Analysis IV : Assignment 3 Solutions
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EXERCISE 1 $(l^2(\mathbb{Z}), \|\cdot\|_2)$ is a complete and separable Hilbert space.

PROOF Let $\{u^{(n)}\}_{n\in\mathbb{N}}$ be a Cauchy sequence. Say $u^{(n)} = (..., u^{(n)}_{-2}, u^{(n)}_{-1}, u^{(n)}_0, u^{(n)}_1, u^{(n)}_2, ...)$. In particular for every fixed $m \in \mathbb{N}$ the sequence $(u^{(n)}_m)_{n=1}^\infty$ is Cauchy in $(\mathbb{C}, |\cdot|)$, so $\exists! u_m \in \mathbb{C}$ such that $u^{(n)}_m \to u_m$ as $n \to \infty$. We claim that $u = (..., u_{-2}, u_{-1}, u_0, u_1, u_2, ...) \in l^2(\mathbb{Z})$ and that $u^{(n)} \to u$ in $(l^2(\mathbb{Z}), \|\cdot\|_2)$.

- Since the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy, it is bounded by say N. Also for all n and M we have $(\sum\limits_{m=-M}^{M} |u_m|^2)^{1/2} \leq (\sum\limits_{m=-M}^{M} |u_m - u^{(n)}_m|^2)^{1/2} + N$. Taking first the limit as $n \to \infty$ and then the limit as $M \to \infty$ shows that $\|u\|_2 \leq N$.
- Now let $\epsilon > 0$ be given. Choose N such that $\|u^{(n)} - u^{(n')}\|_2 < \epsilon$ whenever $n, n' \geq N$. In particular for every $M \in \mathbb{N}$, and $n, n' \geq N$ we have $\sum\limits_{m=-M}^{M} |u^{(n)}_m - u^{(n')}_m|^2 < \epsilon^2$. Taking first the limit as $n' \to \infty$ and then the limit as $M \to \infty$ shows that $\|u^{(n)} - u\|_2 < \epsilon$ whenever $n \geq N$.
- Now for $n \in \mathbb{N}$, let $\mathcal{D}_n = \{u \in (l^2(\mathbb{Z}) : u_i \in \mathbb{Q} \text{ for } |i| \leq n, u_i = 0 \text{ for } |i| > n\}$. We claim that $\bigcup\limits_{n\geq 0} \mathcal{D}_n$ is a countable dense set of $l^2(\mathbb{Z})$. Indeed given $u \in l^2(\mathbb{Z})$ and $\epsilon > 0$, first choose N such that $\sum\limits_{|n|>N} |u_n|^2 < \epsilon$ and then choose $2N+1$ rationals $q_{-N}, ...q_N$ such that $\sum\limits_{|n|\leq N} |u_n - q_n|^2 < \epsilon$. Then letting $q = (..., 0, q_{-N}, ..., q_N, 0, ...) \in \mathcal{D}$, we certainly have $\|u - q\|_2^2 < 2\epsilon$.

$\square$

EXERCISE 2

1. In general, $L^1(\mathbb{R}^n) \not\subset L^2(\mathbb{R}^n)$
2. In general, $L^2(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$.
3. If $f \in L^2(\mathbb{R}^n)$ and f is supported on a set E of finite measure, then $f \in L^1(\mathbb{R}^n)$.
4. If $f \in L^1(\mathbb{R}^n)$ and f is bounded by M (almost everywhere), then $f \in L^2(\mathbb{R}^n)$.

PROOF We are working in n dimensions.

1. Consider $\int\limits_{B(0,1)} |x|^{-\alpha} dx = \int_0^1 r^{-\alpha} r^{n-1} dr \int d\Omega_{n-1} = \int_0^1 r^{n-1-\alpha} dr \int d\Omega_{n-1}$ where $\int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

   Choose α so that $n - 1 - \alpha = -\frac{1}{2}$. Then $\int\limits_{B(0,1)} |x|^{-\alpha} dx = \frac{4\pi^{n/2}}{\Gamma(n/2)}$, but $\int\limits_{B(0,1)} |x|^{-2\alpha} dx = \infty$. i.e. the function $|x|^{-n+1/2} \mathbb{1}_{B(0,1)}$ belongs to $L^1(\mathbb{R}^n)$ but doesn't belong to $L^2(\mathbb{R}^n)$.

