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Eigenfunction restriction bounds for Neumann data along hypersurfaces

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Background

• \((M^n, g)\) compact, smooth manifold (with or without boundary).

• \(H \subset M^n\) an orientable smooth hypersurface. In some cases, \(H\) can be a higher-codimension submanifold.

Cauchy data along \(H\)

• Consider the eigenvalue problem on \(M\)

\[-\Delta_g \phi_j = \lambda_j^2 \phi_j, \quad \langle \phi_j, \phi_k \rangle = \delta_{jk}\]

\[B \phi_j = 0 \text{ on } \partial M,\]

where \(\langle f, g \rangle = \int_M f \bar{g} dV\) (\(dV\) is the volume form of the metric) and where \(B\) is the boundary operator, e.g. \(B \phi = \phi|_{\partial M}\) in the
Dirichlet case or $B\phi = \partial_\nu \phi|_{\partial M}$ in the Neumann case. We also allow $\partial M = \emptyset$.

- Let $h_j = \lambda_j^{-1}$ and $\phi_{h_j}$ be a corresponding orthonormal basis of eigenfunctions with eigenvalue $h_j^{-2}$, so that the eigenvalue problem takes the semi-classical form,

$$(-h^2\Delta_g - 1)\phi_h = 0,$$

$$B\phi_h = 0 \text{ on } \partial M$$

where $B = I$ or $B = hD_\nu$ in the Dirichlet or Neumann cases respectively.

- Semiclassical Cauchy data along $H$:

$$CD(\phi_h) := \{(\phi_h|_H, hD_\nu \phi_h|_H)\}.$$

• **Problem:** Upper (and lower) bounds for

\[ \| \phi_h \|_{L^2(H)} \quad (\text{Dirichlet}), \]

\[ \| hD_{\nu} \phi_h \|_{L^2(H)} \quad (\text{Neumann}). \]

• A lot of recent work on upper bounds for Dirichlet data along \( H \). We considered Neumann data (closely linked with Dirichlet via Rellich identity).

• **Theorem 1 [Christianson-Hassell-T]** Let \( H \subset M \) be any oriented, smooth separating hypersurface with \( H \cap \partial M = \emptyset \). Then,

\[ \| h\partial_{\nu} \phi_h \|_{L^2(H)} = O(1). \]

• Result holds for eigenfunctions \( \phi_h \) of general Schrödinger operators \( P(h) = -h^2 \Delta_g + V(x) \) with \( V \in C^{\infty}(M; \mathbb{R}) \) and

\[ P(h) \phi_h = E(h) \phi_h, \quad E(h) = E + O(h), \]

\( E \) regular energy value and \( H \subset \{ V(x) < E \} \).
Cauchy data along $H$: Rellich identity

- Let $H \subset M$ be an oriented separating hypersurface with exterior unit normal $\nu$ bounding smooth domain $M_H \subset M$.

- **Rellich identity.** Self-adjointness of $\Delta_g$, the eigenfunction equation $-h^2 \Delta_g \phi_h = \phi_h$ and an easy application of Green’s formula gives with any $A(h) \in \Psi^m_h$ and $V \in C^\infty(M, \mathbb{R})$,

  $$\frac{i}{h} \langle [-h^2 \Delta_g + V, A(h)] \phi_h, \phi_h \rangle_{M_H}$$

  $$= \langle A(h) \phi_h, hD_\nu \phi_h \rangle_H + \langle hD_\nu A(h) \phi_h, \phi_h \rangle_H. \quad (*)$$

- Key identity relating Dirichlet and Neumann data along $H$ to interior eigenfunctions on $M$. 
• Let \((x_n, x')\) be Fermi coordinates near \(H\) with \(H = \{x_n = 0\}\) and \(\chi \in C_0^\infty([-\delta, \delta])\) equal to 1 near origin. Idea of proof of [CHT] is to apply Rellich with \(A(h) = \chi(x_n)hD_{x_n}\).

