Quantum Ergodic Restriction Theorems

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Manifolds without boundary

(M^n, g) a compact Riemannian manifold with ergodic geodesic flow $G^t : T^*M - 0 \to T^*M - 0$.

Laplacian: $\Delta_g : C^\infty(M) \longrightarrow C^\infty(M)$

Eigenfunctions: $\Delta_g \phi_{\lambda_j} + \lambda_j^2 \phi_{\lambda_j} = 0$, $j = 0, 1, 2, ...$

$$\langle \phi_{\lambda_j}, \phi_{\lambda_k} \rangle_{L^2(M)} = \delta_j^k.$$

Quantum Ergodicity (Zelditch, Colin de Verdière) Let (M^n, g) be ergodic and $Op(a) \in Op_M(S^0_{cl})$ be a zeroth-order pseudodifferenrtial operator. Then, there exists a density-one subset $S \subset \mathbf{N}$ such that

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op(a) \phi_{\lambda_j}, \phi_{\lambda_j} \rangle = \int_{S^*M} a(x, \xi) \, d\mu_L(x, \xi),$$

where $d\mu_L$ is Liouville measure.

Special case: $\Omega \subset M$ open,

$$\lim_{\lambda_j \to \infty; j \in S} \int_{\Omega} |\phi_{\lambda_j}|^2 dx = vol(\Omega).$$

For a full-density ergodic sequence of eigenfunctions, mass is equidistributed on n-dimensional submanifolds of M.

A more refined question about eigenfunction equidistribution is the following

Basic Question: Given (M, g) ergodic, does quantum ergodicity hold for restrictitons of eigenfunctions to hypersurfaces, $H \subset M$? In particular, is it true that with $u_j := \phi_j|_H$,

$$\lim_{\lambda_j \to \infty; j \in S} \frac{1}{vol(H)} \int_H |u_j(s)|^2 d\sigma_H(s) = 1?$$

Our main result answers this question in the affirmative for generic hypersurfaces, $H \subset M$

Theorem 1 [Zeldtich-T] Let (M, g) be a compact manifold with ergodic geodesic flow, and let $H \subset M$ be a hypersurface. Let $\phi_{\lambda_j}; j = 1, 2, ...$ denote the L^2-normalized eigenfunctions in Δ_g. Then, if H has a zero measure of microlocal symmetry, then there exists a density-one subset S of $\mathbf{N}$ such that for $\lambda_0 > 0$ and $a(s, \tau) \in S^0_{cl}(T^* H)$

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op_H(a) \phi_{\lambda_j}|_H, \phi_{\lambda_j}|_H \rangle_{L^2(H)} = \omega(a_0),$$

where

$$\omega(a_0) = \frac{4}{vol(S^* M)} \int_{B^* H} a_0(y, \eta)\, \rho_H(s, \tau)\, ds d\tau,$$

with $\rho_H(s, \tau) := (1 - |\tau|^2)^{-1/2}$.

The measure-zero microlocal reflection symmetry condition on H is generic. Specific examples include closed geodesics and geodesic circles on compact hyperbolic surfaces, $M = H^2/\Gamma$.

Microlocal reflection symmetry: $H \subset M$ orientable submanifold with two unit normal vector fields $\nu_\pm$ to H. There is the corresponding decomposition

$$T^*_H M = T^*_{H,+} M \cup T^*_{H,-} M.$$

For $(s, \tau) \in B^*H$ define

$$\xi_\pm(s,\tau) = \tau \pm \sqrt{1 - |\tau|^2}\nu_s \in T^*_{H,\pm} M.$$

Given $\xi_+(s,\tau) \in T^*_{H,+}M$, follow the geodesic arc $G^t(\xi_\pm(s,\tau))$ emanating from the two sides of H until it hits H again at time $t = t(s,\tau)$. Assuming $G^t(s,\tau) \in T^*_{H,+}M$ also, we tangentially project back to B^*H. There are corresponding return maps:

$$\mathcal{P}_{\pm,j} : B^*H \to B^*H, \; j \in \mathbf{Z}$$

The indices $j \in Z$ label the intersection number of the geodesic with H.

Definition: $H \subset M$ has zero measure of microlocal reflection symmetry if for all $(j,k) \in \mathbf{Z} \times \mathbf{Z}$,

$$\Big| \{(s,\tau) \in B^* H;\, \mathcal{P}_{\pm,j}(s,\tau) = \mathcal{P}_{\mp,k}(s,\tau)\} \Big| = 0.$$

Here $|\cdot|$ denotes symplectic measure on B^*H.

