Intersection bounds for nodal sets of planar Neumann eigenfunctions with interior analytic curves

John Toth (McGill)
(joint in part with Y. Canzani and L. El-Hajj)

Weyl law at 100 (Fields Institute)

September 19-21, 2012

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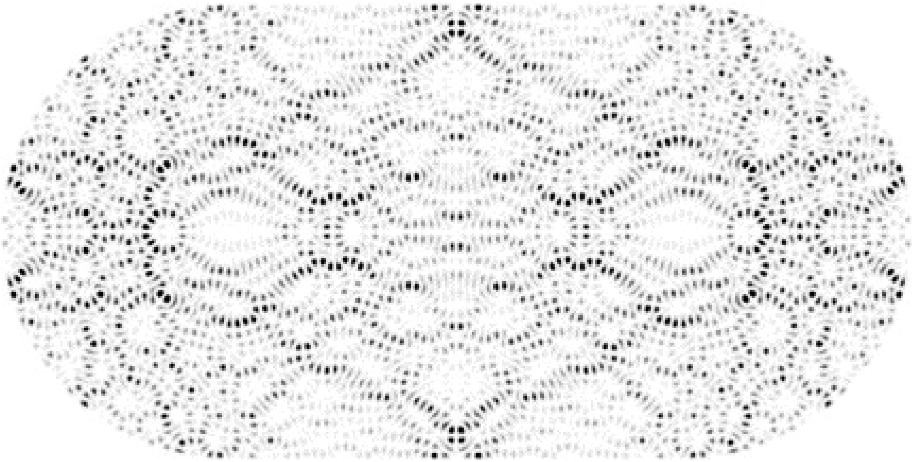
- **QUESTION:** As $\lambda \to \infty$, how many nodal lines (components of the nodal set) <u>intersect</u> a fixed interior real analytic curve H?

Probability density plot of an eigenstate of a Bunimovich stadium



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Theorem

Theorem (Zelditch-T (2009)) Let H be a real analytic interior curve that is good.Then, there is a constant $C_{\Omega,H} > 0$ such that for all Neumann eigenfunctions ϕ_λ,
$$n(\lambda, H) \leq C_{\Omega,H}\lambda.$$
When $H = \partial\Omega$,
$$n(\lambda, \partial\Omega) \leq C_\Omega \lambda.$$

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- For interior H's, the goodness condition is not easy to verify for all eigenfunctions.

- Easy to see that not all interior curves are good. For example, the Neumann eigenfunctions for the disc in polar variables $(r, \theta) \in (0, 1] \times [0, 2\pi]$ are

$$\phi_{m,n}(r, \theta) = C_{m,n} \cos m\theta J_m(j'_{m,n} r) \ (C_{m,n} \sin m\theta J_m(j'_{m,n} r)).$$

Here, J_m is the m-th integral Bessel function and $j'_{m,n}$ is the m-th critical point of J_m. The eigenvalues are $\lambda^2_{m,n} = (j'_{m,n})^2$.

Positive results known

- Fix $m \in Z^+$ and consider

$$H_m = \{(r, \theta); \theta = \frac{2\pi k}{m}; \; k = 0, ..., m-1\}.$$

Then, clearly for all n, $\quad \phi_{m,n}|_{H_m} = 0$, and so H_m is not good.

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- When (M^n, g) is a flat torus with $n = 2, 3$, and $H \subset M$ has strictly positive curvature (Bourgain-Rudnick(2010))

$$\int_H |\phi_\lambda|^2 d\sigma \approx 1.$$

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- Closed horocycles H in finite-volume hyperbolic surfaces are good (Jung(2011)) and so the $O_H(\lambda)$ intersection bound holds.

Theorem 1 [El-Hajj - T (2012)]

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Let Ω be a bounded, piecewise-analytic domain and $H \subset \mathring{\Omega}$ an interior, C^ω curve with restriction map $\gamma_H : C^0(\Omega) \to C^0(H)$.

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Let $H_{\epsilon_0}^{\mathbb{C}}$ denote the complex radius $\epsilon_0 > 0$ Grauert tube containing H as its totally real submanifold and $(\gamma_H \phi_\lambda)^{\mathbb{C}}$ be the holomorphic continuation of $\gamma_H \phi_\lambda$ to $H_{\epsilon_0}^{\mathbb{C}}$.