• In this case, Rellich gives

\[
\frac{i}{h} \langle [-h^2 \Delta + V, \chi hD_\nu] \phi_h, \phi_h \rangle_{MH} = \langle hD_\nu \phi_h, hD_\nu \phi_h \rangle_H + \langle (I + h^2 \Delta_H) \phi_h, \phi_h \rangle_H.
\]

• Formula has other applications such as in the case where \((\phi_h)\) is quantum ergodic (QE); that is, for any \(B(h) \in \Psi^0_h(M)\),

\[
\langle B(h) \phi_h, \phi_h \rangle_{L^2(M)} \sim_{h \to 0^+} \int_{B^*M} b(x, \xi) dx d\xi.
\]
Quantum ergodic restriction for Cauchy data (QERCD)

- **Theorem 1 [Christianson-Zelditch-T]** Suppose $H \subset M$ is a smooth, codimension 1 embedded orientable separating hypersurface and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Assume that $\{\phi_h\}$ is an interior QE sequence. Then the appropriately renormalized Cauchy data $d\Phi_{h}^{CD}$ of $\phi_h$ is quantum ergodic in the sense that for any $a^w \in \Psi^0(H)$, there exists a sub-sequence of eigenvalues of density one so that as $h_j \to 0^+$,

$$
\langle a^w h D_{\nu} \phi_h \vert_H, h D_{\nu} \phi_h \vert_H \rangle_{L^2(H)}
$$

$$
+ \langle a^w (1 + h^2 \Delta_H) \phi_h \vert_H, \phi_h \vert_H \rangle_{L^2(H)}
$$

$$
\rightarrow h \to 0^+ \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x', \xi')(1-|\xi'|^2)^{1/2} d\sigma.
$$
Here, $a_0(x', \xi')$ is the principal symbol of $a^w$, $-h^2 \Delta_H$ is the induced tangential (semi-classical) Laplacian with principal symbol $|\xi'|^2$ and $d\sigma$ is the standard symplectic volume form on $B^*H$.

Result holds for all interior hypersurfaces and generalizes results of Hassell-Zelditch and Burq in the boundary case (ie. $H = \partial \Omega$.)

Idea of Proof in [CTZ]: Apply Rellich with $A(h) = \chi(x_n)hD_\nu$ and compute LHS with the commutator $\frac{i}{h}[-h^2\Delta_g, A(h)] \in \Psi^0_h$ applying QE assumption on the eigenfunction sequence $(\phi_h)$.
Dirichlet Data

• **Problem:** Estimate the $L^2$ restrictions

\[
\int_H |\phi_\lambda(s)|^2 d\sigma(s). \tag{1}
\]

• **Rationale:** Key to understanding the large–$\lambda$ behaviour of the $\phi_\lambda$’s. Pointwise $L^\infty$ results are very hard; difficult to improve on the bound

\[
\|\phi_\lambda\|_{L^\infty(M)} = \mathcal{O}(\lambda^{n-1/2}).
\]

The problem in (1) is easier but still very non-trivial.

2) Restriction bounds naturally arise in study of eigenfunction nodal sets. In particular, *lower* bounds of the form

\[
\int_H |\phi_\lambda|^2 ds \geq e^{-C\lambda}, \quad C > 0
\]

are central to this problem.
3) Quantum ergodicity: Recent results (Zelditch-T, Dyatlov-Zworski) on **Quantum Ergodic Restriction (QER)** show that for generic $H$’s satisfying a geodesic asymmetry condition relative to $H$ and a density-one sub-sequence of eigenfunctions $\phi_\lambda$,

$$
\lim_{\lambda \to \infty} \langle Op_H(a) \phi_\lambda | H, \phi_\lambda | H \rangle_{L^2(H)} = 2 \int_{B^*H} a(s, \sigma)(1 - |\sigma|^2)^{1/2} ds d\sigma.
$$

- Most results extend to semiclassical Schrödinger operators $P(h) = -h^2 \Delta + V(x)$ with eigenfunctions $\phi_h$ satisfying $P(h)\phi_h = E(h)\phi_h$, $|E(h) - E| = o(1)$, $E$ a regular energy level.
General Results for Dirichlet data

• For general Laplace eigenfunctions with \( \|\phi_\lambda\|_{L^2(M)} = 1 \), Burq-Gérard-Tzvetkov [BGT] prove that

\[
\int_H |\phi_\lambda|^2 d\sigma(s) = O(\lambda^{\frac{1}{2}}), \quad (n = 2). \tag{2}
\]

• The universal bound (2) is achieved on \( S^2 \) with \( H = \{(x, y, z) \in S^2; z = 0\} \) the equator and \( \phi_n(x, y, z) = c_0 n^{\frac{1}{4}} (x + iy)^n; n = 1, 2, 3, ..., \) the highest-weight harmonics.

• In the case where \( H \) has positive geodesic curvature, the bound (2) improves to

\[
\int_H |\phi_\lambda|^2 d\sigma(s) = O(\lambda^{\frac{1}{3}}); \quad (n = 2).
\]
BGT also obtain sharp general $L^p$ bounds for $p \neq 2$ in any dimension and Hu generalized the positively-curved results to any dimension. Hassell-Tacy have extended these $L^p$ bounds to the semiclassical case where $P(h) = -h^2 \Delta + V(x)$.