2. Consider $\int\limits_{B(0,1)^c} |x|^{-\alpha}dx = \int_1^\infty r^{n-1-\alpha}dr \int d\Omega_{n-1}$. Choose α so that $n-1-\alpha=-1$. Then $\int\limits_{B(0,1)^c} |x|^{-\alpha}dx = \infty$, but $\int\limits_{B(0,1)^c} |x|^{-2\alpha}dx = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ i.e. the function $|x|^{-n}\mathbb{1}_{B(0,1)^c}$ belongs to $L^2(\mathbb{R}^n)$ but doesn't belong to $L^1(\mathbb{R}^n)$.

3. $\|f\|_1 = \|f\mathbb{1}_E\|_1 \le \|f\|_2\|\mathbb{1}_E\|_2 = \|f\|_2 m(E)^{1/2}$

4. $\|f\|_2 \le \|f\cdot M\|_1^{1/2} = \|f\|_1^{1/2}M^{1/2}$.

$\square$

EXERCISE 3 Let $\{\phi_k\}_{k=1}^\infty$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then $\{\phi_k(x)\phi_j(y)\}_{k,j=1}^\infty$ is an orthonormal basis for $L^2(\mathbb{R}^n\times\mathbb{R}^n)$.

PROOF By Fubini's theorem, $\iint_{\mathbb{R}^n\times\mathbb{R}^n} \overline{\phi_k(x)\phi_j(y)}\phi_{k'}(x)\phi_{j'}(y)dxdy = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\overline{\phi_k(x)}\phi_{k'}(x)dx\right)\overline{\phi_j(y)}\phi_{j'}(y)dy =$ Kronecker(k,k') Kronecker(j,j'). So the $\phi_k(x)\phi_j(y)$ are mutually orthogonal. To show that they span $L^2(\mathbb{R}^n\times\mathbb{R}^n)$, it is equivalent to show that if $f(x,y)\in L^2(\mathbb{R}^n\times\mathbb{R}^n)$ is such that $\iint_{\mathbb{R}^n\times\mathbb{R}^n}\overline{f(x,y)}\phi_k(x)\phi_j(y)dxdy=0$, then $f(x,y)=0$. (In other words, the only vector that is simultaneously orthogonal to all the basis elements is the 0 vector.) Indeed, by Fubini, $0=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\overline{f(x,y)}\phi_k(x)\phi_j(y)dxdy = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\overline{f(x,y)}\phi_k(x)dx\right)\phi_j(y)dy$. Since $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, this implies that for every k $\int_{\mathbb{R}^n}\overline{f(x,y)}\phi_k(x)dx=0$ for $\mathrm{Leb}_{\mathbb{R}^n}$ - a.e. $y\in\mathbb{R}^n$. That is, for every k, there is a set $M_k\subset\mathbb{R}^n$ with $m(M_k)=1$ such that for $y\in M_k$, $\int_{\mathbb{R}^n}\overline{f(x,y)}\phi_k(x)dx=0$. Then obviously $m(\bigcap\limits_{k\ge1}M_k)=1$ and for $y\in\bigcap\limits_{k\ge1}M_k$, $f(x,y)=0$ for $\mathrm{Leb}_{\mathbb{R}^n}$ - a.e. $x\in\mathbb{R}^n$. Hence $f(x,y)=0$ for $\mathrm{Leb}_{\mathbb{R}^n\times\mathbb{R}^n}$ - a.e. $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$. $\square$

EXERCISE 4 Let $L^2([-\pi,\pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product $\langle F,G\rangle = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}F(e^{i\theta})\overline{G(e^{i\theta})}d\theta$. Then the mapping $U: L^2([-\pi,\pi])\ni F\to f\in L^2(\mathbb{R})$, where f is given by $f(x)=\frac{1}{\sqrt{\pi}(i+x)}F(\frac{i-x}{i+x})$ is unitary.