Restriction bounds: General restriction bounds of Burq-Gérard-Tzvetkov give

$$\int_H |\phi_\lambda|_H|^2 d\sigma_H = \mathcal{O}(\lambda^{1/2});\ (n=2).$$

Bound is sharp in general: eg. when $H = \{(x,y,z) \in \mathbf{S^2}; z = 0\}$ is equator on sphere, and $\phi_k(x,y,z) = k^{1/4}(x+iy)^k$ highest weight spherical harmonic. Since $|x+iy| = 1$ when $z = 0$, $|\phi_k|_H|^2 = k^{1/2}$ and so,

$$\int_H |\phi_k|_H|^2 d\sigma_H \sim_{k\to\infty} k^{1/2}.$$

Theorem 1 improves on these general bounds in the ergodic case (asymptotic result, not just upper bound).

Other recent results on eigenfunction restriction bounds (Hassell-Tacy, Sogge, T,...)

Sketch of Proof of Theorem 1: Let γ_H : $f \mapsto f|_H$ be restriction map.

$$\langle Op_H(a)\phi_j|_H, \phi_j|_H\rangle_{L^2(H)}$$

$$= \langle Op_H(a)\gamma_H\phi_j, \gamma_H\phi_j\rangle_{L^2(H)}$$

$$= \langle \gamma_H^* Op_H(a)\gamma_H U(t)\phi_j, U(t)\phi_j\rangle_{L^2(M)}$$

$$= \langle U(-t)\gamma_H^* Op(a)\gamma_H U(t)\phi_j, \phi_j\rangle_{L^2(M)}$$

$$= \langle V(t,a)\phi_j, \phi_j\rangle_{L^2(M)}$$

Upshot:

$$\langle Op_H(a)\phi_j|_H, \phi_j|_H\rangle_{L^2(H)} = \langle V(t,a)\phi_j, \phi_j\rangle_{L^2(M)} \tag{1}$$

with

$$V(t,a) := U(-t)\gamma_H^* Op(a)\gamma_H U(t).$$

Time-average the identity in (1) and get

$$\langle Op_H(a)\phi_j|_H, \phi_j|_H\rangle_{L^2(H)} \;=\; \langle V_T(a)\phi_j, \phi_j\rangle_{L^2(M)}, \tag{2}$$

with

$$V_T(a) := \frac{1}{T}\int_{-\infty}^{\infty} V(t;a)\,\chi(T^{-1}t)\,dt. \tag{3}$$

Here, $\chi \in C_0^\infty(\mathbf{R})$ with $\int_{-\infty}^{\infty}\chi(t)dt = 1$,

Proposition (Generalized Egorov Theorem for $V_T(a)$) There is a decomposition

$$V_T(a) = P_T(a) + F_T(a) + R_T(a).$$

- The operator $P_T(a) \in Op(S^0_{cl})$ with

$$\sigma(P_T(a)) = \frac{1}{T}\int_{-\infty}^{\infty} G_t^* a \; \chi(T^{-1}t)dt,$$

- $F_T(a)$ is a zeroth order FIO with canonical relation

$$\Gamma_T \;\; = \{ \, (x, \xi, x', \xi') \in T^*M \times T^*M : \exists t \in (-T, T) :$$

$$\exp_x t\xi = \exp_{x'} t\xi' = s \in H,$$

$$G^t(x', \xi') = r_H G^t(x, \xi), \;\; |\xi| = |\xi'| \, \},$$

where, $r_H : T^*_H M \to T^*_H M$ is normal reflection in H.

- $R_T(a)$ has tangential operator wavefront. It has no bearing on the QER results for eigenfunctions and we ignore it here.

Variance Estimates: Proof of Theorem 1

To prove QER, one needs to show that

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \Big| \langle V_T(a)\phi_j, \phi_j \rangle_{L^2(M)} - \omega(a) \Big|^2 = o(1), \tag{4}$$

as $\lambda \to \infty$. Ignoring $R_T(a)$, from the Egorov decompostion $V_T(a) = P_T(a) + F_T(a)$,

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \Big| \langle V_T(a)\phi_j, \phi_j \rangle_{L^2(M)} - \omega(a) \Big|^2$$

$$\leq \frac{2}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \Big| \langle P_T(a)\phi_j, \phi_j \rangle_{L^2(M)} - \omega(a) \Big|^2$$

$$+ \frac{2}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \Big| \langle F_T(a)\phi_j, \phi_j \rangle_{L^2(M)} \Big|^2 .$$

By usual QE, the pseudodifferential variance term

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} \left| \langle P_T(a)\phi_j, \phi_j \rangle_{L^2(M)} - \omega(a) \right|^2 = o(1).$$

By Cauchy-Schwarz,

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle F_T(a)\phi_j, \phi_j \rangle_{L^2(M)} \right|^2$$

$$\leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle F_T(a)^* F_T(a)\phi_j, \phi_j \rangle_{L^2(M)}.$$

It suffices to prove that

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle F_T(a)^* F_T(a)\phi_j, \phi_j \rangle_{L^2(M)} = o(1) \tag{5}$$

as $\lambda \to \infty$.