- Improvements for $(M, g)$ non-positive curvature (Chen-Sogge, Sogge-Zelditch), flat tori with dim = 2, 3 (Bourgain-Rudnick), arithmetic surfaces (Jung, Rudnick-Sarnak, Ghosh-Reznikov-Sarnak), quantum completely integrable case (T), ...
Neumann Data along $H$

• Starting point is the Rellich identity with test operator $A(h) = \chi(x_n) hDx_n$. We recall it here (with $V = 0$ for simplicity):

$$\frac{i}{h} \langle [-h^2 \Delta_g, \chi hD_v] \phi_h, \phi_h \rangle_{M_H}$$

$$= \langle hD_v \phi_h, hD_v \phi_h \rangle_H + \langle (I + h^2 \Delta_H) \phi_h, \phi_h \rangle_H.$$

• Let $\gamma_H : C^0(M) \to C^0(H)$ be restriction. Consider spectral projector $N(h) = \sum_j \chi(h^{-1} - h_j^{-1}) \phi_j(x) \overline{\phi_j(y)}$ with supp $\hat{\chi} \subset [\epsilon, 2\epsilon] > 0$. The kernel

$$N(x, y, h) = (2\pi h)^{-(n-1)/2} e^{ir(x, y)/h} a(x, y, h)$$

$$+ O(h^\infty)_{L^2 \to L^2}.$$ 

Writing

$$\gamma_H \phi_h = \gamma_H N(h) \phi_h,$$
it is not hard to show that

\[ \text{WF}_h(\phi_h) \subset B^*H = \{(s, \eta) \in T^*H; |\eta|_{g(s)} \leq 1\}. \quad (*) \]

- **Heuristic:** On the RHS of Rellich, the Dirichlet term \( \langle (I + h^2\Delta_H)\phi_h, \phi_h \rangle_H \) looks like it should be “essentially” non-negative in view of \((*)\) since

\[
\sigma(I + h^2\Delta_H)(s, \eta) = 1 - |\eta|_{g(s)}^2.
\]

- If that is the case, we are done and would simply get

\[
\|hD_\nu\phi_h\|^2_H \leq \frac{i}{h} \langle [-h^2\Delta_g, \chi D_\nu\phi_h] \phi_h, \phi_h \rangle_{M_H} = O(1)
\]

where the last estimate follows by \(L^2\)-boundedness of \(\frac{i}{h}[-h^2\Delta_g, \chi D_\nu\phi_h] \in \Psi^0_h(M)\).

- Unfortunately, we cannot quite prove this. Subtlety lies in mass concentration of \(\phi_h|_H\) near glancing set \(S^*H\) on \(h^\delta\)-scales with \(\delta > 1/2\).
The example of the disc

- Consider Dirichlet eigenfunctions $\phi_\lambda(r, \theta)$ in the unit disc with eigenvalue $\lambda$. They are of the form

\[ \phi_{\lambda, n}(r, \theta) = c_n J_n(\lambda r) e^{in\theta}, \quad J_n(\lambda) = 0. \]

- Let $H = \{ r = \frac{1}{2} \}$, so that

\[ \phi^H_{\lambda, n}(\theta) = c_n J_n \left( \frac{\lambda}{2} \right) e^{in\theta}. \]

- Consider pairs $(\lambda, n)$ with

\[ \lambda = 2nz n^{1/3}, \quad z \in [z_1, z_2]. \]

These eigenfunctions peak near the caustic $H = \{ r = \frac{1}{2} \}$ and can contain semiclassical frequencies $1 + cn^{-2/3}$ with

\[ \lambda^{-1} \partial_r \phi_{\lambda, n} \approx n^{-1/6} Ai(2^{1/3}z). \]
Sketch of proof of Theorem 1

- Choose $a^w \in \Psi^*_h$ with principal symbol principal symbol

  $$a(x, \xi) = \chi(x_n)\xi_n,$$

- We recall Rellich formula

  $$\frac{i}{\hbar} \int_{M_-} \left[ -\hbar^2 \Delta - 1, a^w \right] \phi_h \bar{\phi}_h dx$$

  $$= \int_H (hD_n a^w \phi_h) |_{H} \bar{\phi}_h |_{H} d\sigma_H$$

  $$+ \int_H (a^w \phi_h) |_{H} \bar{hD_n \phi_h} |_{H} d\sigma_H.$$
• It follows that
  \[ \int_H (1 + h^2 \Delta_H) \phi_H^H \overline{\phi_H^H} d\sigma_H + \int_H |\phi_H^{H,\nu}|^2 d\sigma_H = O(1). \]

• In order to bound the Neumann data from above, need to show the first term on the left hand side of (???) is semi-bounded below.

