_*+1)^{-p_i}(\lambda'-\lambda)^{-p_i}q_i^{p_i}\times R_{\delta_i}^{-\frac{p_i}{q_i}}\|u\|^{\gamma_i}_{L^{\gamma_i}\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')\right)}\Big)^{\frac{n}{n-p}},\quad(4.1)$$

where $\delta_i:=1$ if $i\in\mathcal{I}_1$, $\delta_i:=2$ if $i\in\mathcal{I}_2$, and $\gamma_i:=\frac{n-p}{n}\gamma+p_i-p$ for all $i\in\mathcal{I}_1\cup\mathcal{I}_2$.

Preliminary to the proof of Lemma 4.1, we prove the following result.

Lemma 4.2. *Let v be a nonnegative solution in $D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ of*

$$-\Delta_{\overrightarrow{p}}v\le\Lambda v^{p^*-1}\quad\text{in }\mathbb{R}^n,\quad(4.2)$$

for some real number $\Lambda>0$, where the inequality must be understood in the weak sense as in (3.1). Let $\beta>-1$ and $\eta\in C^1(\mathbb{R}^n)$ be such that $0\le\eta\le1$ in $\mathbb{R}^n$, ηv has compact support, and $\eta^{(\beta+p_-)/p_+}\in C^1(\mathbb{R}^n)$, where $p_-:=\min(\{p_i\in\overrightarrow{p}\})$ and $p_+:=\max(\{p_i\in\overrightarrow{p}\})$. Then there exists a constant $C=C(n,\overrightarrow{p})$ such that

$$\left(\int_{\mathbb{R}^n}(\eta v)^{\frac{n(\beta+p)}{n-p}}dx\right)^{\frac{n-p}{n}}\le C\left(|\beta|^p+1\right)\Big(\Lambda(\beta+1)^{-1}\int_{\mathbb{R}^n}\eta^{\beta+p_-}v^{\beta+p^*}dx+\sum_{i=1}^n\min(1,\beta+1)^{-p_i}\int_{\mathbb{R}^n}|\partial_{x_i}\eta|^{p_i}\eta^{\beta+p_--p_i}v^{\beta+p_i}dx\Big).\quad(4.3)$$

The finiteness of the integrals in (4.3) is ensured by the fact that $v\in L^\infty(\mathbb{R}^n)$, ηv has compact support, and $\eta^{(\beta+p_-)/p_+}\in C^1(\mathbb{R}^n)$.

Proof of Lemma 4.2. For any $\varepsilon>0$, we define $v_\varepsilon:=v+\varepsilon\overline{\eta}$, where $\overline{\eta}$ is a cutoff function on a neighborhood of the support of ηv such that $\overline{\eta}^{\beta+1}\in C^1(\mathbb{R}^n)$. Since $v\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $\eta^{(\beta+p_-)/p_+}\in C^1(\mathbb{R}^n)$, and $(\beta+p_-)/p_+\le(\beta+p_-)/p_-=1+\beta/p_-$, we get $(\eta v_\varepsilon)^{\min(1,1+\beta/p_-)}\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. By a generalized version of the anisotropic Sobolev inequality (see Cîrstea–Vétois [7, Lemma A.1]), we then obtain

$$\|\eta v_\varepsilon\|^{\beta+p}_{L^{\frac{n(\beta+p)}{n-p}}(\mathbb{R}^n)}\le C(\beta+p)^p\prod_{i=1}^n\left\|(\eta v_\varepsilon)^{\frac{\beta}{p_i}}\partial_{x_i}(\eta v_\varepsilon)\right\|^{\frac{p}{n}}_{L^{p_i}(\mathbb{R}^n)}<\infty\quad(4.4)$$

for some constant $C=C(n,\overrightarrow{p})$. For any $i=1,\dots,n$, we have

$$\int_{\mathbb{R}^n}(\eta v_\varepsilon)^\beta|\partial_{x_i}(\eta v_\varepsilon)|^{p_i}dx\le2^{p_i-1}\left(\int_{\mathbb{R}^n}|\partial_{x_i}\eta|^{p_i}\eta^\beta v_\varepsilon^{\beta+p_i}dx+\int_{\mathbb{R}^n}|\partial_{x_i}v_\varepsilon|^{p_i}\eta^{\beta+p_i}v_\varepsilon^\beta dx\right).\quad(4.5)$$

Since $v\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, $\overline{\eta}^{\beta+1}\in C^1(\mathbb{R}^n)$, $\eta^{(\beta+p_-)/p_+}\in C^1(\mathbb{R}^n)$, and $(\beta+p_-)/p_+\le\beta+p_-$, we get $\eta^{\beta+p_i}v_\varepsilon^{\beta+1}\in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$. For any $i=1,\dots,n$, since $v_\varepsilon\equiv v+\varepsilon$ on the support of ηv, testing (4.2) with $\eta^{\beta+p_i}v_\varepsilon^{\beta+1}$ gives

$$(\beta+1)\sum_{j=1}^n\int_{\mathbb{R}^n}\left|\partial_{x_j}v\right|^{p_j}\eta^{\beta+p_i}v_\varepsilon^\beta dx\le\Lambda\int_{\mathbb{R}^n}\eta^{\beta+p_i}v^{p^*-1}v_\varepsilon^{\beta+1}dx-(\beta+p_i)\sum_{j=1}^n\int_{\mathbb{R}^n}\left|\partial_{x_j}v\right|^{p_j-2}\left(\partial_{x_j}v\right)\left(\partial_{x_j}\eta\right)\eta^{\beta+p_i-1}v_\varepsilon^{\beta+1}dx\,.\quad(4.6)$$

For any $i, j = 1, \ldots, n$, Youngs inequality yields

$$-\left(\beta+p_i\right)\left|\partial_{x_j} v\right|^{p_j-2}\left(\partial_{x_j} v\right)\left(\partial_{x_j} \eta\right) \eta^{\beta+p_i-1} v_{\varepsilon}^{\beta+1} \leq \frac{p_j-1}{p_j} \cdot(\beta+1)\left|\partial_{x_j} v\right|^{p_j} \eta^{\beta+p_i} v_{\varepsilon}^{\beta}+\frac{1}{p_j} \cdot \frac{\left(\beta+p_i\right)^{p_j}}{(\beta+1)^{p_j-1}}\left|\partial_{x_j} \eta\right|^{p_j} \eta^{\beta+p_i-p_j} v_{\varepsilon}^{\beta+p_j}. \quad (4.7)$$

It follows from (4.6) and (4.7) that

$$\sum_{j=1}^{n} \frac{1}{p_j} \int_{\mathbb{R}^n}\left|\partial_{x_j} v\right|^{p_j} \eta^{\beta+p_i} v_{\varepsilon}^{\beta} d x \leq \Lambda(\beta+1)^{-1} \int_{\mathbb{R}^n} \eta^{\beta+p_i} v^{p^*-1} v_{\varepsilon}^{\beta+1} d x +\sum_{j=1}^{n} \frac{1}{p_j} \cdot\left(\frac{\beta+p_i}{\beta+1}\right)^{p_j} \int_{\mathbb{R}^n}\left|\partial_{x_j} \eta\right|^{p_j} \eta^{\beta+p_i-p_j} v_{\varepsilon}^{\beta+p_j} d x . \quad (4.8)$$

In particular, by (4.5) and (4.8), we obtain

$$\int_{\mathbb{R}^n}\left(\eta v_{\varepsilon}\right)^{\beta}\left|\partial_{x_i}\left(\eta v_{\varepsilon}\right)\right|^{p_i} d x \leq C\Bigg(\Lambda(\beta+1)^{-1} \int_{\mathbb{R}^n} \eta^{\beta+p_i} v^{p^*-1} v_{\varepsilon}^{\beta+1} d x +\sum_{j=1}^{n} \min (1, \beta+1)^{-p_j} \int_{\mathbb{R}^n}\left|\partial_{x_j} \eta\right|^{p_j} \eta^{\beta+p_i-p_j} v_{\varepsilon}^{\beta+p_j} d x+\varepsilon^{\beta+p_i} \int_{\mathbb{R}^n}\left|\partial_{x_i} \overline{\eta}\right|^{p_i} \eta^{\beta+p_i} \overline{\eta}^{\beta} d x\Bigg) \quad (4.9)$$

for some constant $C=C(n, \overrightarrow{p})$. Finally, since $\eta^{p_i} \leq \eta^{p_-}$, we get (4.3) by plugging (4.9) into (4.4) and passing to the limit as $\varepsilon \rightarrow 0$. This ends the proof of Lemma 4.2. □

Now, we can prove Lemma 4.1 by using Lemma 4.2.