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Suppose the curve H satisfies the revised goodness condition

$$\sup_{z \in H_{\epsilon_0}^{\mathbb{C}}} |(\gamma_H \phi_\lambda)^{\mathbb{C}}(z)| \geq e^{-C_0 \lambda} \qquad \textit{for some } C_0 > 0. \quad (*)$$

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Then, there is a constant $C_{\Omega,H} > 0$ such that for all $\lambda \leq \lambda_0$,

$$n(\lambda, H) \leq C_{\Omega,H} \lambda.$$

Key tool: potential layer

- An important point is that $(*)$ can be verified using T^*T-type operator bounds for the holomorphic continuation to $H^{\mathbb{C}}_{\epsilon_0}$ of the potential layer operator $N(\lambda) : C^\infty(\partial\Omega) \to C^\infty(H)$

$$N(\lambda)(x,y) = \int_{\partial\Omega} \partial_{\nu_y} G_0(x,y,\lambda)\, d\sigma(y),$$

where,

$$G_0(x,y,\lambda) = \frac{i}{4} \mathrm{Ha}_0^{(1)}(\lambda|x-y|).$$

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Then,

$$n(h_{j_k}, H) = \mathcal{O}_{H,\Omega}(h_{j_k}^{-1}).$$

Theorem 3 [Canzani-T(2012)]

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Let (M^2, g) be a compact, real-analytic surface with $\partial M = \emptyset$ and ergodic geodesic flow $G^t : S^*M \to S^*M$. Let $H \subset M^2$ be a real-analytic closed curve with strictly-positive geodesic curvature.

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- When (M^2, g) is quantum uniquely ergodic (QUE), the intersection bound in Theorem 3 holds for all eigenfunctions. For example, this is the case when $M = \Gamma / \mathbf{H}$ is arithmetic (Lindenstrauss (2006)).

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- One can formulate a more general version of Theorem 2 in terms of defect measures which need not be ergodic (examples?)

Outline of the proof of Theorem 1

- We want to show that $n(h,H) = \mathcal{O}(h^{-1})$ under the assumption that $sup_{z \in H^{\mathbb{C}}_{\epsilon_0}} |\phi_h^{H,\mathbb{C}}(z)| \geq e^{-C/h}$ for some $C > 0$.

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- Let $q : [-\pi, \pi] \to H$ be a C^ω-parametrization of a closed curve H with $|q'(t)| \neq 0$ and $q(t + 2\pi) = q(t)$.

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- Consider the eigenfunction restriction,

$$u_h^H(t) = \phi_h(q(t)),\ t \in [-\pi, \pi]$$

and complexify u_h^H to a holomorphic function $u_h^{H,\mathbb{C}}(t)$ with $t \in S_{2\epsilon_0}$ where

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- Let $C_{\epsilon_0} \subset S_{2\epsilon_0}$ be a simply-connected domain with C^ω boundary ∂C_{ϵ_0} containing the interval $[-\pi, \pi]$.

Outline of the proof of Theorem 1

- Assuming $u_h^{H,\mathbb{C}}(t) \neq 0$ for all $t \in C_{\epsilon_0}$, frequency function method of Han-Lin gives the upper bound

$$n(h,H) \leq C_1 \left(\frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} \right). \tag{2}$$

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- In (2), $L^2_{\epsilon_0} := L^2(\partial C_{\epsilon_0}, d\sigma(t))$ and ∂_T is the unit tangential derivative along ∂C_{ϵ_0}.

Outline of the proof of Theorem 1

- We h-microlocally decompose the right hand side in (2). Let $\chi_R \in C_0^\infty(T^*\partial C_{\epsilon_0})$ with $\chi_R(s,\sigma) = 1$ for $|\sigma| \leq R+1$ and $\chi_R(s,\sigma) = 0$ for $|\sigma| \geq R+2$ with $R > 1$ sufficiently large.

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  Clearly,

$$\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} \leq \|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} + \|\partial_T (1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}. \tag{3}$$

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- Since $h\partial_T Op_h(\chi_R) \in Op_h(S^{0,0}(T^*\partial C_{\epsilon_0}))$, by L^2-boundedness one estimates the first term on RHS of (3):

$$\frac{\|\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} = h^{-1} \frac{\|h\partial_T Op_h(\chi_R) u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} = \mathcal{O}(h^{-1}). \tag{4}$$

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- To estimate the right hand side of (3), we use potential layer formulas combined with a complex contour deformation argument to show that

$$\|h\partial_T(1 - Op_h(\chi_R))u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} = \mathcal{O}(e^{-C_R/h}).$$

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- Choose the strip $S_{\epsilon_0,\pi} = \{t \in \mathbb{C}; -\pi \leq \Re t \leq \pi, |\Im t| < \epsilon_0\}$ with $S_{\epsilon_0,\pi} \subset \mathrm{Int}(C_{\epsilon_0})$. By Cauchy integral formula, Cauchy-Schwarz and the goodness condition $(*)$,

$$\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}} \geq C \cdot \sup_{t \in S_{\epsilon_0,\pi}} |u_h^{H,\mathbb{C}}(t)| \geq e^{-C_0/h}.$$

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- It follows that

$$\frac{\|h\partial_T(1 - Op_h(\chi_R))u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}}{\|u_h^{H,\mathbb{C}}\|_{L^2_{\epsilon_0}}} = \mathcal{O}(e^{(-C_R+C_0)/h}).$$

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- By choosing R sufficiently large in the radial frequency cutoff χ_R, we get that $C_R - C_0 \gg R > 0$.