PROOF Let $\mathbb{C}_+=\{z\in\mathbb{C}:\mathfrak{Im}(z)>0\}$ (open upper half plane) and $D=\{z\in\mathbb{C}:|z|<1\}$ open unit disc. The well-known conformal map $\mathbb{C}_+\ni z\to\frac{i-z}{i+z}\in D$ extends to a homeomorphism $\overline{\mathbb{C}_+}\ni z\to\frac{i-z}{i+z}\in\overline{D}\setminus\{-1\}$.

Clearly if $z=x\in\mathbb{R}$, then $|\frac{i-z}{i+z}|=1$, and the map $x\to\frac{i-x}{i+x}$ takes $0\to1$, $1\to i$, $-1\to-i$, etc . . . The inverse of the map $\mathbb{R}\ni x\to\frac{i-x}{i+x}\in\partial D\setminus\{-1\}$ is $\partial D\setminus\{-1\}\ni\omega\to\frac{i(1-\omega)}{1+\omega}\in\mathbb{R}$.

$\langle f,g\rangle=\int_{-\infty}^\infty f(x)\overline{g(x)}dx=\frac{1}{\pi}\int_{-\infty}^\infty\frac{1}{(i+x)\overline{(i+x)}}F(\frac{i-x}{i+x})\overline{G(\frac{i-x}{i+x})}dx=\frac{1}{\pi}\int_{-\infty}^\infty\frac{1}{1+x^2}F(\frac{i-x}{i+x})\overline{G(\frac{i-x}{i+x})}dx$

Performing the change of variable $x=\tan(\frac{\theta}{2})$, we have $d\theta=\frac{2dx}{1+x^2}$, $\sin(\theta)=\frac{2x}{1+x^2}$, $\cos(\theta)=\frac{1-x^2}{1+x^2}$, and $\frac{i-x}{i+x}=e^{i\theta}$ Hence $\langle f,g\rangle=\langle F,G\rangle$. We also need to check that U is surjective:

Given a function $f \in L^2(\mathbb{R})$, we have for all $x \in \mathbb{R}$, $f(x) = f\left(\frac{i(1-\omega)}{1+\omega}\right)$, where $\omega = \frac{i-x}{i+x}$.

Also $i + x = \frac{2i}{\frac{i-x}{i+x}+1} = \frac{2i}{\omega+1}$. Consider the function $\partial D \setminus \{-1\} \ni \omega \to F(\omega) = \frac{2i\sqrt{\pi}}{\omega+1} f\left(\frac{i(1-\omega)}{1+\omega}\right) \in \mathbb{C}$. By construction $UF = f$ and so U is surjective.

Note: technically, as a function, F should be defined on ∂D. But since $\{-1\}$ is a set of measure zero, we have defined the equivalence class in which F belongs, and that is what is relevant.

Since $\{e^{inx} : n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2([-\pi, \pi])$, automatically $\{Ue^{inx} : n \in \mathbb{Z}\} = \{\frac{1}{\sqrt{\pi}(i+x)} \left(\frac{i-x}{i+x}\right)^n : n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2(\mathbb{R})$.

□

Exercise 5 Let $\mathcal{H}$ be a Hilbert space; $A : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then $\|AA^*\| = \|A^*A\| = \|A^*\|^2 = \|A\|^2$

Remark If $\|A\| = \sup\limits_{x \in \mathcal{H}, \|x\| \leq 1} \|Ax\| < \infty$, we say that A is a bounded operator with norm $\|A\|$. The following are equal to $\|A\|$ as defined above:

1. $\sup\limits_{x \in \mathcal{H}, \|x\| = 1} \|Ax\|$
2. $\sup\limits_{x \in \mathcal{H}, x \neq 0} \frac{\|Ax\|}{\|x\|}$
3. $\inf\{M \geq 0 : \|Ax\| \leq M\|x\| \text{ for all } x \in \mathcal{H}\}$
4. $\sup\{|\langle x, Ay\rangle| : x, y \in \mathcal{H}, \|x\| \leq 1, \|y\| \leq 1\}$
5. $\sup\{|\langle x, Ay\rangle| : x, y \in \mathcal{H}, \|x\| = 1, \|y\| = 1\}$

The proof of these is obvious for the most part.