Proof of Lemma 4.1. We denote $\beta:=\frac{n-p}{n} \gamma-p$. In particular, $\gamma>p_*-1$ is equivalent to $\beta>-1$. In connexion with the sets $\Omega_{\overrightarrow{q}}\left(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda\right)$, we define test functions of the form

$$\eta(x):=\left[\overline{\eta}_{\lambda, \lambda'}\left(R_1^{-1} \sum_{i \in \mathcal{I}_1}\left|x_i\right|^{q_i}\right) \widetilde{\eta}_{\lambda, \lambda'}\left(R_2^{-1} \sum_{i \in \mathcal{I}_2}\left|x_i\right|^{q_i}\right)\right]^{\max \left(1, \frac{p_+}{\beta+p_-}\right)} \quad (4.10)$$

for all $x \in \mathbb{R}^n$, where $\overline{\eta}_{\lambda, \lambda'}, \widetilde{\eta}_{\lambda, \lambda'} \in C^1(0, \infty)$ satisfy $0 \leq \overline{\eta}_{\lambda, \lambda'}, \widetilde{\eta}_{\lambda, \lambda'} \leq 1$ in $(0, \infty)$, $\overline{\eta}_{\lambda, \lambda'}=1$ in $[0,1+\lambda]$, $\overline{\eta}_{\lambda, \lambda'}=0$ in $\left[1+\lambda', \infty\right)$, $\left|\overline{\eta}_{\lambda, \lambda'}'\right| \leq 2$ in $\left[1+\lambda, 1+\lambda'\right]$, $\widetilde{\eta}_{\lambda, \lambda'}=1$ in $[1-\lambda, 1+\lambda]$, $\widetilde{\eta}_{\lambda, \lambda'}=0$ in $\left[0,1-\lambda'\right] \cup\left[1+\lambda', \infty\right)$, and $\left|\widetilde{\eta}_{\lambda, \lambda'}'\right| \leq 2 /\left(\lambda'-\lambda\right)$ in $\left[1-\lambda', 1-\lambda\right] \cup\left[1+\lambda, 1+\lambda'\right]$. With these properties of $\overline{\eta}_{\lambda, \lambda'}$ and $\widetilde{\eta}_{\lambda, \lambda'}$, we obtain

$$0 \leq \eta \leq 1 \text { in } \mathbb{R}^n,\ \eta=1 \text { in } \Omega_{\overrightarrow{q}}\left(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda\right), \text { and } \eta=0 \text { in } \mathbb{R}^n \backslash \Omega_{\overrightarrow{q}}\left(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda'\right) .$$

Since $\Omega_{\overrightarrow{q}}\left(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda'\right) \cap \operatorname{supp}(u)$ is bounded by assumption, we get that ηu has compact support. Moreover, since $q_i>1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$, we get $\eta^{\left(\beta+p_-\right) / p_+} \in C^1\left(\mathbb{R}^n\right)$ and

$$\left|\partial_{x_i} \eta(x)\right|^{p_i} \eta(x)^{\beta+p_--p_i} \leq\left(\frac{4 q_i}{\lambda'-\lambda} \max \left(1, \frac{p_+}{\beta+p_-}\right)\right)^{p_i} R_{\delta_i}^{-\frac{p_i}{q_i}} \quad (4.11)$$

for all $x \in \operatorname{supp}(\eta)$, where $\delta_i := 1$ if $i \in \mathcal{I}_1$, $\delta_i := 2$ if $i \in \mathcal{I}_2$. By applying Lemma 4.2 with $v = |u|$ and η as in (4.10), and using (4.11), we obtain

$$\left(\int_{\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda)} u^\gamma dx\right)^{\frac{n-p}{n}} \le C\left(|\beta|^p + 1\right)\Bigg(\Lambda(\beta+1)^{-1}\int_{\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')} u^{\beta+p^*}dx + \sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2} \min(1,\beta+1)^{-p_i}(\lambda'-\lambda)^{-p_i} q_i^{p_i} R_{\delta_i}^{-\frac{p_i}{q_i}} \int_{\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')} u^{\beta+p_i}dx\Bigg) \tag{4.12}$$

for some constant $C = C(n, \overrightarrow{p})$.

Now, we estimate the first integral in the right-hand side of (4.12). We claim that there exists a constant $C' = C(n, \overrightarrow{p}, K_0)$ such that

$$u(x)^{p^*-p_{i_0}} \le C' R_2^{-\frac{p_{i_0}}{q_{i_0}}} \quad \text{for all } x \in \Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda'), \tag{4.13}$$

where K_0 is the constant given by Lemma 3.3 and $i_0 \in \mathcal{I}_2$ is such that

$$\frac{q_{i_0}(p^*-p_{i_0})}{p_{i_0}} = \max_{i\in\mathcal{I}_2}\left(\frac{q_i(p^*-p_i)}{p_i}\right). \tag{4.14}$$

We prove this claim. For any $x \in \Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda')$, since $\lambda' \le 1/2$, we obtain

$$\frac{R_2}{2} \le \sum_{i\in\mathcal{I}_2} |x_i|^{q_i} \le n\cdot |x_{i(x)}|^{q_{i(x)}} \le n \cdot d_{\overrightarrow{p}}(x,0)^{\frac{q_{i(x)}(p^*-p_{i(x)})}{\delta p_{i(x)}}}, \tag{4.15}$$

where $i(x) \in \mathcal{I}_2$ is such that $|x_{i(x)}|^{q_{i(x)}} = \max(\{|x_i|^{q_i} : i \in \mathcal{I}_2\})$, and the distance function $d_{\overrightarrow{p}}$ and the real number δ are as in (3.4). Since $\Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda') \cap B_{\overrightarrow{p}}(0, \max(r,1)) = \emptyset$ by assumption, (4.13) follows from (4.14), (4.15), and Lemma 3.3. In particular, (4.13) implies

$$\int_{\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')} u^{\beta+p^*}dx \le C' R_2^{-\frac{p_{i_0}}{q_{i_0}}} \int_{\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda')} u^{\beta+p_{i_0}}dx. \tag{4.16}$$

Finally, (4.1) follows from (4.12), (4.16), and the fact that $\beta + 1 = \frac{n-p}{n}(\gamma - p_* + 1)$ and $\beta + p_i = \gamma_i$. This ends the proof of Lemma 4.1. ☐

5. The iteration scheme

In this section, we describe the iteration scheme which leads to the proofs of our main results.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two disjoint subsets of $\{1,\dots,n\}$, $\mathcal{I}_2 \ne \emptyset$, and $\overrightarrow{q} = (q_i)_{i\in\mathcal{I}_1\cup\mathcal{I}_2}$ be such that $q_i > 1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$. The idea is to apply Lemma 4.1 by induction. For any $\gamma > p_* - 1$, Lemma 4.1 provides an estimate of the L^γ–norm of u with respect to the set of $L^{\gamma_{i_1}}$–norms of u, where $\gamma_{i_1} := \frac{n-p}{n}\gamma + p_{i_1} - p$ for all $i_1 \in \mathcal{I}_1 \cup \mathcal{I}_2$. If $\gamma_{i_1} > p_* - 1$, then another application of Lemma 4.1 gives estimates of the $L^{\gamma_{i_1}}$–norms of u with respect to the set of $L^{\gamma_{i_1 i_2}}$–norms of u, where $\gamma_{i_1 i_2} := \frac{n-p}{n}\gamma_{i_1} + p_{i_2} - p$, etc... By induction, we define

$$\gamma_{i_1,\dots,i_{j+1}} := \frac{n-p}{n}\gamma_{i_1,\dots,i_j} + p_{i_{j+1}} - p \tag{5.1}$$

for all $j \in \mathbb{N}$ and $i_1, \dots, i_{j+1} \in \mathcal{I}_1 \cup \mathcal{I}_2$, with the convention that $\gamma_{i_1,\dots,i_j} := \gamma$ if $j = 0$. In particular, we obtain the formula

$$\gamma_{i_1,\dots,i_k} = \left(\frac{n-p}{n}\right)^k \gamma + \sum_{j=1}^k \left(\frac{n-p}{n}\right)^{k-j}(p_{i_j} - p) \tag{5.2}$$

for all $k \in \mathbb{N}$. The stopping condition in our induction argument is $\gamma_{i_1,\dots,i_k} < \frac{n}{p}(p_\varepsilon - p)$, where

$$p_\varepsilon := (1+\varepsilon)\, p_0\,, \quad p_0 := \max\left(p_*, \{p_i \,:\, i \in \mathcal{I}_1 \cup \mathcal{I}_2\}\right), \tag{5.3}$$

and ε is a fixed real number in $(0,1)$. Note that $\frac{n}{p}(p_\varepsilon - p) > \frac{n}{p}(p_* - p) = p_* - 1$ so that we can apply Lemma 4.1 as long as our stopping condition is not satisfied. For any $k \geq 1$, we let $\Phi_{k,\gamma,\varepsilon}$ be the set of all sequences of indices for which our induction argument stops after exactly k iterations, namely

$$\Phi_{k,\gamma,\varepsilon} := \Big\{ (i_1,\dots,i_k) \in (\mathcal{I}_1 \cup \mathcal{I}_2)^k \,:\quad \gamma_{i_1,\dots,i_j} \geq \frac{n}{p}(p_\varepsilon - p) \text{ for all } j = 0,\dots,k-1 \\ \text{and}\quad \gamma_{i_1,\dots,i_k} < \frac{n}{p}(p_\varepsilon - p) \Big\}. \tag{5.4}$$

The following result provides a control on the number of iterations in our induction argument.

