

PARK CITY LECTURES ON EIGENFUNKTIONS

STEVE ZELDITCH

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1. INTRODUCTION

These lectures are devoted to recent results on the nodal geometry of eigenfunktions

$$(1) \quad \Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda$$

of the Laplacian Δ_g of a Riemannian manifold (M^m, g) of dimension m and to associated problems on L^p norms of eigenfunktions (§2.9 and §12). The manifolds are generally assumed to be compact, although the problems can also be posed on non-compact complete Riemannian manifolds. The emphasis of these lectures is on *real analytic* Riemannian manifolds, but we also mention some new results for general C^∞ metrics. Although we mainly discuss the Laplacian, analogous problems and results exist for Schrödinger operators $-\frac{\hbar^2}{2}\Delta_g + V$ for certain potentials V . Moreover, many of the results on eigenfunktions also hold for *quasi-modes* or approximate eigenfunktions defined by oscillatory integrals.

The study of eigenfunktions of Δ_g and $-\frac{\hbar^2}{2}\Delta_g + V$ on Riemannian manifolds is a branch of harmonic analysis. In these lectures, we emphasize high frequency (or semi-classical)

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asymptotics of eigenfunctions and their relations to the global dynamics of the geodesic flow $G^t : S^*M \rightarrow S^*M$ on the unit cosphere bundle of M . Here and henceforth we identify vectors and covectors using the metric. As in [Ze3] we give the name ‘‘Global Harmonic Analysis’’ to the use of global wave equation methods to obtain relations between eigenfunction behavior and geodesics. The relations between geodesics and eigenfunctions belongs to the general correspondence principle between classical and quantum mechanics. The correspondence principle has evolved since the origins of quantum mechanics [Sch] into a systematic theory of Semi-Classical Analysis and Fourier integral operators, of which [HoI, HoII, HoIII, HoIV] and [Zw] give systematic presentations; see also §2.11 for further references. Quantum mechanics provides not only the intuition and techniques for the study of eigenfunctions, but in large part also provides the motivation. Readers who are unfamiliar with quantum mechanics are encouraged to read standard texts such as Landau-Lifschitz [LL] or Weinberg [Wei]. Atoms and molecules are multi-dimensional and difficult to visualize, and there are many efforts to do so in the physics and chemistry literature. Some examples may be found in [KP, He, Th, SHM]. Nodal sets of the hydrogen atom have recently been observed using quantum microscopes [St]. Here we concentrate on eigenfunctions of the Laplacian; some results on nodal sets of eigenfunctions of Schrödinger operators can be found in [Jin, HZZ].

Among the fundamental tools in the study of eigenfunctions are parametrix constructions of wave and Poisson kernels, and the method of stationary phase in the real and complex domain. The plan of these lectures is to concentrate at once on applications to nodal sets and L^p norms and refer to Appendices or other texts for the techniques. Parametrix constructions and stationary phase are techniques whose role in spectral asymptotics are as basic as the maximum principle is in elliptic PDE. We do include Appendices on the geodesic flow (§13), on parametrix constructions for the wave group (§14), on general facts and definitions concerning oscillatory integrals (§15) and on spherical harmonics (§16).

1.1. The eigenvalue problem on a compact Riemannian manifold. The (negative) Laplacian Δ_g of (M^m, g) is the unbounded essentially self-adjoint operator on $C_0^\infty(M) \subset L^2(M, dV_g)$ defined by the Dirichlet form

$$D(f) = \int_M |\nabla f|^2 dV_g,$$

where ∇f is the metric gradient vector field and $|\nabla f|$ is its length in the metric g . Also, dV_g is the volume form of the metric. In terms of the metric Hessian Dd ,

$$\Delta f = \text{trace } Ddf.$$

In local coordinates,

$$(2) \quad \Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right),$$

in a standard notation that we assume the reader is familiar with (see e.g. [BGM, Ch] if not).

Remark: We are not always consistent on the sign given to Δ_g . When we work with $\sqrt{-\Delta_g}$ we often define Δ_g to be the opposite of (2) and write $\sqrt{\Delta}$ for notational simplicity. We

often omit the subscript g when the metric is fixed. We hope the notational conventions do not cause confusion.