By Cauchy-Schwarz, we have $\sup\{|\langle x, Ay\rangle| : x, y \in \mathcal{H}, \|x\| \leq 1, \|y\| \leq 1\} \leq \|A\|$. Conversely,

$$\begin{aligned}
\sup\{|\langle x, Ay\rangle| : x, y \in \mathcal{H}, \|x\| \leq 1, \|y\| \leq 1\} &\geq \sup\{|\langle \frac{Ay}{\|Ay\|}, Ay\rangle| : y \in \mathcal{H}, \|y\| \leq 1\} \\
&= \sup\{\|Ay\| : y \in \mathcal{H}, \|y\| \leq 1\} \\
&= \|A\|.
\end{aligned}$$

Proof Clearly $|\langle x, Ay\rangle| = |\langle y, A^*x\rangle|$ for all x, y so this gives $\|A\| = \|A^*\|$.

Next, $\|A^*A\| = \sup\limits_{x \in \mathcal{H}, \|x\|=1} \|A^*(Ax)\| \leq \sup\limits_{x \in \mathcal{H}, \|x\|=1} \|A^*\|\|Ax\| = \|A^*\|\|A\|$.

Conversely $\|A\|^2 = \left(\sup\limits_{x \in \mathcal{H}, \|x\|=1} \|Ax\|\right)^2 = \sup\limits_{x \in \mathcal{H}, \|x\|=1} \|Ax\|^2$ (since $x \to x^2$ is increasing and continuous on $[0, \infty)$) $= \sup\limits_{x \in \mathcal{H}, \|x\|=1} \langle Ax, Ax\rangle = \sup\limits_{x \in \mathcal{H}, \|x\|=1} \langle x, A^*Ax\rangle \leq \|A^*A\|$ by Cauchy-Schwarz. □

Exercise 6 Let $\mathcal{H}$ be an infinite dimensional (separable) Hilbert space, $(f_k)_{k=1}^{\infty}$ a sequence in $\mathcal{H}$, with $\|f_k\| = 1$ for all k. Then there exists a subsequence $(f_{k_j})_{j=1}^{\infty}$ of $(f_k)_{k=1}^{\infty}$ and a unique $f \in \mathcal{H}$ such that $f_{k_j} \xrightarrow{w} f$, that is $\langle f_{k_j}, g\rangle \to \langle f, g\rangle$ for all $g \in \mathcal{H}$.

Proof This is a special case of the famous theorem by Alaoglu. Let $\{g_n\}$ be an orthonormal basis for $\mathcal{H}$. We are assuming that $\mathcal{H}$ is separable so that the basis is countable. Consider the (infinite) array with k, n^{th} entry equal to $\langle f_k, g_n\rangle$. We describe the diagonalization procedure which produces a subsequence $(f_{k_j})_{j=1}^{\infty}$ of $(f_k)_{k=1}^{\infty}$ such that $\langle f_{k_j}, g_n\rangle$ converges in $(\mathbb{C}, |\cdot|)$ as $j \to \infty$ for every $n \in \mathbb{N}$.

Consider the first column of the array, namely the sequence $(\langle f_k, g_1\rangle)_{k=1}^{\infty}$. This is a bounded sequence (C-S) in $\mathbb{C}$ so it has a convergent subsequence denoted by $(\langle f_{k_{1j}}, g_1\rangle)_{j=1}^{\infty}$. Denote this limit c_1. Now we move on to the second column of the array, and consider the sequence $(\langle f_{k_{1j}}, g_2\rangle)_{j=1}^{\infty}$. Notice that we are restricting our attention only to the indices of the subsequence obtained in the previous column. This is a bounded sequence and we may therefore extract a converging subsequence $(\langle f_{k_{2j}}, g_2\rangle)_{j=1}^{\infty}$. Denote the limit c_2. In general, suppose that in the n^{th} column we have found a subsequence $(\langle f_{k_{nj}}, g_n\rangle)_{j=1}^{\infty}$ of $(\langle f_k, g_n\rangle)_{k=1}^{\infty}$ converging to c_n. Then we pass to the $(n+1)^{th}$ column and extract a subsequence $(\langle f_{k_{(n+1)j}}, g_{n+1}\rangle)_{j=1}^{\infty}$ of $(\langle f_{k_{nj}}, g_{n+1}\rangle)_{j=1}^{\infty}$ converging to c_{n+1}.

