Analysis IV, Assignment 4

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Exercise 1

Let $f \in C^0(\mathbb{R})$ and periodic with $f(x+2\pi) = f(x)$. Let $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$ and $(S_N f)(x) = \sum_{n=-N}^{N} a_n e^{inx}$. If f is continuously differentiable at $x_o \in [-\pi, \pi]$, then $f(x_0) = \lim_{N\to\infty}(S_N f)(x_0)$.

Solution

Define $F(t) = \begin{cases} \frac{f(x_0-t)-f(x_0)}{t} & t \neq 0 \text{ and } |t| < \pi \\ -f'(x_0) & t = 0 \end{cases}$

By the assumptions on f, F is bounded on $[-\pi, \pi]$. Recall the Dirichlet Kernel D_N defined by $\frac{\sin((N+1/2)t)}{\sin(t/2)}$. One can verify that $(S_N f)(x) = (f * D_N)(x)$ and that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N = 1$.

Then

$$\begin{aligned}
(S_N f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t)D_N(t)dt - f(x_0) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x_0 - t) - f(x_0)\right) D_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\frac{t}{\sin(t/2)}\sin((N+1/2)t)dt
\end{aligned}$$

From here we write $\sin((N + 1/2)t) = \sin(Nt)\cos(t/2) + \cos(Nt)\sin(t/2)$ and take the limit as $N \to \infty$. Since the terms in the integrand not depending on N are bounded this integral converges to 0 by the Riemann-Lebesgue lemma.

Exercise 2

The Volterra integral operator $T : L^2([0, 1]) \to L^2([0, 1])$ given by $(Tf)(x) = \int_0^x f(y)dy$ for $x \in [0, 1]$ is compact with spectrum $= \{0\}$.

Solution

T is a bounded operator since

$$\begin{aligned}\|Tf\|_2^2 &= \int_0^1 \left(\int_0^x f(y)dy\right)^2 dx \\ &\leq \int_0^1 \|f\|_1^2 dx \\ &\leq \|f\|_2^2\end{aligned}$$

Notice that we can rewrite T as $(Tf)(x) = \int_0^1 k(x,y)f(y)dy$ where $k(x,y) = \begin{cases} 1 & y < x \\ 0 & y \geq x \end{cases}$

Since $k : [0,1]^2 \to \mathbb{C} \in L^2([0,1]^2)$, T is Hilbert-Schmidt and therefore compact.

From the general theory of compact operators, $\lambda \neq 0 and \lambda \in \sigma(T) \longrightarrow \lambda$ is an eigenvalue, and since $L^2([0,1])$ is infinite dimensional, $0 \in \sigma(T)$. Suppose that $\int_0^1 f(y)dy = \lambda f(x)$. Since $f \in L^1([0,1])$, by the fundamental theorem of calculus we may differentiate the equation to obtain $f(x) = \lambda f'(x)$ which has solution $f(x) = ce^{\frac{x}{\lambda}}$ for $\lambda \neq 0$. Putting back into the integral equation will give that $c = 0$, and we conclude that $\lambda \neq 0$ is not possible.

Exercise 3

The Medellin transform defined as $\mathcal{M} : L^2(\mathbb{R}^+, dt/t) \to L^2(\mathbb{R})$, $(\mathcal{M}f)(x) = \frac{1}{\sqrt{2\pi}} \int_0 \infty f(t)t^{ix-1}dt$ is a unitary operator.

Solution

An alternate definition, perhaps more standard, of the Medellin transform is $\tilde{\mathcal{M}} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$. $(\tilde{\mathcal{M}}f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)t^{-1/2+ix}dt$.

Consider the map $\tilde{\phi} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ $(\tilde{\phi}f)(z) = e^{z/2}f(e^z)$. $\tilde{\phi}$ is unitary since $\|\tilde{\phi}f\|_2^2 = \int_{-\infty}^\infty e^z|f(e^z)|^2dz = \int_0^\infty |f(u)|^2du = \|f\|_2^2$ and is easily seen to be surjective.

Now notice that $\tilde{\mathcal{M}} = \mathcal{F} \circ \tilde{\phi}$, where $\mathcal{F}$ denotes the Fourier transform. Indeed, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{z/2}f(e^z)e^{-izx}dz = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)t^{ix-1}dt$

Similarly one checks that $\mathcal{M} = \mathcal{F}^{-1} \circ \phi$ where $\phi : L^2(\mathbb{R}^+, dt/t) \to L^2(\mathbb{R})$ $(\phi f)(z) = f(e^z)$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. ϕ is unitary since $\|\phi f\|_2^2 = \int_{-\infty}^\infty |f(e^z)|^2dz = \int_0^\infty |f(u)|^2\frac{1}{u}du = \|f\|_2^2$.