Lemma 5.1. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two disjoint subsets of $\{1,\dots,n\}$, $\mathcal{I}_2 \neq \emptyset$, and $\overrightarrow{q} = (q_i)_{i \in \mathcal{I}_1 \cup \mathcal{I}_2}$ be such that $q_i > 1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$. Then for any $\varepsilon > 0$, $\gamma \geq \frac{n}{p}(p_\varepsilon - p)$, $k \in \mathbb{N}$, and $(i_1,\dots,i_k) \in (\mathcal{I}_1 \cup \mathcal{I}_2)^k$, we have*

$$\gamma_{i_1,\dots,i_k} > \frac{n}{p}(p_\varepsilon - p) \quad \text{if } k < k^-_{\gamma,\varepsilon} \quad \text{and} \quad \gamma_{i_1,\dots,i_k} < \frac{n}{p}(p_\varepsilon - p) \quad \text{if } k \geq k^+_{\gamma,\varepsilon}\,, \tag{5.5}$$

where $\gamma_{i_1,\dots,i_k}$ is as in (5.1), p_ε is as in (5.3), and $k^-_{\gamma,\varepsilon}$ and $k^+_{\gamma,\varepsilon}$ are the smallest and largest natural numbers, respectively, such that

$$\frac{n}{p}\left(\frac{n}{n-p}\right)^{k^+_{\gamma,\varepsilon}-1} \varepsilon p_0 < \gamma < \frac{n}{p}\left(\frac{n}{n-p}\right)^{k^-_{\gamma,\varepsilon}} (p_\varepsilon - p_-)\,, \tag{5.6}$$

where $p_- := \min\left(\{p_i \in \overrightarrow{p}\}\right)$. In particular, we have $\Phi_{k,\gamma,\varepsilon} = \emptyset$ for all $k < k^-_{\gamma,\varepsilon}$ and $k > k^+_{\gamma,\varepsilon}$, where $\Phi_{k,\gamma,\varepsilon}$ is as in (5.4).

Proof of Lemma 5.1. Since $p_- \leq p_{i_j} \leq p_0$ for all $j = 1,\dots,k$, it follows from (5.2) that

$$-\sum_{j=1}^{k}\left(\frac{n-p}{n}\right)^{k-j}(p - p_-) \leq \gamma_{i_1,\dots,i_k} - \left(\frac{n-p}{n}\right)^k \gamma \leq \sum_{j=1}^{k}\left(\frac{n-p}{n}\right)^{k-j}(p_0 - p)\,. \tag{5.7}$$

Moreover, by a simple calculation, we obtain

$$\sum_{j=1}^{k}\left(\frac{n-p}{n}\right)^{k-j} = \frac{n}{p}\left(1 - \left(\frac{n-p}{n}\right)^k\right) < \frac{n}{p}\,. \tag{5.8}$$

It follows from (5.7) and (5.8) that

$$-\frac{n}{p}(p - p_-) < \gamma_{i_1,\dots,i_k} - \left(\frac{n-p}{n}\right)^k \gamma < \frac{n}{p}(p_0 - p)\,. \tag{5.9}$$

Finally, (5.5) follows from (5.9) together with the definitions of $k^-_{\gamma,\varepsilon}$ and $k^+_{\gamma,\varepsilon}$. ☐

Now, we can prove the main result of this section.

Lemma 5.2. *Assume that $p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that (1.2) holds true, u be a solution of (1.1), and κ, r, and K_0 be as in Lemma 3.3. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two disjoint subsets of $\{1,\dots,n\}$, $\mathcal{I}_2 \neq \emptyset$, and $\overrightarrow{q} = (q_i)_{i \in \mathcal{I}_1 \cup \mathcal{I}_2}$ be such that $q_i > 1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$. Then there exists a constant $c_1 = c_1(n, \overrightarrow{p}, \Lambda, K_0) > 1$ such that for any $\varepsilon \in (0,1)$,*

$\gamma > \frac{n}{p}(p_\varepsilon - p)$, and $R_1, R_2 > 0$ such that $\Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r,1)) = \emptyset$ and $\Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, 1/2) \cap \operatorname{supp}(u)$ is bounded, we have

$$\|u\|_{L^\gamma\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda_{0,\gamma,\varepsilon})\right)} \le c_1^{\frac{1}{\varepsilon}} \max(q_i)^{\frac{1}{\varepsilon}\cdot\frac{n}{n-p}} (p_\varepsilon - p_*)^{-\frac{1}{\varepsilon}\cdot\frac{n}{n-p}} \times \max_{(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}} \left(\left(\prod_{j=1}^k R_{\delta_{i_j}}^{-\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^j \frac{p_{i_j}}{q_{i_j}}}\right) \|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(\Omega_{\overrightarrow{q}}\left(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}\right), \tag{5.10}$$

where $\delta_{i_j} := 1$ if $i_j \in \mathcal{I}_1$, $\delta_{i_j} := 2$ if $i_j \in \mathcal{I}_2$, $\gamma_{i_1,\dots,i_k}$ is as in (5.1), p_ε is as in (5.3), $\Omega_{\overrightarrow{q}}(\mathcal{I}_1, R_1, \mathcal{I}_2, R_2, \lambda_k)$ is as in (1.10), and

$$\lambda_{k,\gamma,\varepsilon} := \frac{1}{4}\left(1 + 2^{k-k_{\gamma,\varepsilon}^+-1}\right) \quad \text{and} \quad \Phi_{\gamma,\varepsilon} := \bigcup_{k=k_{\gamma,\varepsilon}^-}^{k_{\gamma,\varepsilon}^+} \Phi_{k,\gamma,\varepsilon} \tag{5.11}$$

with $k_{\gamma,\varepsilon}^-$ and $k_{\gamma,\varepsilon}^+$ as in Lemma 5.1, and $\Phi_{k,\gamma,\varepsilon}$ as in (5.4).

Proof of Lemma 5.2. Applying Lemma 4.1 by induction with the stopping condition $\gamma_{i_1,\dots,i_k} < \frac{n}{p}(p_\varepsilon - p)$ gives

$$\|u\|_{L^\gamma\left(\Omega_{\overrightarrow{q}}(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda_{0,\gamma,\varepsilon})\right)} \le \max_{(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}} \Bigg(\mathcal{A}_{k,\gamma} \times \mathcal{B}_{i_1,\dots,i_{k-1},\gamma} \times \mathcal{C}_{i_1,\dots,i_k,\gamma} \times \mathcal{D}_{i_1,\dots,i_k,\gamma,\varepsilon} \times \left(\prod_{j=1}^k R_{\delta_{i_j}}^{-\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^j \frac{p_{i_j}}{q_{i_j}}}\right) \|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(\Omega_{\overrightarrow{q}}\left(\mathcal{I}_1,R_1,\mathcal{I}_2,R_2,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}\Bigg), \tag{5.12}$$

where $\lambda_{k,\gamma,\varepsilon}$ and $\Phi_{\gamma,\varepsilon}$ are as in (5.11), and

$$\mathcal{A}_{k,\gamma} := \left(c_0 \cdot \max\left(q_i^{\frac{np_i}{n-p}}\right)\right)^{\frac{1}{\gamma}\sum_{j=0}^{k-1}\left(\frac{n}{n-p}\right)^j}, \quad \mathcal{B}_{i_1,\dots,i_{k-1},\gamma} := \prod_{j=1}^k \gamma_{i_1,\dots,i_{j-1}}^{\frac{p}{\gamma}\left(\frac{n}{n-p}\right)^j},$$

$$\mathcal{C}_{i_1,\dots,i_{k-1},\gamma} := \prod_{j=1}^k \min\left(1, \gamma_{i_1,\dots,i_{j-1}} - p_* + 1\right)^{-\frac{p_{i_j}}{\gamma}\left(\frac{n}{n-p}\right)^j},$$

$$\mathcal{D}_{i_1,\dots,i_k,\gamma,\varepsilon} := \prod_{j=1}^k \left(\lambda_{j,\gamma,\varepsilon} - \lambda_{j-1,\gamma,\varepsilon}\right)^{-\frac{p_{i_j}}{\gamma}\left(\frac{n}{n-p}\right)^j}.$$

Now, we fix $(i_1,\dots,i_k) \in \Phi_{\gamma,\varepsilon}$ and estimate each of the terms in the right-hand side of (5.12).

Estimate of $\mathcal{A}_{k,\gamma}$. By using the fact that $k \le k_{\gamma,\varepsilon}^+$ and applying (5.6), we obtain

$$\sum_{j=0}^{k-1}\left(\frac{n}{n-p}\right)^j = \frac{n-p}{p}\left[\left(\frac{n}{n-p}\right)^k - 1\right] < \frac{n}{p}\left(\frac{n}{n-p}\right)^{k_{\gamma,\varepsilon}^+-1} < \frac{\gamma}{\varepsilon p_0}. \tag{5.13}$$

Since $c_0 > 1$, $q_i > 1$, and $p_i \le p_0$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$, it follows from (5.13) that

$$\mathcal{A}_{k,\gamma} < c_0^{\frac{1}{\varepsilon p_0}} \max(q_i)^{\frac{1}{\varepsilon}\cdot\frac{n}{n-p}}. \tag{5.14}$$

Estimate of $\mathcal{B}_{i_1,\dots,i_{k-1},\gamma}$. For any $j = 1,\dots,k$, since $p_i \le p_0$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$, by (5.2), (5.6), and (5.8), we obtain

$$\gamma_{i_1,\dots,i_{j-1}} \le \left(\frac{n-p}{n}\right)^{j-1}\gamma + \frac{n}{p}(p_0 - p) \le C\max\left(1, \left(\frac{n}{n-p}\right)^{k_{\gamma,\varepsilon}^- - j}\right) \tag{5.15}$$

for some constant $C = C(n, \overrightarrow{p}) > 1$. It follows from (5.15) that

$$\mathcal{B}_{i_1,\dots,i_{k-1},\gamma} \le C^{\frac{p}{\gamma}\sum_{j=1}^{k}\left(\frac{n}{n-p}\right)^j} \left(\frac{n}{n-p}\right)^{\frac{p}{\gamma}\sum_{j=1}^{k^-_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^-_{\gamma,\varepsilon}-j\right)}. \tag{5.16}$$