Now set $k_j = k_{jj}$ for $j = 1, 2, 3, \ . \ . \ .$ Then for every n, $(\langle f_{k_j}, g_n\rangle)_{j=1}^{\infty}$ is a subsequence of $(\langle f_{k_{nj}}, g_n\rangle)_{j=1}^{\infty}$ starting from $j = n$. Hence for every n, $(\langle f_{k_j}, g_n\rangle)_{j=1}^{\infty} \to c_n$.

Finally letting $f = \sum\limits_{n=1}^{\infty} c_n g_n$, we have that $(\langle f_{k_j}, g_n\rangle)_{j=1}^{\infty} \to \langle f, g_n\rangle$ for every n. If $g = \sum\limits_{i=1}^{N} a_i g_i$, Then by linearity of the inner product $(\langle f_{k_j}, g\rangle)_{j=1}^{\infty} \to \langle f, g\rangle$. □

Exercise 7

Consider a Hilbert space $\mathcal{H}$ and the collection of bounded operators on $\mathcal{H}$ equipped with the operator norm.

1. Strong convergence doesn't imply convergence in norm.
2. Weak convergence doesn't imply strong convergence
3. For any bounded operator T on $\mathcal{H}$ there exists a sequence of finite rank operators $(T_n)_{n=1}^{\infty}$ such that $T_n \to T$ strongly.

Proof

1. Consider the Hilbert space $l^2(\mathbb{N}), \|\cdot\|_2)$ and the shift operator to the left $(Lx)_n = x_{n+1}$ for $n \geq 0$, $x \in l^2(\mathbb{N})$. We claim that the sequence $(L^n)_{n=1}^{\infty}$ (L composed with itself n times) converges strongly to the 0 operator but does not converge in norm to the 0 operator. Given $\epsilon > 0$ and $x \in l^2(\mathbb{N})$ choose N such that $\sum\limits_{i=N}^{\infty} |x_i|^2 < \epsilon$. Then $\|L^n x\|_2^2 < \epsilon$ for every $n \geq N$. Since ϵ was arbitrary we have $\|L^n x\|_2 \to 0$, i.e. $L^n x \to 0$. On the other hand we clearly have $\|L^n\| \leq 1$ and $\|L^n \delta_{n+1}\|_2 = 1$, where δ_{n+1} is the element $(0, 0, ..., 1, 0, 0, ...)$ where the 1 occurs in the $n+1$ position. Hence $\|L^n\| = 1$ for all n, and L^n cannot converge in norm to the 0 operator.

2. Now consider the shift operator to the right $(Rx)_{n+1} = x_n$ for $n \geq 0$, $(Rx)_0 = 0$, $x \in l^2(\mathbb{N})$. We claim that the sequence $(R^n)_{n=1}^{\infty}$ converges weakly to the 0 operator but does not converge strongly to the 0 operator.

   Clearly $\|R^n x\|_2 = \|x\|_2$ for all x and so $(R^n)_{n=1}^{\infty}$ cannot converge strongly to 0 operator. However, since $(l^2(\mathbb{N}))^* = l^2(\mathbb{N})$, every linear functional on $l^2(\mathbb{N})$ is of the form $\langle a, \cdot \rangle = \sum_{n=0}^{\infty} a_n \bar{\cdot}_n$ for some $a \in l^2(\mathbb{N})$. Hence given $\epsilon > 0$, $a, x \in l^2(\mathbb{N})$, choose N such $\sqrt{\sum_{i=N}^{\infty} |a_i|^2} < \epsilon$. Then for $n \geq N$, $|\langle a, R^n x \rangle| \leq \sum_{i=N}^{\infty} |a_i \overline{x_{i-N}}| \leq \sqrt{\sum_{i=N}^{\infty} |a_i|^2} \|x\|_2 \leq \epsilon \|x\|_2$.