In either case, we have the composition of two unitary maps, so the Medellin transform is unitary.

Exercise 4

Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{H}$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω. Then $\mathcal{H}$ is closed subspace of $L^2(\Omega)$ and hence a Hilbert space with inner product $\langle f, g \rangle = \int_\Omega f(x+iy)\overline{g(x+iy)}dxdy$

Solution

Obviously the second part of the statement follows from the first as closed subspaces of Hilbert spaces are Hilbert spaces with the same inner product. We shall need the following result from complex analysis:

Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of complex-valued functions analytic on an open connected set D $\subset \mathbb{C}$ which converge uniformly to $f(z)$. Then $f(z)$ is analytic on D.

We also need the following inequality for finite measure spaces:

If $\mu(\Omega) < \infty$, then for $0 < p < q < \infty$, $\|f\|_p \leq \|f\|_q \mu(\Omega)^{1/p-1/q}$ which is proved by using Holder's inequality with conjugate exponents $\frac{q}{p}$ and $\frac{q}{q-p}$:

$$\|f\|p^p = \int |f|^p \cdot 1 \leq \||f|^p\|_{q/p}\|1\|_{\frac{q}{q-p}} = \|f\|_q^p \mu(\Omega)^{\frac{q-p}{q}}$$

Let g be a holomorphic function on Ω. Then by the mean value property of holomorphic functions for any $z \in \Omega$ and any $r = r(z) > 0$ sufficiently small so that $B(z,r) \subset \Omega$ we have $f(z) = \frac{1}{\pi r^2}\int_{B(z,r)} f(\zeta)dA$ Then $|f(z)| \leq \frac{1}{\pi r^2}\|f\|_{1,B(z,r)} \leq \frac{1}{\sqrt{\pi} r}\|f\|_{2,B(z,r)} \leq \frac{1}{\sqrt{\pi} r}\|f\|_2$ In particular, if $K \subset \mathbb{C}$ is compact, and $z \in K$, then $|f(z)| \leq \frac{1}{\sqrt{\pi} r_o}\|f\|_2$ where $r_o = d(K, \Omega^c)$. Hence if $\{f_n\}$ is a Cauchy sequence in $L^2(\Omega)$, then $\{f_n\}$ is uniformly Cauchy on all compacts $K \subset \Omega$, and therefore converges uniformly on all compacts $K \subset \Omega$. Combining this with previous result stated earlier completes the proof.

Exercise 5

Let $\{\varphi_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then:

1. $\forall z \in \Omega, \sum\limits_{n=0}^{\infty} |\varphi_n(z)|^2 \leq \frac{1}{\pi d(z,\Omega^c)^2}$

2. $B(z,w) = \sum\limits_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)}$ called the Bergman kernel converges absolutely for $(z,w) \in \Omega \times \Omega$ and is independent of the choice of orthonormal basis.

3. The linear transformation $T : L^2(\Omega) \to \mathcal{H}$, $(Tf)(x) = \int_\Omega B(z,w)f(w)dw$ is the orthogonal projection onto $\mathcal{H}$.

4. In the special case where Ω is the unit disc, $f \in \mathcal{H}$ iff $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ for some a_n satisfying $\sum\limits_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$. Also $\{z^n\sqrt{\frac{n+1}{\pi}}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ and $B(z,w) = \frac{1}{\pi(1-z\overline{w})^2}$.

Solution

We start with a lemma: Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of complex numbers. Then $\sqrt{\sum\limits_{n=0}^{\infty} |b_n|^2} = \sup\limits_{\sum |a_n|^2 \leq 1} |\sum\limits_{n=0}^{\infty} a_n b_n|$.

By Cauchy-Schwarz, we have $\sqrt{\sum\limits_{n=0}^{\infty}|b_n|^2} \geq \sup\limits_{\sum|a_n|^2\leq 1} |\sum\limits_{n=0}^{\infty} a_n b_n|$. By taking $a_n^{(N)} = \frac{1}{\sqrt{\sum\limits_{j=0}^{N}|b_j|^2}} \begin{cases} \overline{b_n} & n \leq N \\ 0 & n > N \end{cases}$ the supremum attains $\sqrt{\sum\limits_{n=0}^{\infty}|b_n|^2}$.