FIO Weyl law: Let $F : C^\infty(M) \to C^\infty(M)$ be a homogeneous FIO of order zero with canonical relation $\Gamma_F = \text{graph}(\kappa_F)$, $\kappa_F : T^*M \to T^*M$

symplectic. An old result of Zelditch says that

$$\lim_{\lambda\to\infty}\frac{1}{N(\lambda)}\sum_{j:\lambda_j\leq\lambda}\langle F\phi_{\lambda_j},\phi_{\lambda_j}\rangle=\int_{\Gamma_F\cap\Delta_{T^*M}}\sigma_\Delta(F)d\mu_L. \tag{6}$$

The integral on the RHS of (6) is over the intersection of the canonical relation Γ_F of F with the diagonal of $T^*M\times T^*M$ and $d\mu_L$ is Liouville Measure. The right side of (6) is zero unless the intersection has dimension $m = \dim M$, i.e. it sifts out the 'pseudo-differential part' of F.

Apply (6) with

$$F=F_T(a)^*F_T(a).$$

By wavefront calculus,

$$|\Gamma_F\cap\Delta_{T^*M\times T^*M}|=0\Longleftrightarrow$$

H satisfies zero measure microlocal reflection symmetry condition.

Manifolds with boundary

Theorem 2 [Zelditch-T] Let $M \subset \mathbf{R}^n$ be a piecewise-smooth billiard with totally ergodic billiard flow and let $H \subset int(M)$ be a smooth interior hypersurface satisfying the measure zero microlocal reflection condition. Let $\phi_{\lambda_j}; j = 1, 2, ...$ denote the L^2-normalized Neumann eigenfunctions in Ω. Then, there exists a density-one subset S of $\mathbf{N}$ such that for $a(s, \tau) \in S^0_{cl}(T^*H)$,

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op_H(a) \phi_{\lambda_j}|_H, \phi_{\lambda_j}|_H \rangle_{L^2(H)} = \omega(a_0).$$

- Similar results for Dirichlet eigenfunctions with suitable limiting measures $\omega(a_0)$.

- Method of proof is simliar to case $\partial M = \emptyset$ but is more complicated. Since the wave operator $U(t)$ is complicated when $\partial M \neq \emptyset$

in [Zelditch-T] we use potential layers instead. Microlocal analysis is then semiclassical (inhomogeneous) and the corresponding semiclassical FIO Weyl law is more complicated than in the homogeneous case.

Quantum ergodic restriction for Cauchy data

Instead of Dirichlet data $DD_H := (\phi_\lambda|_H)$ consider (normalized) Cauchy data

$$CD_H(\phi_\lambda) : (\phi_\lambda|_H, \lambda^{-1}\partial_\nu \phi_\lambda|H). \tag{7}$$

Theorem 3 [Zelditch-T, Christianson-Hezari-Zelditch-T] Let $H \subset M$ be **any** interior hypersurface. Then, there exists a measure $d\mu_\infty$ on B^*H so that along a subsequence of eigenvalues of density one we have,

$$\begin{aligned} &\langle Op_\lambda((1-|\tau|^2)a(s,\tau))\phi_\lambda|_H, \phi_\lambda|_H\rangle \\ &+\lambda^{-2}\langle Op_\lambda(a(s,\tau))\partial_{\nu_H}\phi_\lambda|_H, \partial_{\nu_H}\phi_\lambda|_H\rangle \qquad (8) \\ &= \int_{B^*H} 2(1-|\tau|^2) a\, d\mu_\infty + o(1). \end{aligned}$$

Remarks:

- Cauchy data result holds for all interior hypersurfaces H. No microlocal reflection symmetry assumption required.

- In the case of Cauchy data: Quantum Unique Ergodicity (QUE)
$\Longrightarrow$ Quantum Ergodic Restriction (QER) in Theorem 3 for **all** eigenfunctions $\phi_{\lambda_j}; j = 1, 2, \ldots$