A simple calculation gives

$$\sum_{j=1}^{k^-_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^-_{\gamma,\varepsilon}-j\right) = \frac{n^2}{p^2}\left[\left(\frac{n}{n-p}\right)^{k^-_{\gamma,\varepsilon}-1} - \frac{p}{n}\left(k^-_{\gamma,\varepsilon}-1\right)-1\right] < \frac{n^2}{p^2}\left(\frac{n}{n-p}\right)^{k^-_{\gamma,\varepsilon}-1},$$

and hence by definition of $k^-_{\gamma,\varepsilon}$, we obtain

$$\sum_{j=1}^{k^-_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^-_{\gamma,\varepsilon}-j\right) \le \frac{n}{p}\cdot\frac{\gamma}{p_\varepsilon-p_-} < \frac{n}{p}\cdot\frac{\gamma}{p_*-p_-}. \tag{5.17}$$

It follows from (5.13), (5.16), and (5.17) that

$$\mathcal{B}_{i_1,\dots,i_{k-1},\gamma} \le C^{\frac{p^*}{\varepsilon p_0}}\left(\frac{n}{n-p}\right)^{\frac{n}{p_*-p_-}}. \tag{5.18}$$

Estimate of $\mathcal{C}_{i_1,\dots,i_{k-1},\gamma}$. Since $p_*-1 = \frac{n}{p}(p_*-p)$ and $\gamma_{i_1,\dots,i_{j-1}} > \frac{n}{p}(p_\varepsilon-p)$ for all $j=1,\dots,k$, we obtain

$$\mathcal{C}_{i_1,\dots,i_{k-1},\gamma} \le \min\left(1, \frac{n}{p}(p_\varepsilon-p_*)\right)^{-\frac{1}{\gamma}\sum_{j=1}^{k}p_{i_j}\left(\frac{n}{n-p}\right)^j}. \tag{5.19}$$

Since $p_{i_j} \le p_0$ for all $j=1,\dots,k$, it follows from (5.13) and (5.19) that

$$\mathcal{C}_{i_1,\dots,i_{k-1},\gamma} \le \min\left(1, \frac{n}{p}(p_\varepsilon-p_*)\right)^{-\frac{1}{\varepsilon}\cdot\frac{n}{n-p}}. \tag{5.20}$$

Estimate of $\mathcal{D}_{i_1,\dots,i_k,\gamma,\varepsilon}$. By (5.11) and since $k \le k^+_{\gamma,\varepsilon}$ and $p_{i_j} \le p_0$ for all $j=1,\dots,k$, we obtain

$$\mathcal{D}_{i_1,\dots,i_k,\gamma,\varepsilon} \le 2^{\frac{1}{\gamma}\sum_{j=1}^{k}p_{i_j}\left(\frac{n}{n-p}\right)^j\left(k^+_{\gamma,\varepsilon}-j+4\right)} \le 2^{\frac{p_0}{\gamma}\sum_{j=1}^{k^+_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^+_{\gamma,\varepsilon}-j+4\right)}. \tag{5.21}$$

We find

$$\begin{aligned}\sum_{j=1}^{k^+_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^+_{\gamma,\varepsilon}-j+4\right) &= \frac{n}{p}\left[\left(\frac{n}{n-p}\right)^{k^+_{\gamma,\varepsilon}}\left(3+\frac{n}{p}\right) - k^+_{\gamma,\varepsilon} - 3 - \frac{n}{p}\right]\\ &< \frac{n}{p}\left(\frac{n}{n-p}\right)^{k^+_{\gamma,\varepsilon}}\left(3+\frac{n}{p}\right),\end{aligned}$$

and hence by (5.6), we obtain

$$\sum_{j=1}^{k^+_{\gamma,\varepsilon}}\left(\frac{n}{n-p}\right)^j\left(k^+_{\gamma,\varepsilon}-j+4\right) < \frac{\gamma}{\varepsilon p_0}\cdot\frac{n}{n-p}\left(3+\frac{n}{p}\right). \tag{5.22}$$

It follows from (5.21) and (5.22) that

$$\mathcal{D}_{i_1,\dots,i_k,\gamma,\varepsilon} \le 2^{\frac{1}{\varepsilon}\cdot\frac{n}{n-p}\left(3+\frac{n}{p}\right)}. \tag{5.23}$$

End of proof of Lemma 5.2. The estimate (5.10) follows from (5.12), (5.14), (5.18), (5.20), and (5.23). ∎

6. The vanishing result

In this section, we prove a vanishing result which will give Point (i) in Theorem 1.3. We define

$$\overline{p}_0 := \max\left(p_*, \{p_i \in \overrightarrow{p} \, : \, i \in \Theta\}\right), \tag{6.1}$$

where p_* is as in (1.4) and Θ is the set of all indices i such that

$$\Big(p_i - p_- - \frac{n}{p}\left(p_i - p_*\right)\Big) \sum_{j=1}^{n} \max\left(\frac{p_i - p_j}{p_j}, 0\right) \geq \left(p_* - 1\right)\left(p_i - p_-\right) \tag{6.2}$$

with $p_- := \min\left(\{p_i \in \overrightarrow{p}\}\right)$. We define $\mathcal{I}_0$ as the set of all indices i such that $p_i > \overline{p}_0$.

When $p_+ > p_*$, one easily sees that the condition (6.2) does not hold true for $p_i = p_+$, and hence we have $\overline{p}_0 < p_+$ and $\mathcal{I}_0 \neq \emptyset$.

We prove the following result.

Theorem 6.1. *Assume that $p_* < p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true and u be a solution of* (1.1). *Then there exists a constant $R_0 = R_0\left(n, \overrightarrow{p}, \Lambda, u\right)$ such that $u\left(x\right) = 0$ for all $x \in \mathbb{R}^n$ such that $\sum_{i \in \mathcal{I}_0} |x_i| \geq R_0$.*

The proof of Theorem 6.1 is based on the following result, which we obtain by applying the iteration scheme in Section 5.

Lemma 6.2. *Assume that $p_* < p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true, u be a solution of* (1.1), *and κ, r, and K_0 be as in Lemma 3.3. Let $\overline{p}_0$ be as in* (6.1) *and $p_0 \in \overrightarrow{p}$ be such that*

$$p_0 > \overline{p}_0 \quad \text{and} \quad R_i\left(u\right) < \infty \text{ for all indices } i \text{ such that } p_i > p_0\,, \tag{6.3}$$

where

$$R_i\left(u\right) := \sup\left(\{|x_i| \, : \, x \in \operatorname{supp}\left(u\right)\}\right). \tag{6.4}$$

Let $\mathcal{I}_1$, $\mathcal{I}_2$ be the sets of indices i such that $p_i < p_0$, $p_i = p_0$, respectively. For any $\varepsilon, \lambda \in (0,1)$ and $R > 1$, we define

$$A_\varepsilon\left(R, \lambda\right) := \Omega_{\overrightarrow{q}}\left(\mathcal{I}_1, R^{1/\varepsilon}, \mathcal{I}_2, R, \lambda\right) \quad \text{with} \quad q_i := \begin{cases} \dfrac{p_\varepsilon p_i}{p_\varepsilon - p_i} & \text{if } i \in \mathcal{I}_1\,, \\ p_\varepsilon & \text{if } i \in \mathcal{I}_2\,, \end{cases} \tag{6.5}$$

where $p_\varepsilon := \left(1 + \varepsilon\right) p_0$. If $A_\varepsilon\left(R, 1/2\right) \cap B_{\overrightarrow{p}}\left(0, \max\left(r, 1\right)\right) = \emptyset$, then

$$\|u\|_{L^\infty(A_\varepsilon(R,1/4))} \leq \left(c_2 R^{-\frac{1}{p_\varepsilon}}\right)^{\frac{1}{\varepsilon}} \tag{6.6}$$

for some constant $c_2 = c_2(n, \overrightarrow{p}, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}, R_0\left(u\right))$, where

$$R_0\left(u\right) := \max\left(\{R_i\left(u\right) : i \in \{1, \dots, n\} \setminus \left(\mathcal{I}_1 \cup \mathcal{I}_2\right)\}\right). \tag{6.7}$$