3. Let $(g_k)_{k=1}^{\infty}$ be a orthonormal basis of $\mathcal{H}$. Let $T_n g_k = T g_k$ for $k \leq n$ and $T_n g_k = 0$ for $k > n$. Then each T_n has finite rank (i.e. dimension of range is finite) and $\|T_n g_k - T g_k\| = 0$ for n sufficiently large. For an arbitrary vector g, applying the triangle inequality gives $\|T_n g - T g\| = 0$ for n sufficiently large.

□

EXERCISE 9 Suppose w is a measurable function on $\mathbb{R}^d$ with $0 < w(x) < \infty$ for a.e. x, K is a measurable function on $\mathbb{R}^{2d}$ that satisfies:

1. $\int_{\mathbb{R}^d} |K(x,y)| w(y) dy \leq A w(x)$ for almost every $x \in \mathbb{R}^d$
2. $\int_{\mathbb{R}^d} |K(x,y)| w(x) dx \leq A w(y)$ for almost every $y \in \mathbb{R}^d$

Then the integral operator $L^2(\mathbb{R}^d) \ni f(x) \to Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy \in L^2(\mathbb{R}^d)$ is bounded with operator norm less than A.

PROOF First,

$$\begin{aligned}
\int_{\mathbb{R}^d} |K(x,y)||f(y)| dy &= \int_{\mathbb{R}^d} |K(x,y)|^{1/2} w(y)^{1/2} |K(x,y)|^{-1/2} |f(y)| w(y)^{-1/2} dy \\
&\leq \left(\int_{\mathbb{R}^d} |K(x,y)| w(y) dy \right)^{1/2} \left(\int_{\mathbb{R}^d} |K(x,y)||f(y)|^2 w(y)^{-1} dy \right)^{1/2} \\
&\leq A^{1/2} w(x)^{1/2} \left(\int_{\mathbb{R}^d} |K(x,y)||f(y)|^2 w(y)^{-1} dy \right)^{1/2}
\end{aligned}$$

Next,

$$\begin{aligned}
\|Tf\|_2^2 &= \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} K(x,y)f(y)dy|^2 dx \\
&\leq \int_{\mathbb{R}^d} Aw(x) \left(\int_{\mathbb{R}^d} |K(x,y)||f(y)|^2 w(y)^{-1} dy \right) dx \\
&= \text{(Fubini)} \int_{\mathbb{R}^d} A|f(y)|^2 w(y)^{-1} \left(\int_{\mathbb{R}^d} |K(x,y)|w(x)dx \right) dy \\
&\leq A^2 \int_{\mathbb{R}^d} |f(y)|^2 w(y)^{-1} w(y) dy \\
&= A^2 \|f\|_2^2
\end{aligned}$$

Somehow it all works out nicely and $\|T\| \leq A$.

$\square$

Exercise 10 The operator $L^2(0,\infty) \ni f(x) \to Tf(x) = \int_0^\infty \frac{1}{\pi(x+y)} f(y)dy \in L^2(0,\infty)$ is bounded and $\|T\| \leq 1$.

Proof Let $w(x) = \frac{1}{\sqrt{x}}$. By the previous exercise it is enough to check that $\int_0^\infty \frac{1}{\pi(x+y)} w(y)dy \leq Aw(x)$.