A more geometric definition uses at each point p an orthonormal basis $\{e_j\}_{j=1}^m$ of T_pM and geodesics γ_j with $\gamma_j(0) = p$, $\gamma_j'(0) = e_j$. Then

$$\Delta f(p) = \sum_j \frac{d^2}{dt^2} f(\gamma_j(t)).$$

We refer to [BGM] (G.III.12).

EXERCISE 1. Let $m = 2$ and let γ be a geodesic arc on M . Calculate $(\Delta f)(s, 0)$ in Fermi normal coordinates along γ .

Background: Define Fermi normal coordinates (s, y) along γ by identifying a small ball bundle of the normal bundle $N\gamma$ along $\gamma(s)$ with its image (a tubular neighborhood of γ) under the normal exponential map, $\exp_{\gamma(s)} y\nu_{\gamma(s)}$. Here, $\nu_{\gamma(s)}$ is the unit normal at $\gamma(s)$ (fix one of the two choices) and $\exp_{\gamma(s)} y\nu_{\gamma(s)}$ is the unit speed geodesic in the direction $\nu_{\gamma(s)}$ of length y .

The focus of these lectures is on the eigenvalue problem (1). As mentioned above, we sometimes multiply Δ and the eigenvalue by -1 for notational simplicity. We assume throughout that φ_λ is L^2 -normalized,

$$\|\varphi_\lambda\|_{L^2}^2 = \int_M |\varphi_\lambda|^2 dV = 1.$$

When M is compact, the spectrum of eigenvalues of the Laplacian is discrete there exists an orthonormal basis of eigenfunctions. We fix such a basis $\{\varphi_j\}$ so that

$$(3) \quad \Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle_{L^2(M)} := \int_M \varphi_j \varphi_k dV_g = \delta_{jk}$$

If $\partial M \neq \emptyset$ we impose Dirichlet or Neumann boundary conditions. Here dV_g is the volume form. When M is compact, the spectrum of Δ_g is a discrete set

$$(4) \quad \lambda_0 = 0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$$

repeated according to multiplicity. Note that $\{\lambda_j\}$ denote the *frequencies*, i.e. square roots of Δ -eigenvalues. We mainly consider the behavior of eigenfunctions in the ‘high frequency’ (or high energy) limit $\lambda_j \rightarrow \infty$.

The Weyl law asymptotically counts the number of eigenvalues less than λ ,

$$(5) \quad N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = \frac{|B_n|}{(2\pi)^n} \text{Vol}(M, g) \lambda^n + O(\lambda^{n-1}).$$

Here, $|B_n|$ is the Euclidean volume of the unit ball and $\text{Vol}(M, g)$ is the volume of M with respect to the metric g . The size of the remainder reflects the measure of closed geodesics [DG, HoIV]. It is a basic example of global the effect of the global dynamics on the spectrum. See §2.9 and §12 for related results on eigenfunctions.

- (1) In the *aperiodic* case where the set of closed geodesics has measure zero, the Duistermaat-Guillemin-Ivrii two term Weyl law states

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = c_m \operatorname{Vol}(M, g) \lambda^m + o(\lambda^{m-1})$$

where $m = \dim M$ and where c_m is a universal constant.

- (2) In the *periodic* case where the geodesic flow is periodic (Zoll manifolds such as the round sphere), the spectrum of $\sqrt{\Delta}$ is a union of eigenvalue clusters C_N of the form

$$C_N = \left\{ \left(\frac{2\pi}{T} \right) \left(N + \frac{\beta}{4} \right) + \mu_{Ni}, i = 1 \dots d_N \right\}$$

with $\mu_{Ni} = o(N^{-1})$. The number d_N of eigenvalues in C_N is a polynomial of degree $m - 1$.

Remark: The proof that the spectrum is discrete is based on the study of spectral kernels such as the heat kernel or Green's function or wave kernel. The standard proof is to show that Δ_g^{-1} (whose kernel is the Green's function, defined on the orthogonal complement of the constant functions) is a compact self-adjoint operator. By the spectral theory for such operators, the eigenvalues of Δ_g^{-1} are discrete, of finite multiplicity, and only accumulate at 0. Although we concentrate on parametrix constructions for the wave kernel, one can construct the Hadamard parametrix for the Green's function in a similar way. Proofs of the above statements can be found in [GSj, DSj, Zw, HoIII].