Proof of Lemma 6.2. As is easily seen, we have $1 < q_i < s_0$ for some constant $s_0 = s_0\left(\overrightarrow{p}\right)$. Moreover, by (6.3), we obtain that $p_\varepsilon - p_* > p_0 - p_* > 0$ and $A_\varepsilon\left(R, 1/2\right) \cap \operatorname{supp}\left(u\right)$ is bounded. By Lemma 5.2, we then get that there exists a constant $\widetilde{c}_1 = \widetilde{c}_1\left(n, \overrightarrow{p}, \Lambda, K_0\right)$ such that for any $\gamma > \frac{n}{p}\left(p_\varepsilon - p\right)$, we have

$$\|u\|_{L^\gamma(A_\varepsilon(R,\lambda_{0,\gamma,\varepsilon}))} \leq \widetilde{c}_1^{\frac{1}{\varepsilon}} \max_{(i_1,\dots,i_k) \in \Phi_{\gamma,\varepsilon}} \left(R^{-\sigma_{i_1,\dots,i_k,\gamma,\varepsilon}} \|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}\right) \tag{6.8}$$

provided that $A_\varepsilon(R,1/2)\cap B_{\overrightarrow{p}}(0,\max(r,1))=\emptyset$, where $\gamma_{i_1,\dots,i_k}$ is as in (5.1), $\lambda_{k,\gamma,\varepsilon}$ and $\Phi_{\gamma,\varepsilon}$ are as in (5.11), and

$$\sigma_{i_1,\dots,i_k,\gamma,\varepsilon}:=\frac{1}{\varepsilon\gamma p_\varepsilon}\sum_{j=1}^{k}\left(\frac{n}{n-p}\right)^j\left(p_\varepsilon-p_{i_j}\right). \tag{6.9}$$

We claim that for any $\nu\in(0,1)$ and $(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}$, there exists a constant $c_\nu=c\big(n,\overrightarrow{p},K_0,\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)},R_0(u),\nu\big)$ such that

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}\le c_\nu^{\frac{1}{\varepsilon}}R^{\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}}, \tag{6.10}$$

where

$$\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}:=\max\left(0\,,\frac{1}{\varepsilon\gamma p_\varepsilon}\left(\frac{n}{n-p}\right)^k\left(1-\frac{\gamma_{i_1,\dots,i_k}}{p_*-1+\nu}\right)\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{p_\varepsilon-p_i}{p_i}\right). \tag{6.11}$$

We separate two cases:

- Case 1: $p_*-1+\nu\le\gamma_{i_1,\dots,i_k}<\frac{n}{p}(p_\varepsilon-p)$ (in which case $\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}=0$).
- Case 2: $\gamma_{i_1,\dots,i_k}<p_*-1+\nu$ (in which case $\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}>0$).

We begin with proving (6.10) in Case 1. By interpolation (see, for instance, Grafakos [18, Proposition 1.1.14]), and by Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned}\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}&\le\left(\frac{\gamma_{i_1,\dots,i_k}}{\gamma_{i_1,\dots,i_k}-p_*+1}\right)^{\frac{1}{\gamma_{i_1,\dots,i_k}}}\|u\|_{L^{p_*-1,\infty}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{p_*-1}{\gamma_{i_1,\dots,i_k}}}\|u\|_{L^{\infty}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{1-\frac{p_*-1}{\gamma_{i_1,\dots,i_k}}}\\&\le C\nu^{\frac{-1}{\gamma_{i_1,\dots,i_k}}}\le C\nu^{\frac{-1}{p_*-1}}\end{aligned} \tag{6.12}$$

for some constant $C=C\big(n,\overrightarrow{p},K_0,\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. Moreover, since $\gamma_{i_1,\dots,i_k}<\frac{n}{p}(p_\varepsilon-p)$ and $k\le k_{\gamma,\varepsilon}^{+}$, by (5.6), we get

$$\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k\le\frac{1}{\varepsilon p_0}\cdot\frac{n}{n-p}\,(p_\varepsilon-p)\,. \tag{6.13}$$

Then (6.10) follows from (6.12) and (6.13).

Now, suppose that we are in Case 2. By (5.1) and since $\gamma_{i_1,\dots,i_{k-1}}\ge\frac{n}{p}(p_\varepsilon-p)$ and $p_\varepsilon>p_*$, we obtain

$$\gamma_{i_1,\dots,i_k}\ge\frac{n}{p}(p_\varepsilon-p)+p_--p_\varepsilon>\frac{n}{p}(p_*-p)+p_--p_*=p_--1\,. \tag{6.14}$$

By Hölder's inequality, we then get

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}\le\left|A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\cap\operatorname{supp}(u)\right|^{\frac{1}{\gamma_{i_1,\dots,i_k}}-\frac{1}{p_*-1+\nu}}\|u\|_{L^{p_*-1+\nu}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}\,. \tag{6.15}$$

Direct computations yield

$$\left|A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\cap\operatorname{supp}(u)\right|\le CR^{\frac{1}{\varepsilon p_\varepsilon}\cdot\sum\limits_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{p_\varepsilon-p_i}{p_i}} \tag{6.16}$$

for some constant $C=C\big(n,\overrightarrow{p},R_0(u)\big)$, where $R_0(u)$ is as in (6.7). Similarly to (6.12) and (6.13), we obtain

$$\|u\|_{L^{p_*-1+\nu}\left(A_\varepsilon\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}\le C\nu^{\frac{-1}{p_*-1}} \tag{6.17}$$

for some constant $C = C\big(n, \overrightarrow{p}, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$, and

$$\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k < \frac{1}{\varepsilon p_0}\cdot\frac{p}{n-p}\left(p_*-1+\nu\right). \tag{6.18}$$

Then (6.10) follows from (6.15)–(6.18).

By (6.8) and (6.10), we obtain

$$\|u\|_{L^\gamma(A_\varepsilon(R,\lambda_{0,\gamma,\varepsilon}))} \le \left(\widetilde{c}_1 c_\nu\right)^{\frac{1}{\varepsilon}} \max_{(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}} R^{\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}-\sigma_{i_1,\dots,i_k,\gamma,\varepsilon}} \tag{6.19}$$

for all $\nu\in(0,1)$, where $\sigma_{i_1,\dots,i_k,\gamma,\varepsilon}$ and $\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}$ are as in (6.9) and (6.11).

We claim that there exists a constant $\nu_0=\nu_0\left(n,\overrightarrow{p}\right)$ such that for any $\nu\in(0,\nu_0)$, we have

$$\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}-\sigma_{i_1,\dots,i_k,\gamma,\varepsilon} \le -\frac{1}{\varepsilon p_\varepsilon}\left(1-\frac{n}{\gamma p}\left(p_\varepsilon-p\right)\right). \tag{6.20}$$

We prove this claim. By (5.2), we obtain

$$\begin{aligned}\sigma_{i_1,\dots,i_k,\gamma,\varepsilon} &= \frac{1}{\varepsilon p_\varepsilon}\left(1-\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k+\frac{1}{\gamma}\sum_{j=1}^k\left(\frac{n}{n-p}\right)^j\left(p_\varepsilon-p\right)\right)\\ &= \frac{1}{\varepsilon p_\varepsilon}\left(1+\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^k\left(\frac{n}{p}\left(p_\varepsilon-p\right)-\gamma_{i_1,\dots,i_k}\right)-\frac{n}{\gamma p}\left(p_\varepsilon-p\right)\right).\end{aligned} \tag{6.21}$$

In case $p_*-1+\nu\le\gamma_{i_1,\dots,i_k}<\frac{n}{p}\left(p_\varepsilon-p\right)$, since $\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}=0$, we deduce (6.20) directly from (6.21). In the remaining case $\gamma_{i_1,\dots,i_k}<p_*-1+\nu$, by (6.11) and (6.21), we obtain

$$\begin{aligned}\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}-\sigma_{i_1,\dots,i_k,\gamma,\varepsilon} \le -\frac{1}{\varepsilon p_\varepsilon}\Bigg(1+\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^k\Bigg(\frac{n}{p}\left(p_\varepsilon-p\right)-\gamma_{i_1,\dots,i_k}\\ -\left(1-\frac{\gamma_{i_1,\dots,i_k}}{p_*-1+\nu}\right)\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{p_\varepsilon-p_i}{p_i}\Bigg)-\frac{n}{\gamma p}\left(p_\varepsilon-p\right)\Bigg).\end{aligned} \tag{6.22}$$

If ν is small enough so that $p_*-1+\nu<\frac{n}{p}\left(p_0-p\right)$, i.e. $\nu<\frac{n}{p}\left(p_0-p_*\right)$, then

$$1-\frac{1}{p_*-1+\nu}\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{p_\varepsilon-p_i}{p_i}<\frac{p}{n\left(p_0-p\right)}\sum_{i\in(\mathcal{I}_1\cup\mathcal{I}_2)^c}\frac{p_0-p_i}{p_i}<0, \tag{6.23}$$

where $\left(\mathcal{I}_1\cup\mathcal{I}_2\right)^c:=\{1,\dots,n\}\setminus\left(\mathcal{I}_1\cup\mathcal{I}_2\right)$. It follows from (6.14), (6.22), and (6.23) that

$$\tau_{i_1,\dots,i_k,\gamma,\varepsilon,\nu}-\sigma_{i_1,\dots,i_k,\gamma,\varepsilon} \le -\frac{1}{\varepsilon p_\varepsilon}\left(1+\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^k\varphi_\nu\left(p_\varepsilon\right)-\frac{n}{\gamma p}\left(p_\varepsilon-p\right)\right) \tag{6.24}$$

for all $\nu\in\left(0,\frac{n}{p}\left(p_0-p_*\right)\right)$, where

$$\begin{aligned}\varphi_\nu\left(q\right) &:= q-p_--\left(1-\frac{\frac{n}{p}\left(q-p\right)+p_--q}{p_*-1+\nu}\right)\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{q-p_i}{p_i}\\ &= q-p_--\frac{q-p_--\frac{n}{p}\left(q-p_*\right)}{p_*-1+\nu}\sum_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\frac{q-p_i}{p_i}\end{aligned}$$

for all $q\in\mathbb{R}$. By (6.3) and by definition of $\overline{p}_0$, we obtain $\varphi_0\left(p_0\right)>0$. Moreover, it can easily be seen that $\varphi_0\left(p_*\right)\le 0$. Observing that φ_0 is a quadratic polynomial with positive leading coefficient, we then get that φ_0 is increasing in $[p_0,\infty)$. By continuity of φ_ν with respect to ν, it follows that $\varphi_\nu(p_\varepsilon) \le 0$ provided that $\nu < \nu_0$ for some constant $\nu_0 = \nu_0(n, \overrightarrow{p})$. By (6.24), we then get (6.20).