By the change of variable $\sqrt{y} = u$ one verifies that $\int_0^\infty \frac{1}{\pi(x+y)} \frac{1}{\sqrt{y}} dy = \frac{1}{\sqrt{x}}$. So the desired inequality holds with $A = 1$. $\square$

Exercise 11 Consider the sawtooth function defined on $[-\pi, \pi)$ by $K(x) = i(\pi sgn(x) - x)$ and extended to $\mathbb{R}$ with period 2π. Suppose that $f \in L^1([-\pi,\pi])$ is extended to $\mathbb{R}$ with period 2π, and consider the operator $Tf(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x-y)f(y)dy$. Then

1. $F(x) = Tf(x)$ is absolutely continuous and $F'(x) = if(x)$ a.e. x whenever $\int_{-\pi}^{\pi} f(y)dy = 0$
2. The map $f \to Tf$ is compact and symmetric on $L^2([-\pi,\pi])$.
3. $\varphi(x) \in L^2([-\pi,\pi])$ is an eigenfunction for T iff $\varphi(x) \sim e^{inx}$ for some integer $n \neq 0$ with eigenvalue $1/n$ or $\varphi(x) = 1$ with eigenvalue 0.
4. $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2([-\pi,\pi])$

Remark First we remind two important results

1. If $f : [a,b] \to \mathbb{R}$ is absolutely continuous then f' exists for a.e. $x \in [a,b]$.

2. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition with Lipschitz constant M iff f is absolutely continuous and $|f'(x)| \le M$ for a.e. x.

PROOF

1. To show that $F(x) = Tf(x)$ is absolutely continuous we could hope to show that $F(x)$ is Lipshitz. We will do a first calculation and realize that $F(x)$ may not necessarily be Lipschitz, but the calculation will shed some light on how to get around the problem.

$$\begin{aligned}|Tf(x) - Tf(y)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi}(K(x-s) - K(y-s))f(s)ds| \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|\pi(sgn(x-s) - \pi sgn(y-s) + y - x||f(s)|ds\end{aligned}$$

WLOG we assume that $x < y$ then

$$|\pi sgn(x-s) - \pi sgn(y-s) + y - x| = \begin{cases} |y-x| & \text{if } s < x \text{ or } s > y \\ |-2\pi + y - x| & \text{if } x < s < y \end{cases}$$

We don't need to specify the function at $s = x$ and $s = y$ since this doesn't change the integration.So

$$\begin{aligned}|Tf(x) - Tf(y)| &\le \frac{1}{2\pi}\int_{-\pi}^{x}|y-x||f(s)|ds + \int_{x}^{y}|-2\pi+y-x||f(s)|ds + \int_{y}^{\pi}|y-x||f(s)|ds \\ &\le \frac{1}{2\pi}2|y-x|\int_{-\pi}^{\pi}|f(s)|ds + \frac{1}{2\pi}\int_{x}^{y}|-2\pi+y-x||f(s)|ds\end{aligned}$$

Now $f \in L^1([-\pi,\pi])$, so from here we see here that we almost have Lipschitzity, but the second term on the RHS with the -2π is problematic. Indeed as the distance between x and y gets smaller, the first term on the RHS gets better while the while the second term gets worse. This is precisely what happens to the difference of two identical sawtooth functions as the phase between them gets smaller. In general the difference gets smaller, but it gets larger at the point where there is a jump discontinuity. This problem is inherent to the sawtooth function so to bypass the problem we have to use the fact that $f \in L^1([-\pi,\pi])$.

Indeed we choose a bounded (simple) function g such that $\int_{-\pi}^{\pi}|f(s) - g(s)|ds < \epsilon$.

Then $\int_x^y |-2\pi + y - x||f(s)|ds \le 2\pi\int_x^y |f(s) - g(s)|ds + 2\pi M|y-x|$ where M is an upper bound for g.

Let $\epsilon > 0$ be given and suppose that (x_i, y_i) are disjoint intervals of $[-\pi,\pi]$ such that $\sum_{i=1}^{n}|x_i - y_i| < \epsilon$. Choose g such that $|g(x)| \le M$ and $\int_{-\pi}^{\pi}|f(s) - g(s)|ds < \epsilon$.

Then $\sum_{i=1}^{n}|Tf(x_i) - Tf(y_i)| \le \sum_{i=1}^{n}C_1|x_i - y_i| + \int_{\cup_i(x_i,y_i)}|f(s) - g(s)|ds \le C\epsilon + \epsilon$, where $C = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(s)|ds + M$.