1. For $\sum\limits_{n=0}^{\infty}|a_n|^2 \leq 1$, from the previous exercise we have $|\sum\limits_{n=0}^{\infty} a_n\varphi_n(z)| \leq \frac{1}{\sqrt{\pi}d(z,\Omega^c)}\|\sum\limits_{n=0}^{\infty} a_n\varphi_n(z)\|_2 = \frac{1}{\sqrt{\pi}d(z,\Omega^c)}$, Now taking $a_n = a_n^{(N)} = \frac{1}{\sqrt{\sum\limits_{j=0}^{N}|\varphi_j(z)|^2}} \begin{cases} \overline{\varphi_n(z)} & n \leq N \\ 0 & n > N \end{cases}$ and applying the lemma yields the result.

2. The fact that $B(z,w)$ converges absolutely follows from applying Cauchy-Schwarz and invoking the previous result, (and remembering that for complex series, convergence implies absolute convergence).

3. Since $\mathcal{H}$ is a Hilbert subspace of $L^2(\Omega)$ we may complete $\{\varphi_n\}_{n=0}^{\infty}$ with $\{\psi_k\}_{k=0}^{\infty}$ to an orthonormal basis of $L^2(\Omega)$. For $f \in L^2(\Omega)$, say $f = \sum\limits_{n=0}^{\infty} a_n\varphi_n + \sum\limits_{k=0}^{\infty} b_k\psi_k$ we have

$$\begin{aligned}(Tf)(Z) &= \int_\Omega \left(\sum_{n=0}^{\infty}\varphi_n(z)\overline{\varphi_n(w)}\right)\left(\sum_{m=0}^{\infty} a_m\varphi_m(w) + \sum_{k=0}^{\infty} b_k\psi_k(w)\right)dw \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_m\varphi_n(z)\int_\Omega \overline{\varphi_n(w)}\varphi_m(w)dw + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_k\varphi_n(z)\int_\Omega \overline{\varphi_n(w)}\psi_k(w)dw \\ &= \sum_{n=0}^{\infty} a_n\varphi_n(z)\end{aligned}$$

   The fact that the kernel $B(z,w)$ is independent of the choice of basis is now easy to see, because if $B(z,w)$ corresponds to $\{\varphi_n\}$ and $\tilde{B}(z,w)$ corresponds to $\{\tilde{\varphi_n}\}$, then for all $f \in L^2(\Omega)$ we have $\int_\Omega \left(B(z,w) - \tilde{B}(z,w)\right) f(w)dw$ which implies that for all z, $B(z,w) - \tilde{B}(z,w) = 0$.

4. If $n = m$,

$$\begin{aligned}\langle\varphi_n,\varphi_n\rangle &= \frac{n+1}{\pi}\int_{B(0,1)} z^n\overline{z^n}dz \\ &= \frac{n+1}{\pi}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^{2n}rdrd\theta = 1\end{aligned}$$

   If $n \neq m$,

$$\langle \varphi_n, \varphi_m \rangle = \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_{B(0,1)} z^n \overline{z^m} dz$$
$$= \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^{n+m} e^{i\theta(n-m)} r dr d\theta = 0$$

If $f \in \mathcal{H}$, then $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ for some a_n. Then $f = \sum\limits_{n=0}^{\infty} a_n \sqrt{\frac{\pi}{n+1}} \varphi_n$ and so $\sum\limits_{n=0}^{\infty} |a_n|^2 \frac{\pi}{n+1} < \infty$.

Finally $B(z,w) = \sum\limits_{n=0}^{\infty} \sqrt{\frac{n+1}{\pi}} z^n \overline{\sqrt{\frac{n+1}{\pi}} w^n} = \frac{1}{\pi} \sum\limits_{n=0}^{\infty} (n+1)(z\overline{w})^n = \frac{1}{\pi(1-z\overline{w})^2}$ where we used the fact that for $|\zeta| < 1$, $\frac{1}{1-\zeta} = \sum\limits_{n=0}^{\infty} \zeta^n$ implies $\frac{1}{(1-\zeta)^2} = \sum\limits_{n=0}^{\infty} (n+1)\zeta^n$

Exercise 6

Let L be a linear partial differential operator with constant coefficients. Then the space of solutions u of $Lu = 0$ with $u \in C^\infty(\mathbb{R}^d)$ is infinite dimensional for $d \geq 2$.