Finally, we fix $\nu = \nu_0/2$, and we obtain (6.6) by passing to the limit as $\gamma \to \infty$ into (6.19) and (6.20) and using the fact that $p_\varepsilon > p_0$ and $R > 1$. This ends the proof of Lemma 6.2. □

Now, we can conclude the proof of Theorem 6.1.

Proof of Theorem 6.1. We proceed by contradiction. Suppose that there exists a solution u of (1.1) such that

$$p_0 := \max\left(\{p_i \in \overrightarrow{p} \,:\, R_i(u) = \infty\}\right) > \overline{p}_0\,, \tag{6.25}$$

where $R_i(u)$ is as in (6.4). Then we can apply Lemma 6.2. For any $\varepsilon \in (0,1)$ and $x \in \mathbb{R}^n$, it follows from (6.6) that

$$|u(x)| \le \left(c_2 R_\varepsilon(x)^{-\frac{1}{p_\varepsilon}}\right)^{\frac{1}{\varepsilon}} \quad \text{where} \quad R_\varepsilon(x) := \sum_{i \in \mathcal{I}_2} |x_i|^{p_\varepsilon} \tag{6.26}$$

provided that $\sum_{i\in\mathcal{I}_1} |x_i|^{\frac{p_\varepsilon p_i}{p_\varepsilon - p_i}} < \frac{5}{4} R_\varepsilon(x)^{1/\varepsilon}$ and $A_\varepsilon(R_\varepsilon(x), 1/2) \cap B_{\overrightarrow{p}}(0, \max(r,1)) = \emptyset$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are as in Lemma 6.2, and r is as in Lemma 3.3. One easily gets that there exists a constant $R_r = R(n, \overrightarrow{p}, r) > 1$ such that for any $\varepsilon \in (0,1)$ and $R > R_r$, we have $A_\varepsilon(R, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r,1)) = \emptyset$. By passing to the limit as $\varepsilon \to 0$ into (6.26), we then obtain that $u(x) = 0$ for all $x \in \mathbb{R}^n$ such that

$$\sum_{i\in\mathcal{I}_2} |x_i|^{p_0} > \max\left(R_r, c_2^{p_0}\right),$$

and hence $R_i(u) < \infty$ for all $i \in \mathcal{I}_2$, which is in contradiction with (6.25). This ends the proof of Theorem 6.1. □

Remark 6.3. *As one can see from the above proof, the constant R_0 that we obtain in Theorem 6.1 depends on n, $\overrightarrow{p}$, Λ, κ, r, $r_\kappa(u)$, and $\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}$.*

7. The decay estimates

In this section, we prove Theorem 1.1 in case $p_+ < p_*$ and Theorem 7.1 below in case $p_* \le p_+ < p^*$. The latter implies Theorem 1.2 in case $p_+ = p_*$ and allows us to complete the proof of Theorem 1.3 in case $p_* < p_+ < p^*$.

We let $\overline{p}_0$ and $\mathcal{I}_0$ be as in Section 6. We define q_0 as the largest real number such that for any $q > q_0$, we have

$$\left(q - p_- - \frac{n}{p}(q - p_*)\right) \sum_{i\in\mathcal{I}_0^c} \frac{q - p_i}{p_i} < (p_* - 1)(q - p_-)\,, \tag{7.1}$$

where $\mathcal{I}_0^c := \{1, \dots, n\} \backslash \mathcal{I}_0$. It easily follows from the definition of $\overline{p}_0$ and the fact that $\overline{p}_0 < p_+$ in case $p_+ > p_*$ that

$$\begin{cases} q_0 = \overline{p}_0 = p_* & \text{in case } p_+ \le p_*\,, \\ \overline{p}_0 \le q_0 < p_+ & \text{in case } p_+ > p_*\,. \end{cases}$$

We prove the following result.

Theorem 7.1. *Assume that $p_* \le p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true and u be a solution of* (1.1)*. Let q_0 be defined as above. Then for any $q > q_0$, there exists a constant $C_q = C(n, \overrightarrow{p}, \Lambda, u, q)$ such that*

$$|u(x)|^q + \sum_{i=1}^n |\partial_{x_i} u(x)|^{p_i} \le C_q \Bigg(1 + \sum_{i \in \mathcal{I}_0^c} |x_i|^{\frac{qp_i}{q-p_i}}\Bigg)^{-1} \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{7.2}$$

We conclude the proofs of Theorems 1.2 and 1.3 as follows.

Proof of Theorem 1.2. In case $p_+ = p_*$, since $q_0 = \overline{p}_0 = p_*$, we get that (7.2) holds true for all $q > p_*$. Since in this case we have $\mathcal{I}_0^c = \{1, \dots, n\}$, this is exactly the result in Theorem 1.2. □

Proof of Theorem 1.3. In case $p_* < p_+ < p^*$, Points (i) and (ii) in Theorem 1.3 follow directly from Theorems 6.1 and 7.1 and the fact that $\overline{p}_0 \le q_0 < p_+$. □

Now, it remains to prove Theorems 1.1 and 7.1. By another application of the iteration scheme in Section 5, we prove the following result.

Lemma 7.2. *Assume that $p_+ < p^*$. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that* (1.2) *holds true, u be a solution of* (1.1)*, and κ, r, and K_0 be as in Lemma 3.3. Let $q = p_*$ in case $p_+ < p_*$ and $q \in (q_0, p^*)$ in case $p_* \le p_+ < p^*$. For any $\lambda \in (0,1)$ and $R > 1$, we define*

$$A_q(R, \lambda) := \Omega_{\overrightarrow{q}}(\emptyset, 1, \mathcal{I}_0^c, R, \lambda) \quad \textit{with} \quad q_i := \frac{qp_i}{q - p_i} \textit{ for all } i \in \mathcal{I}_0^c. \tag{7.3}$$

If $A_q(R, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r, 1)) = \emptyset$, then

$$\|u\|_{L^\infty(A_q(R,1/4))} \le c_q R^{-\frac{1}{q}}, \tag{7.4}$$

for some constant $c_q = c\big(n, \overrightarrow{p}, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}, R_0, q\big)$, where R_0 is as in Theorem 6.1.

Proof of Lemma 7.2. By Theorem 6.1, we obtain that $A_q(R, \lambda) \cap \operatorname{supp}(u)$ is bounded. By Lemma 5.2, we then get that for any $\varepsilon \in (0,1)$, there exists a constant $c_{q,\varepsilon} = c(n, \overrightarrow{p}, \Lambda, K_0, q, \varepsilon)$ such that for any $\gamma > \frac{n}{p}(\overline{p}_\varepsilon - p)$, where $\overline{p}_\varepsilon := (1+\varepsilon)\overline{p}_0$, we have

$$\|u\|_{L^\gamma(A_q(R,\lambda_{0,\gamma,\varepsilon}))} \le c_{q,\varepsilon} \max_{(i_1,\dots,i_k) \in \Phi_{\gamma,\varepsilon}} \bigg(R^{-\sigma_{i_1,\dots,i_k,q,\gamma}} \|u\|^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}\bigg), \tag{7.5}$$

provided that $A_q(R, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r,1)) = \emptyset$, where $\gamma_{i_1,\dots,i_k}$ is as in (5.1), $\lambda_{k,\gamma,\varepsilon}$ and $\Phi_{\gamma,\varepsilon}$ are as in (5.11), and

$$\sigma_{i_1,\dots,i_k,q,\gamma} := \frac{1}{\gamma q} \sum_{j=1}^k \left(\frac{n}{n-p}\right)^j (q - p_{i_j}). \tag{7.6}$$

End of proof of Lemma 7.2 in case $p_* \le p_+ < p^*$ and $q_0 < q < p^*$. In this case, we follow in large part the same arguments as in the proof of Lemma 6.2. We set $\varepsilon := (q - \overline{p}_0)/\overline{p}_0$ so that $q = \overline{p}_\varepsilon$. Since $q < p^*$ and $\overline{p}_0 \ge p_*$, we get $\varepsilon < (p^* - p_*)/p_* \le 1$. Similarly to (6.10), we then obtain that for any $(i_1, \dots, i_k) \in \Phi_{\gamma,\varepsilon}$ and $\nu \in (0,1)$, there exists a constant $c_\nu = c\big(n, \overrightarrow{p}, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}, R_0, \nu\big)$ such that

$$\|u\|^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k}_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \le c_\nu^{\frac{1}{\varepsilon}} R^{\tau_{i_1,\dots,i_k,q,\gamma,\nu}}, \tag{7.7}$$

where

$$\tau_{i_1,\dots,i_k,q,\gamma,\nu} := \max\left(0\,, \frac{1}{q\gamma}\left(\frac{n}{n-p}\right)^k \left(1 - \frac{\gamma_{i_1,\dots,i_k}}{p_* - 1 + \nu}\right) \sum_{i \in \mathcal{I}_0^c} \frac{q - p_i}{p_i}\right). \tag{7.8}$$

It follows from (7.5) and (7.7) that

$$\|u\|_{L^\gamma(A_q(R,\lambda_{0,\gamma,\varepsilon}))} \le c_{q,\varepsilon} c_\nu^{\frac{1}{\varepsilon}} \max_{(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}} R^{\tau_{i_1,\dots,i_k,q,\gamma,\nu}-\sigma_{i_1,\dots,i_k,q,\gamma}} \tag{7.9}$$

for all $\nu \in (0,1)$, where $\sigma_{i_1,\dots,i_k,q,\gamma}$ and $\tau_{i_1,\dots,i_k,q,\gamma,\nu}$ are as in (7.6) and (7.8).