So $F(x) = Tf(x)$ is absolutely continuous and its derivative exists almost everywhere.

Suppose that $\int_{-\pi}^{\pi}f(s)ds = 0$. We want to calculate $\lim_{h\downarrow 0}\frac{Tf(x+h) - Tf(x)}{h} = \lim_{h\downarrow 0}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{K(x+h-s) - K(x-s)}{h}f(s)ds$.

$$K(x+h-s)-K(x-s) = i\left(\pi sgn(x+h-s)-\pi sgn(x-s)-h\right) = \begin{cases} -ih & \text{if } s<x \text{ or } s>x+h \\ i(2\pi-h) & \text{if } x<s<x+h \end{cases}$$

So $\lim_{h\downarrow 0}\frac{Tf(x+h)-Tf(x)}{h} = \lim_{h\downarrow 0}\int_{-\pi}^{\pi} i\frac{\mathbb{1}_{(x,x+h)}(s)}{h}f(s)ds - \frac{i}{2\pi}\int_{-\pi}^{\pi}f(s)ds = \lim_{h\downarrow 0}\frac{i}{h}\int_{x}^{x+h}f(s)ds = if(x)$ by the Fundamental theorem of calculus. Similarly $\lim_{h\uparrow 0}\frac{Tf(x+h)-Tf(x)}{h} = if(x)$. So $(Tf)' = if$. Another way of deriving this way would have been to use DCT and the fact that the derivative of the sign function is two times the delta function.

2. The integral operator T is easily seen to be Hilbert-Schmidt. Indeed, since the sawtooth function is bounded and we are on a finite measure space, $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|K(x,y)|^2dxdy < \infty$. From the general theory of bounded operators, Hilbert-Schmidt operators are compact (see p.190 of Stein & Shakarchi).

   We also check that T is symmetric:

$$\begin{aligned}\langle Tf, g\rangle &= \int_{-\pi}^{\pi}\overline{Tf(x)}g(x)dx \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\overline{i(\pi sgn(x-s)-(x-s))f(s)}ds\; g(x)dx \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} i(\pi sgn(s-x)-(s-x))\overline{f(s)}ds\; g(x)dx \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} i(\pi sgn(s-x)-(s-x))g(x)dx\; \overline{f(s)}ds \\ &= \langle f, Tg\rangle\end{aligned}$$

3. One may compute the integrals to verify that $\{e^{inx}\}_{n\in\mathbb{Z}\setminus\{0\}}$ are indeed eigenfunctions of T with eigenvalue $\frac{1}{n}$ and that 1 is an eigenfunction with eigenvalue 0. Conversely, if $Tf = \lambda f$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}K(x-y)f(y)dy\;dx = \lambda\int_{-\pi}^{\pi}f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\int_{-\pi}^{\pi}K(x-y)dx\;dy = 0$. Hence by 1., $(Tf)' = \lambda f' = if$, so $f(x) = e^{\frac{i}{\lambda}x}$, when $\lambda \neq 0$. Going back to eigenvalue problem will give us a condition on λ:
   $\frac{1}{2\pi}\int_{-\pi}^{\pi}K(x-y)e^{\frac{i}{\lambda}y}dy = \lambda e^{\frac{1}{\lambda}x} - \frac{\lambda^2}{\pi}e^{\frac{1}{\lambda}x}\sin\frac{\pi}{\lambda}$. Hence we must have $\sin\frac{\pi}{\lambda} = 0$, that is $\lambda = \frac{1}{n}$. For the case $\lambda = 0$, we have $(Tf)' = 0 = if$, so f is a constant function.

4. T has been checked to be compact and symmetric. By the spectral theorem, a subset of the eigenvectors of T forms an orthonormal basis for $L^2([-\pi,\pi])$. Since $\{e^{inx}\}_{n\in\mathbb{Z}}$ are the only eigenvectors of T and are mutually orthogonal, they must be a basis for $L^2([-\pi,\pi])$.

$\square$