The proof of the integrated and pointwise Weyl law are based on wave equation techniques and Fourier Tauberian theorems. The wave equation techniques mainly involve the construction of parametrices for the fundamental solution of the wave equation and the method of stationary phase. In §15 we review We refer to [DG, HoIV] for detailed background.

1.2. Nodal and critical point sets. The main focus of these lectures is on nodal hypersurfaces

$$(6) \quad \mathcal{N}_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}.$$

The nodal domains are the components of the complement of the nodal set,

$$M \setminus \mathcal{N}_{\varphi_\lambda} = \bigcup_{j=1}^{\mu(\varphi_\lambda)} \Omega_j.$$

For generic metrics, 0 is a regular value of φ_λ of all eigenfunctions, and the nodal sets are smooth non-self-intersecting hypersurfaces [U]. Among the main problems on nodal sets are to determine the hypersurface volume $\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda})$ and ideally how the nodal sets are distributed. Another well-known question is to determine the number $\mu(\varphi_\lambda)$ of nodal domains in terms of the eigenvalue in generic cases. One may also consider the other level sets

$$(7) \quad \mathcal{N}_{\varphi_j}^a = \{x \in M : \varphi_j(x) = a\}$$

and sublevel sets

$$(8) \quad \{x \in M : |\varphi_j(x)| \leq a\}.$$

The zero level is distinguished since the symmetry $\varphi_j \rightarrow -\varphi_j$ in the equation preserves the nodal set.

Remark: Nodal sets belong to individual eigenfunctions. To the author’s knowledge there do not exist any results on averages of nodal sets over the spectrum in the sense of (5)-(33). That is, we do not know of any asymptotic results concerning the functions

$$Z_f(\lambda) := \sum_{j:\lambda_j \leq \lambda} \int_{\mathcal{N}_{\varphi_j}} f dS,$$

where $\int_{\mathcal{N}_{\varphi_j}} f dS$ denotes the integral of a continuous function f over the nodal set of φ_j . When the eigenvalues are multiple, the sum Z_f depends on the choice of orthonormal basis.

Randomizing by taking Gaussian random combinations of eigenfunctions simplifies nodal problems profoundly, and are studied in many articles (see e.g. [NS]).

One would also like to know the “number” and distribution of critical points,

$$(9) \quad \mathcal{C}_{\varphi_j} = \{x \in M : \nabla \varphi_j(x) = 0\}.$$

In fact, the critical point set can be a hypersurface in M , so for counting problems it makes more sense to count the number of critical values,

$$(10) \quad \mathcal{V}_{\varphi_j} = \{\varphi_j(x) : \nabla \varphi_j(x) = 0\}.$$

At this time of writing, there exist few rigorous upper bounds on the number of critical values, so we do not spend much space on them here. If we ‘randomize’ the problem and consider the average number of critical points (or equivalently values) or random spherical harmonics on the standard \mathbb{S}^m , one finds that the random spherical harmonics of degree N (eigenvalue $\simeq N^2$) has $C_m N^m$ critical points in dimension m . This is not surprising since spherical harmonics are harmonic polynomials. In the non-generic case that the critical manifolds are of co-dimension one, the hypersurface volume is calculated in the real analytic case in [Ba]. The upper bound is also given in [Ze0].

The frequency λ of an eigenfunction (i.e. the square root of the eigenvalue) is a measure of its “complexity”, similar to specifying the degree of a polynomial, and the high frequency limit is the large complexity limit. A sequence of eigenfunctions of increasing frequency oscillates more and more rapidly and the problem is to find its “limit shape”. Sequences of eigenfunctions often behave like “Gaussian random waves” but special ones exhibit highly localized oscillation and concentration properties.

