Solutions to Assignment 3

1. Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$

show that the zero set of $\sin \pi z$ is $\mathbb{Z}$, and that they are all of order 1.
Solution:

$$\sin \pi z = 0 \implies e^{2\pi i z} = 0 \implies z \in \mathbb{Z}$$

For the multiplicity, we calculate the residue of $1/\sin \pi z$.

$$\text{Res}_n f(z) := \lim_{z \to n} \frac{z - n}{\sin \pi z} = \lim_{z \to n} \frac{1}{\pi \cos \pi z} = \frac{(-1)^n}{\pi}$$

where the second equality follows from l'Hopital's rule. ♣

2. Evaluate

$$\int_{\mathbb{R}} \frac{dx}{1 + x^4}$$

What are the poles of $1/(1 + z^4)$?
Solution: Set up the contour integral as a semi-circle Γ parametrized as

$$\Gamma(r, \theta) = \begin{cases} r & \text{for} - R < r < R, \\ Re^{i\theta} & \text{for } 0 \leq \theta \leq \pi. \end{cases}$$

Then,

$$\int_{\Gamma} \frac{dz}{1 + z^4} = 2\pi i \sum \text{Residues}$$

The zeroes of $1 + z^4$ are the 4^{th} roots of unity which sit on the unit circle. Of these, precisely two will lie inside the contour. Namely,

$$z = \frac{\pm 1 + i}{\sqrt{2}}$$

We obtain the residues

$$\frac{\sqrt{2}}{4i \pm 4}$$

Whence,

$$\int_{\mathbb{R}} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \qquad \clubsuit$$

3. Show that

$$\int_{\mathbb{R}} \frac{\cos x}{x^2+a^2} dx = \pi e^{-a}, \text{for } a > 0$$

Solution: Consider the function

$$f(z) = \frac{e^{iz}}{z^2+a^2}$$

which has poles at $z = \pm ia$. Using the same contour integral as the previous question, we calculate the residue for the only pole inside. That is, the one at $z = ia$.

$$\text{Res}_{ia} f = (z - ia) f(ia) = \frac{e^{-a}}{2ia}$$

Whence,

$$\int_{\Gamma} f(z) dz = \pi e^{-a}/a$$

We obtain the proof by equating the real parts of

$$\int_{\mathbb{R}} \frac{\cos x + i \sin x}{x^2+a^2} dx = \pi e^{-a}/a \qquad \clubsuit$$

6. Show that

$$\int_{\mathbb{R}} \frac{\cos x}{(1+x^2)^{n+1}} dx = \prod_{k=1}^{n} \frac{2k-1}{2k} \pi$$

Solution: Same contour as before. Consider

$$f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{[(z-i)(z+i)]^{n+1}}$$

So we have poles of order $n+1$ at $\pm i$. Only the one at i is inside the contour. The residue at i is given by

$$\text{Res}_i(f) = \frac{1}{n!} \frac{d^n}{dz^n} (z+i)^{-(n+1)} \bigg|_{z=i} = \frac{(-1)^n (n+1)(n+2)\cdots 2n}{n!(2i)^{2n+1}} = \frac{1}{2i} \prod_{k=1}^{n} \frac{2k-1}{2k} \pi$$

Whence,

$$\int_{\Gamma} f(z) dz = \prod_{k=1}^{n} \frac{2k-1}{2k} \pi$$

and we obtain the result by equating the real part as usual. ♣

7. Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}, \quad \text{whenever } a > 1$$

Solution: Parametrize the unit circle γ by $z = e^{i\theta}, 0 \le \theta \le 2\pi$ and use $\cos\theta = (z + z^{-1})/2$. Letting

$$f(z) = \frac{1}{iz\left(a + (z + z^{-1})/2\right)}$$

we have

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2}$$

The poles of f are at $-a \pm \sqrt{a^2 - 1}$ of which the only one inside the unit circle is $z_0 := -a + \sqrt{a^2 - 1}$ since $a > 1$. We have

$$f(z) = \frac{h(z)}{\left(z + a - \sqrt{a^2 - 1}\right)^2}$$

where

$$h(z) = \frac{4z}{i\left(z + a - \sqrt{a^2 - 1}\right)^2}$$

The residue is therefore

$$\mathrm{Res}_{z_0}(f) = \frac{a}{i\,(a^2 - 1)^{3/2}}$$

so that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2\pi a}{(a^2 - 1)^{3/2}} \qquad \clubsuit$$

8. Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if $a > |b|$, and $a, b \in \mathbb{R}$.
Solution: Using the same setup as the previous question we have in this case

$$f(z) = \frac{1}{iz\left(a + b\left(z + z^{-1}\right)/2\right)}$$

with poles at $z = -a \pm \sqrt{a^2 - b^2}$. Only $z_0 := -a + \sqrt{a^2 - b^2}$ is in the unit circle.

$$h(z) = \frac{2}{i\left(z + a + \sqrt{a^2 - b^2}\right)}$$

so that

$$\mathrm{Res}_{z_0}(f) = \frac{1}{i\sqrt{a^2 - b^2}}$$

Whence,

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \qquad \clubsuit$$

13. Suppose $f(z)$ is hol'c on a punctured disc $D_r(z_0) - \{z_0\}$ and

$$|f(z)| \leq \frac{A}{|z - z_0|^{1-\varepsilon}}$$

for some $\varepsilon > 0, \forall z$ near z_0. Show that the singularity of f at z_0 is removable.
Solution: Let

$$g(z) = (z - z_0)f(z)$$

so that

$$|g(z)| \leq A|z - z_0|^{\varepsilon} < Ar^{\varepsilon}$$

so g is hol'c and bounded on the punctured disk, hence has a removable singularity at z_0 by Riemann's theorem. Extending g to the entire disk and setting

$$f(z) = \begin{cases} f(z) = \frac{g(z)}{z - z_0} & \text{for } z \in D_r(z_0) - \{z_0\} \\ \frac{d}{dz} g(z)\Big|_{z=z_0} & \text{for } z = z_0 \end{cases}$$

we see that the singularity of f at z_0 is indeed removable. ♣

14. Prove that every injective entire function is linear and nonconstant.
Proof. If f is a constant it cannot be injective so it is nonconstant. Consider

$$g(z) = f(1/z)$$

Since f is injective so is g. Since f is entire, g has an isolated singularity at zero. Suppose that zero is an essential singularity of g. Then by the Casorati-Weierstrass theorem, the image of a punctured neighborhood of zero under g is dense in the complex plane. Since it is the image of a hol'c function it is also an open set. This contradicts the assumption that g is injective. Therefore, g has a pole or a removable singularity at the origin. This implies that f is a polynomial. Since f is injective, it's zero set is a singleton so f takes the form

$$f(z) = a(z - z_0)^n$$

but if $n > 1$, then $z_0 + \zeta_n$ map to a:

$$f(z_0 + \zeta_n) = a\zeta_n^n = a$$

where ζ_n is an n^{th} root of unity. Whence, $n = 1$ and f is indeed linear. ♣