Solution

First we argue that a non-constant polynomial in two variables has uncountably many zeroes. If $p(x,y) \in \mathbb{C}[x,y]$ then we can find polynomials $q_o(y), ..., q_n(y)$ with $q_n(y)$ not identically zero such that $p(x,y) = \sum\limits_{i=0}^{n} q_i(y) x^i$. Let Δ_n denote the set of zeroes of $q_n(y)$ in $\mathbb{C}$ and for fixed λ, $\mathcal{Z}_\lambda$ the set of zeroes of $p(x,\lambda)$ in $\mathbb{C}$. Since $q_n(y)$ is not identically zero, Δ_n is a finite collection and so $\mathbb{C} \setminus \Delta_n$ is uncountable. Now for fixed $\lambda \in \mathbb{C} \setminus \Delta_n$, the polynomial $p(x,\lambda)$ is of degree n, and therefore has at least one zero. It follows that $\bigcup\limits_{\lambda \in \mathbb{C}\setminus\Delta_n} \mathcal{Z}_\lambda \times \{\lambda\}$ is an uncountable distinct union of zeroes of $p(x,y)$. This same argument can be done for higher dimensions. For simplicity of notation, we continue in 2d.

Now we need to show that we can find infinitely many solutions to $P(\xi)\hat{u}(\xi) = 0$ with $u \in C^\infty(\mathbb{R}^d)$. Most of the work is already done because we can find an infinite sequence of distinct roots $(r_1^{(k)}, r_2^{(k)})_{k=0}^{\infty}$, i.e. $P(r_1^{(k)}, r_2^{(k)}) = 0$. Take then $\hat{u}^{(k)}(\xi_1, \xi_2) = e^{r_1^{(k)}\xi_1 + r_2^{(k)}\xi_2}$. It is easy to check that $u^{(k)} \in C^\infty(\mathbb{R}^d)$ and that the collection $\{\hat{u}^{(k)} : 0 \leq k \leq N\}$ is linearly independent for all N.

Exercise 7

Let $f \in L^2(\mathbb{R}^d)$. Then there exists $g \in L^2(\mathbb{R}^d)$ such that $\left(\frac{\partial}{\partial x}\right)^\alpha f(x) = g(x)$ in the weak sense iff $(2\pi i\xi)^\alpha \hat{f}(\xi) = \hat{g}(\xi) \in L^2(\mathbb{R}^d)$.

Solution

We let $L = \left(\frac{\partial}{\partial x}\right)^\alpha$ so that $L^* = (-1)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{alpha}$. Notice that $\overline{\hat{L^*}\psi(\xi)} = \overline{(-1)^{|\alpha|}(2\pi i\xi)^\alpha\hat{\psi}(\xi)} = (2\pi i\xi)^\alpha\overline{\hat{\psi}(\xi)}$

If $(2\pi i\xi)^\alpha\hat{f}(\xi) = \hat{g}(\xi)$, then by Plancherel's formula we have for any $\psi \in C_0^\infty$:

$$
\begin{aligned}
\int_{\mathbb{R}^d} g(x)\overline{\psi(x)} &= \int_{\mathbb{R}^d} \hat{g}(\xi)\overline{\hat{\psi}(\xi)} \\
&= \int_{\mathbb{R}^d} (2\pi i\xi)^\alpha \hat{f}(\xi)\overline{\hat{\psi}(\xi)} \\
&= \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{L^*}\psi(\xi)} \\
&= \int_{\mathbb{R}^d} f(x)\overline{L^*\psi(x)}
\end{aligned}
$$

so $g = Lf$ weakly. Conversely, suppose $g = Lf$ weakly. Then

$$
\begin{aligned}
\int_{\mathbb{R}^d} \hat{g}(\xi)\overline{\hat{\psi}(\xi)} &= \int_{\mathbb{R}^d} g(x)\overline{\psi(x)} \\
&= \int_{\mathbb{R}^d} f(x)\overline{L^*\psi(x)} \\
&= \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{L^*}\psi(\xi)} \\
&= \int_{\mathbb{R}^d} (2\pi i\xi)^\alpha \hat{f}(\xi)\overline{\hat{\psi}(\xi)}
\end{aligned}
$$

which implies that $(2\pi i\xi)^\alpha\hat{f}(\xi) = \hat{g}(\xi)$