1.3. Motivation. Before stating the problems and results, let us motivate the study of eigenfunctions and their high frequency behavior. The eigenvalue problem (1) arises in many areas of physics, for example the theory of vibrating membranes. But renewed motivation to study eigenfunctions comes from quantum mechanics. As is discussed in any textbook on quantum physics or chemistry (see e.g. [LL, Wei]), the Schrödinger equation resolves the problem of how an electron can orbit the nucleus without losing its energy in radiation. The classical Hamiltonian equations of motion of a particle in phase space are orbits of Hamilton’s equations

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j}, \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}, \end{cases}$$

where the Hamiltonian

$$H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x) : T^*M \rightarrow \mathbb{R}$$

is the total Newtonian kinetic + potential energy. The idea of Schrödinger is to model the electron by a wave function φ_j which solves the eigenvalue problem

$$(11) \quad \hat{H}\varphi_j := \left(-\frac{\hbar^2}{2}\Delta + V\right)\varphi_j = E_j(\hbar)\varphi_j,$$

for the Schrödinger operator \hat{H} , where V is the potential, a multiplication operator on $L^2(\mathbb{R}^3)$. Here \hbar is Planck's constant, a very small constant. The semi-classical limit $\hbar \rightarrow 0$ is mathematically equivalent to the high frequency limit when $V = 0$. The time evolution of an 'energy state' is given by

$$(12) \quad U_\hbar(t)\varphi_j := e^{-i\frac{t}{\hbar}\left(-\frac{\hbar^2}{2}\Delta + V\right)}\varphi_j = e^{-i\frac{tE_j(\hbar)}{\hbar}}\varphi_j.$$

The unitary operator $U_\hbar(t)$ is often called the *propagator*. In the Riemannian case with $V = 0$, the factors of \hbar can be absorbed in the t variable and it suffices to study

$$(13) \quad U(t) = e^{it\sqrt{\Delta}}.$$

An L^2 -normalized energy state φ_j defines a probability amplitude, i.e. its modulus square is a probability measure with

$$(14) \quad |\varphi_j(x)|^2 dx = \text{the probability density of finding the particle at } x.$$

According to the physicists, the *observable quantities* associated to the energy state are the probability density (14) and 'more generally' the matrix elements

$$(15) \quad \langle A\varphi_j, \varphi_j \rangle = \int \varphi_j(x) A\varphi_j(x) dV$$

of observables (A is a self adjoint operator, and in these lectures it is assumed to be a pseudo-differential operator). Under the time evolution (12), the factors of $e^{-i\frac{tE_j(\hbar)}{\hbar}}$ cancel and so the particle evolves as if "stationary", i.e. observations of the particle are independent of the time t .

Modeling energy states by eigenfunctions (11) resolves the paradox¹ of particles which are simultaneously in motion and are stationary, but at the cost of replacing the classical model of particles following the trajectories of Hamilton's equations by 'linear algebra', i.e. evolution by (12). The quantum picture is difficult to visualize or understand intuitively. Moreover, it is difficult to relate the classical picture of orbits with the quantum picture of eigenfunctions.

The study of nodal sets was historically motivated in part by the desire to visualize energy states by finding the points where the quantum particle is least likely to be. In fact, just recently (at this time or writing) the nodal sets of the hydrogen atom energy states have become visible to microscopes [St].

2. RESULTS

We now introduce the results whose proofs we sketch in the later sections of this article.

¹This is an over-simplified account of the stability problem; see [LS] for an in-depth account

2.1. Nodal hypersurface volumes for C^∞ metrics. In the late 70's, S. T. Yau conjectured that for general $C^\infty (M, g)$ of any dimension m there exist c_1, C_2 depending only on g so that

$$(16) \quad \lambda \lesssim \mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda}) \lesssim \lambda.$$

Here and below \lesssim means that there exists a constant C independent of λ for which the inequality holds. The upper bound of (16) is the analogue for eigenfunctions of the fact that the hypersurface volume of a real algebraic variety is bounded above by its degree. The lower bound is specific to eigenfunctions. It is a strong version of the statement that 0 is not an “exceptional value” of φ_λ . Indeed, a basic result is the following classical result, apparently due to R. Courant (see [Br]). It is used to obtain lower bounds on volumes of nodal sets:

PROPOSITION 1. *For any (M, g) there exists a constant $A > 0$ so that every ball of (M, g) of radius greater than $\frac{A}{\lambda}$ contains a nodal point of any eigenfunction φ_λ .*

We sketch the proof in §3.2 for completeness, but leave some of the proof as an exercise to the reader.