In the same way as in the proof of (6.20), we then obtain

$$\tau_{i_1,\dots,i_k,q,\gamma,\nu} - \sigma_{i_1,\dots,i_k,q,\gamma} \le -\frac{1}{q}\left(1-\frac{n}{\gamma p}(q-p)\right) \tag{7.10}$$

provided that

$$q-p_- - \frac{q-p_- - \frac{n}{p}(q-p_*)+\nu}{p_*-1+\nu}\sum_{i\in\mathcal{I}_0^c}\frac{q-p_i}{p_i} > 0\,. \tag{7.11}$$

By (7.1), we get that (7.11) holds true provided that $\nu < \nu_0$ for some constant $\nu_0 = \nu_0(n,\overrightarrow{p})$.

Finally, we fix $\nu = \nu_q/2$, and we obtain (7.4) by passing to the limit as $\gamma\to\infty$ into (7.9) and (7.10). This ends the proof of Lemma 7.2. $\square$

Proof of Lemma 7.2 in case $p_+ < p_*$ and $q = p_*$. In this case, we have $\overline{p}_0 = p^*$ and $\mathcal{I}_0^c = \{1,\dots,n\}$. We claim that there exists a constant $\varepsilon_0 = \varepsilon_0(n,\overrightarrow{p}) \in (0,1)$ such that for any $\varepsilon\in(0,\varepsilon_0)$ and $(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}$, we have

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\frac{\gamma_{i_1,\dots,i_k}}{\gamma}\left(\frac{n}{n-p}\right)^k} \le c_\varepsilon R^{\tau_{i_1,\dots,i_k,p_*,\gamma}} \tag{7.12}$$

for some constant $c_\varepsilon = c\big(n,\overrightarrow{p},\Lambda,K_0,\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)},\varepsilon\big)$, where

$$\tau_{i_1,\dots,i_k,p_*,\gamma} := \frac{p_*-1-\gamma_{i_1,\dots,i_k}}{p_*\gamma}\left(\frac{n}{n-p}\right)^k. \tag{7.13}$$

We assume that $(1-\varepsilon)p_* > p$, i.e. $\varepsilon < \frac{p-1}{n-1}$, and we separate two cases:

- Case 1: $\gamma_{i_1,\dots,i_k} \le \frac{n}{p}((1-\varepsilon)p_*-p)$,
- Case 2: $\frac{n}{p}((1-\varepsilon)p_*-p) < \gamma_{i_1,\dots,i_k} < \frac{n}{p}((1+\varepsilon)p_*-p)$.

We begin with proving (7.12) in Case 1. By a generalized version of Hölder's inequality (see for instance Grafakos [18, Exercise 1.1.11]), we obtain

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\gamma_{i_1,\dots,i_k}} \le \frac{p_*-1}{p_*-1-\gamma_{i_1,\dots,i_k}}\left|A_q(R,\lambda_{k,\gamma,\varepsilon})\right|^{1-\frac{\gamma_{i_1,\dots,i_k}}{p_*-1}}\|u\|_{L^{p_*-1,\infty}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\gamma_{i_1,\dots,i_k}}. \tag{7.14}$$

Direct computations give

$$\left|A_q(R,\lambda_{k,\gamma,\varepsilon})\right| \le CR^{\sum_{i=1}^n\frac{p_*-p_i}{p_*p_i}} = CR^{\frac{p_*-1}{p_*}} \tag{7.15}$$

for some constant $C = C(n,\overrightarrow{p})$. Since $\gamma_{i_1,\dots,i_k} \le \frac{n}{p}((1-\varepsilon)p_*-p)$ and $u\in L^{p_*-1,\infty}(\mathbb{R}^n)$, it follows from (7.14) and (7.15) that

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}^{\gamma_{i_1,\dots,i_k}} \le C\varepsilon^{-1}R^{\frac{p_*-1-\gamma_{i_1,\dots,i_k}}{p_*}} \tag{7.16}$$

for some constant $C = C\big(n,\overrightarrow{p},\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. Moreover, since $k\le k_{\gamma,\varepsilon}^+$, by (5.6), we get

$$\frac{1}{\gamma}\left(\frac{n}{n-p}\right)^k < \frac{1}{\varepsilon(n-1)}\,. \tag{7.17}$$

Then (7.12) follows from (7.16) and (7.17).

Now, suppose that we are in case 2. By interpolation, we obtain

$$\|u\|_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \le \|u\|^{\theta}_{L^{\frac{n}{p}((1-\varepsilon)p_*-p)}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \|u\|^{1-\theta}_{L^{\frac{n}{p}((1+\varepsilon)p_*-p)}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)}, \tag{7.18}$$

where $\theta \in (0,1)$ is such that

$$\frac{\theta}{\frac{n}{p}\left((1-\varepsilon)\,p_*-p\right)} + \frac{1-\theta}{\frac{n}{p}\left((1+\varepsilon)\,p_*-p\right)} = \frac{1}{\gamma_{i_1,\dots,i_k}}. \tag{7.19}$$

Similarly to (7.16), we get

$$\|u\|^{\frac{n}{p}((1-\varepsilon)p_*-p)}_{L^{\frac{n}{p}((1-\varepsilon)p_*-p)}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \le C\varepsilon^{-1}R^{\frac{n}{p}\varepsilon} \tag{7.20}$$

for some constant $C = C\big(n, \overrightarrow{p}, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. On the other hand, Lemma 4.1 gives

$$\|u\|^{\frac{n}{p}((1+\varepsilon)p_*-p)}_{L^{\frac{n}{p}((1+\varepsilon)p_*-p)}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \le C \max_{i=1,\dots,n}\left(\varepsilon^{-p_i}R^{\frac{p_i-p_*}{p_*}}\|u\|^{\varepsilon(n-1)+p_i-1}_{L^{\varepsilon(n-1)+p_i-1}(A_q(R,1/2))}\right)^{\frac{n}{n-p}} \tag{7.21}$$

for some constant $C = C\left(n, \overrightarrow{p}, \Lambda, K_0\right)$. We define $\varepsilon_0 := \left(p_*-p_+\right)/\left(p_*+2n-2\right)$ so that for any $\varepsilon \in (0,\varepsilon_0)$ and $i = 1,\dots,n$, we have

$$\varepsilon\,(n-1) + p_i - 1 < \frac{n}{p}\left((1-\varepsilon)\,p_* - p\right).$$

Similarly to (7.16), we then get

$$\|u\|^{\varepsilon(n-1)+p_i-1}_{L^{\varepsilon(n-1)+p_i-1}\left(A_q\left(R,\lambda_{k+1,\gamma,\varepsilon}\right)\right)} \le CR^{\frac{p_*-p_i-\varepsilon(n-1)}{p_*}} \tag{7.22}$$

for some constant $C = C\big(n, \overrightarrow{p}, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$. By putting together (7.18)–(7.22), we obtain

$$\|u\|^{\gamma_{i_1,\dots,i_k}}_{L^{\gamma_{i_1,\dots,i_k}}\left(A_q\left(R,\lambda_{k,\gamma,\varepsilon}\right)\right)} \le C\varepsilon^{-s}R^{\frac{p_*-1-\gamma_{i_1,\dots,i_k}}{p_*}} \tag{7.23}$$

for some constants $C = C\big(n, \overrightarrow{p}, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}\big)$ and $s = s\left(n, \overrightarrow{p}\right) > 0$. Then (7.12) follows from (7.17) and (7.23).

By (7.5) and (7.12), we obtain that for any $\varepsilon \in (0,\varepsilon_0)$, there exists a constant $\widetilde{c}_\varepsilon = c\big(n, \overrightarrow{p}, \Lambda, K_0, \|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}, \varepsilon\big)$ such that

$$\|u\|_{L^\gamma(A_q(R,\lambda_{0,\gamma,\varepsilon}))} \le \widetilde{c}_\varepsilon \max_{(i_1,\dots,i_k)\in\Phi_{\gamma,\varepsilon}} R^{\tau_{i_1,\dots,i_k,p_*,\gamma}-\sigma_{i_1,\dots,i_k,p_*,\gamma}}, \tag{7.24}$$

where $\sigma_{i_1,\dots,i_k,p_*,\gamma}$ and $\tau_{i_1,\dots,i_k,p_*,\gamma}$ are as in (7.6) and (7.13).