• Use small scale decomposition of \( T^* H \) with \( \chi_{in}, \chi_{tan}, \chi_{out} \) cutoffs supported in sets \( |\sigma| < 1 - h^\delta, 1 - h^\delta < |\sigma| < 1 + h^\delta, |\sigma| > 1 + h^\delta \) respectively, satisfying
  \[ 1 = (\chi_{in})_{h,\delta}^w + (\chi_{tan})_{h,\delta}^w + (\chi_{out})_{h,\delta}^w \]

• Here, we need to choose 2-microlocal scales with \( \delta \in (1/2, 2/3) \).
• By Proposition on mass concentration of $\phi_h^H$:

$$\int_H (1 + h^2 \Delta_H) \phi_h^H \phi_h^H d\sigma_H$$

$$= \int_H (1 + h^2 \Delta_H) (\chi_{in})_{h,\delta}^w \phi_h^H \phi_h^H d\sigma_H$$

$$+ \int_H (1 + h^2 \Delta_H) (\chi_{tan})_{h,\delta}^w \phi_h^H \phi_h^H d\sigma_H + O(h^\infty).$$

• On the support of $\chi_{in}$, we have $1 - |\sigma|^2 \geq h^\delta$ and Gårding inequality gives

$$\int_H (1 + h^2 \Delta_H) (\chi_{in})_{h,\delta}^w \phi_h^H \phi_h^H d\sigma_H$$

$$\geq C_1 h^\delta \int_H (\chi_{in})_{h,\delta}^w \phi_h^H \phi_h^H d\sigma_H.$$
On the support of $\chi_{\tan}$, we have $|1 - |\sigma|^2| \leq C_2 h^\delta$, so that

$$\left| \int_H (1 + h^2 \Delta_H) (\chi_{\tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right| \leq C_2 h^\delta \left| \int_H (\chi_{\tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right|.$$  

Combining these two estimates,  

$$\int_H (1 + h^2 \Delta_H) \phi_h^H \overline{\phi_h^H} d\sigma_H \geq C_1 h^\delta \int_H (\chi_{\text{in}})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H$$

$$- C_2 h^\delta \left| \int_H (\chi_{\tan})^w_{h,\delta} \phi_h^H \overline{\phi_h^H} d\sigma_H \right| + O(h^\infty) \geq -C h^\delta \int_H |\phi_h^H|^2 d\sigma_H,$$

exterior term is $O(h^\infty)$, so adding it back in is harmless.
• Use the $\|\phi_h^H\|_{L^2(H)} = O(h^{-1/4})$ bound of Burq-Gérard-Tzvetkov to get

$$\int_H (1 + h^2 \Delta_H) \phi_h^H \overline{\phi_h^H} d\sigma_H \geq -C h^{\delta-1/2}.$$  

• Choosing $\delta > 1/2$ gives

$$-C h^{\delta-1/2} + \int_H |hD_{\nu} \phi_h|^2 d\sigma_H = O(1)$$

and so,

$$\int_H |hD_{\nu} \phi_h|^2 d\sigma_H = O(1).$$

• Choosing $\delta \sim 2/3$ gives the best estimate

$$\|hD_{\nu} \phi_h\|_H^2 \leq \frac{1}{h} \left| \langle [-h^2 \Delta_g, \chi h D_n] \phi_h, \phi_h \rangle_M \right| + C \epsilon h^{\delta+\epsilon}.$$
• **Corollary** If \((\phi_h)\) is QE sequence, then for any \(\epsilon > 0\), and \(h \in (0, h_0(\epsilon)]\),

\[
\| hD_\nu \phi_h \|_H^2 \leq |S^*_H M| + \epsilon.
\]

**Open problems/questions**

• When is it true that for \(h < h_0\),

\[
\| hD_\nu \phi_h \|_H^2 < |S^*_H M|?
\]

An immediate corollary would be

\[
\| \phi_h \|_H^2 \geq C > 0
\]

ie. strong unique continuation for the eigenfunction restrictions \(\phi_h|_H\).

• Run Rellich with other test operators such as \(A(h) = x_n \chi(x_n) hD_n\) with \(H = \{x_n = 0\}\) to try to decouple the Dirichlet and Neumann data along \(H\).