The lower bound of (16) was proved for all C^∞ metrics for surfaces, i.e. for $m = 2$ by Brüning [Br]. For general C^∞ metrics in dimensions ≥ 3 , the known upper and lower bounds are far from the conjecture (16). At present the best lower bound available for general C^∞ metrics of all dimensions is the following estimate of Colding-Minicozzi [CM]; a somewhat weaker bound was proved by Sogge-Zelditch [SoZ] and the later simplification of the proof [SoZa] turned out to give the same bound as [CM]. We sketch the proof from [SoZa].

THEOREM 2.

$$\lambda^{1-\frac{n-1}{2}} \lesssim \mathcal{H}^{m-1}(\mathcal{N}_\lambda),$$

The original result of [SoZ] is based on lower bounds on the L^1 norm of eigenfunctions. Further work of Hezari-Sogge [HS] shows that the Yau lower bound is correct when one has $\|\varphi_\lambda\|_{L^1} \geq C_0$ for some $C_0 > 0$. It is not known for which (M, g) such an estimate is valid. At the present time, such lower bounds are obtained from upper bounds on the L^4 norm of φ_λ . The study of L^p norms of eigenfunctions is of independent interest and we discuss some recent results which are not directly related to nodal sets in §12 and in §2.9. The study of L^p norms splits into two very different cases: there exists a critical index p_n depending on the dimension of M , and for $p \geq p_n$ the L^p norms of eigenfunctions are closely related to the structure of geodesic loops (see §2.9). For $2 \leq p \leq p_n$ the L^p norms are governed by different geodesic properties of (M, g) which we discuss in §12.

We also recall in §4.4 an interesting upper bound due to R. T. Dong (and Donnelly-Fefferman) in dimension 2, since the techniques of proof of [Dong] seem capable of further development.

THEOREM 3. *For $C^\infty (M, g)$ of dimension 2,*

$$\mathcal{H}^1(\mathcal{N}_\lambda) \lesssim \lambda^{3/2}.$$

2.2. Nodal hypersurface volumes for real analytic (M, g) . In 1988, Donnelly-Fefferman [DF] proved the conjectured bounds for real analytic Riemannian manifolds (possibly with boundary). We re-state the result as the following

THEOREM 4. *Let (M, g) be a compact real analytic Riemannian manifold, with or without boundary. Then*

$$c_1 \lambda \lesssim \mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \lesssim \lambda.$$

See also [Ba] for a similar proof that the \mathcal{H}^{m-1} measure of the critical set is $\simeq \lambda$ in the real analytic case.

2.3. Number of intersections of nodal sets with geodesics and number of nodal domains. The Courant nodal domain theorem (see e.g. [Ch, Ch3]) asserts that the n th eigenfunction φ_{λ_n} has $\leq n$ nodal domains. This estimate is not sharp (see [Po] and §11.2 for recent results) and it is possible to find sequences of eigenfunctions with $\lambda_n \rightarrow \infty$ and with a bounded number of nodal domains [L]. In fact, it has been pointed out [Hof] that we do not even know if a given (M, g) possesses any sequence of eigenfunctions φ_n with $\lambda_n \rightarrow \infty$ for which the number of nodal domains tends to infinity. A nodal domain always contains a local minimum or maximum, so a necessary condition that the number of nodal domains increases to infinity along sequence of eigenfunctions is that the number of their critical points (or values) also increases to infinity; see [JN] for a sequence in which the number of critical points is uniformly bounded.

Some new results give lower bounds on the number of nodal domains on surfaces with an orientation reversing isometry with non-empty fixed point set. The first result is due to Ghosh-Reznikov-Sarnak [GRS].