From (5.2), we derive

$$\sigma_{i_1,\dots,i_k,p_*,\gamma} = \frac{1}{p_*}\left(1 + \frac{1}{\gamma}\left(\frac{n}{n-p}\right)^k\left(p_*-1-\gamma_{i_1,\dots,i_k}\right) - \frac{p_*-1}{\gamma}\right),$$

and hence

$$\tau_{i_1,\dots,i_k,p_*,\gamma} - \sigma_{i_1,\dots,i_k,p_*,\gamma} = -\frac{1}{p_*}\left(1 - \frac{p_*-1}{\gamma}\right). \tag{7.25}$$

Finally, we fix $\varepsilon = \varepsilon_0/2$, and we obtain (7.4) by passing to the limit as $\gamma \to \infty$ into (7.24) and (7.25). This ends the proof of Lemma 7.2 in case $p_+ < p_*$ and $q = p_*$. □

Now, we can prove Theorems 7.1 and 1.1.

Proof of Theorem 7.1. As is easily seen, it is sufficient to prove (7.2) for $q \in (q_0, p^*)$. Let u be a solution of (1.1) and $q > q_0$. We define

$$u_R(y) := R^{\frac{1}{q}} \cdot u(\tau_R(y)), \quad \text{where} \quad \tau_R(y) := \left(R^{\frac{q-p_1}{qp_1}} y_1, \dots, R^{\frac{q-p_n}{qp_n}} y_n\right)$$

for all $R > 1$ and $y \in \mathbb{R}^n$. By Lemma 7.2, we obtain

$$\|u_R\|_{L^\infty(A_q(1,1/4))} \le c_q \,. \tag{7.26}$$

provided that $A_q(R, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r, 1)) = \emptyset$, where r be as in Lemma 3.3. One easily gets the existence of a constant $R_r = R(n, \overrightarrow{p}, r) > 1$ such that $A_q(R, 1/2) \cap B_{\overrightarrow{p}}(0, \max(r, 1)) = \emptyset$ for all $R > R_r$. Moreover, by (1.1), we obtain

$$-\Delta_{\overrightarrow{p}} u_R = R^{\frac{q-1}{q}} \cdot f\left(\tau_R(y), R^{-\frac{1}{q}} \cdot u_R\right) \quad \text{in } \mathbb{R}^n, \tag{7.27}$$

and (1.2) gives

$$\left|R^{\frac{q-1}{q}} \cdot f\left(\tau_R(y), R^{-\frac{1}{q}} \cdot u_R\right)\right| \le \Lambda \cdot R^{\frac{q-p^*}{q}} \cdot |u_R|^{p^*-1} \,. \tag{7.28}$$

Since $q - p^* \le 0$, by (7.26)–(7.28) and Lieberman's gradient estimates [22], we get that there exists a constant $c'_q = c(n, \overrightarrow{p}, \Lambda, c_q)$ such that

$$\|\nabla u_R\|_{L^\infty(A_q(1,1/8))} \le c'_q \,. \tag{7.29}$$

For any $x \in \mathbb{R}^n$, it follows from (7.26) and (7.29) that

$$|u(x)|^q + \sum_{i=1}^n |\partial_{x_i} u(x)|^{p_i} \le c''_q R(x)^{-1} \quad \text{where} \quad R(x) := \sum_{i \in \mathcal{I}_0^c} |x_i|^{\frac{qp_i}{q-p_i}}$$

for some constant $c''_q = c(n, \overrightarrow{p}, \Lambda, c_q)$, provided that $R(x) > R_r$. This ends the proof of Theorem 7.1. □

Proof of Theorem 1.1. We fix $q = p^*$ in this case and we follow the same arguments as in the above proof of Theorem 7.1. □

Remark 7.3. *As one can see from the above proofs, the constants C_0 and C_q that we obtain in* (1.5) *and* (7.2) *depend on n, $\overrightarrow{p}$, q, Λ, κ, r, $r_\kappa(u)$, $\|u\|_{L^{p_*-1,\infty}(\mathbb{R}^n)}$, and $\|u\|_{W^{1,\infty}(\mathbb{R}^n \backslash \Omega_r)}$, where $\Omega_r := \left\{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^{\frac{qp_i}{q-p_i}} > R_r\right\}$ for some constant $R_r = R(n, \overrightarrow{p}, r)$.*

Appendix A. Kato-type inequality

In this section, we prove a weak version of Kato's inequality [21] for the operator $\Delta_{\overrightarrow{p}}$. This result is used in Sections 3 and 4. A similar result has been proven by Cuesta Leon [9] in the context of the p–Laplace operator.

For any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we say that a function $u \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ is a solution of the inequality

$$-\Delta_{\overrightarrow{p}} u \le f \quad \text{in } \mathbb{R}^n$$

if we have

$$\sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i-2} (\partial_{x_i} u)(\partial_{x_i} \varphi)\, dx \le \int_{\mathbb{R}^n} f \varphi \, dx$$

for all nonnegative, smooth function φ with compact support in $\mathbb{R}^n$.

We state our result as follows.

Proposition A.1. *Let $f_1, f_2 \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u_1, u_2 \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$ be solutions of the inequalities*

$$-\Delta_{\overrightarrow{p}} u_j \le f_j \quad \text{in } \mathbb{R}^n \tag{A.1}$$

for $j = 1, 2$. Then the function $u := \max(u_1, u_2)$ is a solution of the inequality

$$-\Delta_{\overrightarrow{p}} u \le f \quad \text{in } \mathbb{R}^n, \tag{A.2}$$

where $f(x) := f_1(x)$ if $u_1(x) > u_2(x)$, $f(x) := f_2(x)$ if $u_1(x) \le u_2(x)$ for all $x \in \mathbb{R}^n$.

Proof of Proposition A.1. We essentially follow the lines of Cuesta Leon [9, Proposition 3.2]. For any $\varepsilon > 0$ and $x \in \mathbb{R}^n$, we define

$$\eta_{1,\varepsilon}(x) := \eta_\varepsilon(u_1(x) - u_2(x)) \quad \text{and} \quad \eta_{2,\varepsilon}(x) := 1 - \eta_{1,\varepsilon}(x),$$

where $\eta_\varepsilon \in C^1(R)$ is such that $\eta_\varepsilon \equiv 0$ in $(-\infty, 0]$, $\eta_\varepsilon \equiv 1$ in $[1, \infty)$, $0 \le \eta_\varepsilon \le 1$ and $\eta'_\varepsilon \ge 0$ in $(0, 1)$. In particular, for $j = 1, 2$, we have $\eta_{j,\varepsilon} \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, $0 \le \eta_{j,\varepsilon} \le 1$ in $\mathbb{R}^n$, and

$$\eta_{j,\varepsilon}(x) \longrightarrow \begin{cases} 1 & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega_j \end{cases} \tag{A.3}$$

as $\varepsilon \to 0$ for all $x \in \mathbb{R}^n$, where $\Omega_1 := \{x \in \mathbb{R}^n : u_1(x) > u_2(x)\}$ and $\Omega_2 := \mathbb{R}^n \backslash \Omega_1$. For any nonnegative, smooth function φ with compact support in $\mathbb{R}^n$, testing (A.1) with $\varphi \eta_{j,\varepsilon}$ gives

$$\begin{aligned} &\sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_{x_i} u_j|^{p_i - 2} (\partial_{x_i} u_j)(\partial_{x_i} \varphi) \eta_{j,\varepsilon} dx \\ &+ (-1)^{j-1} \sum_{i=1}^n \int_{\mathbb{R}^n} |\partial_{x_i} u_j|^{p_i-2} (\partial_{x_i} u_j)(\partial_{x_i} u_1 - \partial_{x_i} u_2) \eta'_\varepsilon(u_1 - u_2) \varphi \, dx \le \int_{\mathbb{R}^n} f_j \varphi \eta_{j,\varepsilon} dx. \end{aligned} \tag{A.4}$$

By (A.3) and since $f_j \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u_j \in D^{1,\overrightarrow{p}}(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} f_j \varphi \eta_{j,\varepsilon} dx \longrightarrow \int_{\Omega_j} f_j \varphi \, dx, \tag{A.5}$$

$$\int_{\mathbb{R}^n} |\partial_{x_i} u_j|^{p_i-2} (\partial_{x_i} u_j)(\partial_{x_i} \varphi) \eta_{j,\varepsilon} dx \longrightarrow \int_{\Omega_j} |\partial_{x_i} u_j|^{p_i-2} (\partial_{x_i} u_j)(\partial_{x_i} \varphi) \, dx \tag{A.6}$$

as $\varepsilon \to 0$ for all $i = 1, \dots, n$. Moreover, since $\eta'_\varepsilon \ge 0$ and $\varphi \ge 0$, we get

$$\sum_{i=1}^n \int_{\mathbb{R}^n} \left(|\partial_{x_i} u_1|^{p_i-2} \partial_{x_i} u_1 - |\partial_{x_i} u_2|^{p_i-2} \partial_{x_i} u_2 \right) (\partial_{x_i} u_1 - \partial_{x_i} u_2) \eta'_\varepsilon(u_1 - u_2) \varphi \, dx \ge 0. \tag{A.7}$$

It follows from (A.4)–(A.7) that

$$\sum_{j=1}^2 \sum_{i=1}^n \int_{\Omega_j} |\partial_{x_i} u_j|^{p_i-2} (\partial_{x_i} u_j)(\partial_{x_i} \varphi) \, dx \le \sum_{j=1}^2 \int_{\Omega_j} f_j \varphi \, dx \tag{A.8}$$

and hence (A.2) holds true since $u \equiv u_j$ and $f \equiv f_j$ on Ω_j for $j = 1, 2$. □

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