Outline of proof of Theorem 2

- We consider here the case where Ω is a bounded, convex planar domain with ergodic billiards and that (ϕ_{h_j}) is a sequence of QE interior eigenfunctions. We want to show that $\sup_{z \in H^{\mathbb{C}}_{\epsilon_0}} |\phi_h^{H,\mathbb{C}}(z)| \geq e^{-C/h}$ in the case where $H \subset \Omega$ is an interior curve with $\kappa_H > 0$. We do this by proving some weighted-L^2 lower bounds.

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- Let $H^{\mathbb{C}}(\epsilon_0)$ be a complex Grauert tube of radius $\epsilon_0 > 0$ with totally-real part H and $\zeta_{\epsilon_0} \in C^\infty(H^{\mathbb{C}}(\epsilon_0); [0,1])$ be a cutoff on the Grauert tube equal to 1 on the annulus $H^{\mathbb{C}}(\epsilon_0/2) - H^{\mathbb{C}}(\epsilon_0/3)$ and vanishing outside.

Outline of proof of Theorem 2

- The main technical part of the proof of Theorem 2 consists of showing that under the non-vanishing curvature condition on H and for $\epsilon_0 > 0$ small, there is an order-zero semiclassical pseudodifferential operator

$$P(h) \in Op_h(S^{0,0}(T^*\partial\Omega))$$

and a weight function

$$\rho \in C^\omega(\operatorname{supp} \zeta_{\epsilon_0}; \mathbb{R}^+)$$

such that

$$h^{-1/2} \int \int_{\mathbb{C}} e^{-2\rho(t)/h} |u_h^{H,\mathbb{C}}(t)|^2 \, \zeta_{\epsilon_0}(t) \, dt d\bar{t} \sim_{h \to 0^+} \langle P(h) \phi_h^{\partial\Omega}, \phi_h^{\partial\Omega} \rangle. \tag{5}$$

Outline of proof of Theorem 2

- The potential layer formula gives

$$\phi_h^H = \gamma_H N(h)\phi_h^{\partial\Omega}, \qquad (u_h^H(t) = \phi_h^H(q(t))),$$

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- Writing the LHS of (5) as a composition, reduced to proving that $P(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ with

$$P(h) = (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} N^{\mathbb{C}}(h))^* \circ (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} N^{\mathbb{C}}(h))$$

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- It is this point that the curvature assumption $\kappa_H > 0$ on H is used.
- Here, $N^{\mathbb{C}}(h)$ is holomorphic continuation of the potential layer operator $N(h) : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ and $\gamma_H^{\mathbb{C}} : \Omega_{\epsilon_0}^{\mathbb{C}} \to H_{\epsilon_0}^{\mathbb{C}}$ is restriction.

Outline of proof of Theorem 2

- The principal symbol $\sigma(P(h))$ satisfies

$$\int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dyd\eta \geq C_{H,\Omega,\epsilon_0} > 0$$

where $\gamma(y,\eta) = \sqrt{1-|\eta|^2}$.

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- Given a quantum ergodic sequence $(\phi_{h_{j_k}})_{k=1}^{\infty}$, the boundary restrictions $(\phi_{h_{j_k}}^{\partial\Omega})_{k=1}^{\infty}$ are themselves quantum ergodic (Burq,Hassell-Zelditch) in the sense that

$$\langle P(h)\phi_h^{\partial\Omega}, \phi_h^{\partial\Omega}\rangle \sim_{h\to 0^+} \int_{B^*\partial\Omega} \sigma(P(h))\gamma^{-1}\, dyd\eta. \tag{6}$$

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- It follows that

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- The lower bound in (7) implies that the revised goodness condition $(*)$ must be satisfied.

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- Main point is to prove that $P(h) : C^\infty(M) \to C^\infty(M)$ with

$$P(h) = (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} W(h)^{\mathbb{C}})^* \circ (e^{-\rho/h}\zeta_{\epsilon_0}\gamma_H^{\mathbb{C}} W(h)^{\mathbb{C}})$$

is h-pseudodifferential.

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- **(ii)** Upper bounds for $n(H, \lambda)$ for more general (non-ergodic) domains when H is curved.

- **(iii)** Polynomial lower bounds for $n(H, \lambda)$ when H is either an interior curve or $H = \partial\Omega$.