THEOREM 2.1. *Let φ be an even Maass-Hecke L^2 eigenfunction on $\mathbb{X} = SL(2, \mathbb{Z}) \backslash \mathbb{H}$. Denote the nodal domains which intersect a compact geodesic segment $\beta \subset \delta = \{iy \mid y > 0\}$ by $N^\beta(\varphi)$. Assume β is sufficiently long and assume the Lindelof Hypothesis for the Maass-Hecke L -functions. Then*

$$N^\beta(\varphi) \gg_\epsilon \lambda_\varphi^{\frac{1}{24} - \epsilon}.$$

We refer to [GRS] for background definitions. The strategy of the proof is to first prove that there are many intersections of the nodal set with the vertical geodesic of the modular domain and that the eigenfunction changes sign at many intersections. It follows that the nodal lines intersect the geodesic orthogonally. Using a topological argument, the authors show that the nodal lines must often close up to bound nodal domains.

As is seen from this outline, the main analytic ingredient is to prove that there are many intersections of the nodal line with the geodesic. The study of such intersections in a more general context is closely related to a new series of quantum ergodicity results known as QER (quantum ergodic restriction) theorems [TZ, TZ2, CTZ]. We discuss these ingredients below.

In recent work, the author and J. Jung have proved a general kind result in the same direction. The setting is that of a Riemann surface (M, J, σ) with an orienting-reversing involution σ whose fixed point set $\text{Fix}(\sigma)$ is separating, i.e. $M \setminus \text{Fix}(\sigma)$ consists of two connected components. The result is that for any σ -invariant negatively curved metric, and for almost the entire sequence of even or odd eigenfunctions, the number of nodal domains tends to infinity. In fact, the argument only uses ergodicity of the geodesic flow.

. We first explain the hypothesis. When a Riemann surface possesses an orientation-reversing involution $\sigma : M \rightarrow M$, Harnack's theorem says that the fixed point set $\text{Fix}(\sigma)$ is

a disjoint union

$$(17) \quad H = \gamma_1 \cup \dots \cup \gamma_k$$

of $0 \leq k \leq g + 1$ of simple closed curves. It is possible that $\text{Fix}(\sigma) = \emptyset$ but we assume $k \neq 0$. We also assume that H (17) is a separating set, i.e. $M \setminus H = M_+ \cup M_-$ where $M_+^0 \cap M_-^0 = \emptyset$ (the interiors are disjoint), where $\sigma(M_+) = M_-$ and where $\partial M_+ = \partial M_- = H$.

If $\sigma^*g = g$ where g is a negatively curved metric, then $\text{Fix}(\sigma)$ is a finite union of simple closed geodesics. We denote by $L_{\text{even}}^2(M)$ the set of $f \in L^2(M)$ such that $\sigma f = f$ and by $L_{\text{odd}}^2(M)$ the f such that $\sigma f = -f$. We denote by $\{\varphi_j\}$ an orthonormal eigenbasis of Laplace eigenfunctions of $L_{\text{even}}^2(M)$, resp. $\{\psi_j\}$ for $L_{\text{odd}}^2(M)$.

We further denote by

$$\Sigma_{\varphi_\lambda} = \{x \in \mathcal{N}_{\varphi_\lambda} : d\varphi_\lambda(x) = 0\}$$

the singular set of φ_λ . These are special critical points $d\varphi_j(x) = 0$ which lie on the nodal set Z_{φ_j} .

THEOREM 2.1. *Let (M, g) be a compact negatively curved C^∞ surface with an orientation-reversing isometric involution $\sigma : M \rightarrow M$ with $\text{Fix}(\sigma)$ separating. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of $L_{\text{even}}^2(M)$, resp. $\{\psi_j\}$ of $L_{\text{odd}}^2(M)$, one can find a density 1 subset A of \mathbb{N} such that*

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} N(\varphi_j) = \infty,$$

resp.

$$\lim_{\substack{j \rightarrow \infty \\ j \in A}} N(\psi_j) = \infty,$$

For odd eigenfunctions, the conclusion holds as long as $\text{Fix}(\sigma) \neq \emptyset$. A density one subset $A \subset \mathbb{N}$ is one for which

$$\frac{1}{N} \#\{j \in A, j \leq N\} \rightarrow 1, \quad N \rightarrow \infty.$$

As the image indicates, the surfaces in question are complexifications of real algebraic curves, with $\text{Fix}(\sigma)$ the underlying real curve.



The strategy of the proof is similar to that of Theorem 2.1: we prove that for even or odd eigenfunctions, the nodal sets intersect $\text{Fix}(\sigma)$ in many points with sign changes, and then use a topological argument to conclude that there are many nodal domains. We note that the study of sign changes has been used in [NPS] to study nodal sets on surfaces in a different way.

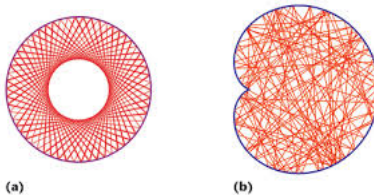
In §2.8 we also discuss upper bounds on nodal intersections in the real analytic case.

2.4. Dynamics of the geodesic or billiard flow. Theorem 2.1 used the hypothesis of ergodicity of the geodesic flow. It is not obvious that nodal sets of eigenfunctions should bear any relation to geodesics, but one of our central themes is that in some ways they do.

In general, there are two broad classes of results on nodal sets and other properties of eigenfunctions:

- Local results which are valid for any local solution of (1), and which often use local arguments. For instance the proof of Proposition 1 is local.
- Global results which use that eigenfunctions are global solutions of (1), or that they satisfy boundary conditions when $\partial M \neq \emptyset$. Thus, they also satisfy the unitary evolution equation (12). For instance the relation between closed geodesics and the remainder term of Weyl's law is global (5)-(33).

Global results often exploit the relation between classical and quantum mechanics, i.e. the relation between the eigenvalue problem (1)-(12) and the geodesic flow. Thus the results often depend on the dynamical properties of the geodesic flow. The relations between eigenfunctions and the Hamiltonian flow are best established in two extreme cases: (i) where the Hamiltonian flow is completely integrable on an energy surface, or (ii) where it is ergodic. The extremes are illustrated below in the case of (i) billiards on rotationally invariant annulus, (ii) chaotic billiards on a cardioid.



A random trajectory in the case of ergodic billiards is uniformly distributed, while all trajectories are quasi-periodic in the integrable case.

We do not have the space to review the dynamics of geodesic flows or other Hamiltonian flows. We refer to [HK] for background in dynamics and to [Ze, Ze3, Zw] for relations between dynamics of geodesic flows and eigenfunctions.

We use the following basic construction: given a measure preserving map (or flow) $\Phi : (X, \mu) \rightarrow (T, \mu)$ one can consider the translation operator

$$(18) \quad U_\Phi f(x) = \Phi^* f(x) = f(\Phi(x)),$$

sometimes called the Koopman operator or Perron-Frobenius operator (cf. [RS, HK]). It is a unitary operator on $L^2(X, \mu)$ and hence its spectrum lies on the unit circle. Φ is ergodic if and only if U_Φ has the eigenvalue 1 with multiplicity 1, corresponding to the constant functions.

The geodesic (or billiard) flow is the Hamiltonian flow on T^*M generated by the metric norm Hamiltonian or its square,

$$(19) \quad H(x, \xi) = |\xi|_g^2 = \sum_{i,j} g^{ij} \xi_i \xi_j.$$

In PDE one most often uses the \sqrt{H} which is homogeneous of degree 1. The geodesic flow is ergodic when the Hamiltonian flow Φ^t is ergodic on the level set $S^*M = \{H = 1\}$.

2.5. Quantum ergodic restriction theorems and nodal intersections. One of the main themes of these lectures is that ergodicity of the geodesic flow causes eigenfunctions to oscillate rapidly everywhere and in all directions, and hence to have a ‘maximal’ zero set. We will see this occur both in the real and complex domain. In the real domain (i.e. on M), ergodicity ensures that restrictions of eigenfunctions in the two-dimensional case to geodesics have many zeros along the geodesic. In §5 we will show that such oscillations and zeros are due to the fact that under generic assumptions, restrictions of eigenfunctions to geodesics are ‘quantum ergodic’ along the geodesic. Roughly this means that they have uniform oscillations at all frequencies below the frequency of the eigenfunction.

We prove that this QER (quantum ergodic restriction) property has the following implications. First, any arc $\beta \subset H$,

$$(20) \quad \left| \int_\beta \varphi_{\lambda_j} ds \right| \leq C \lambda_j^{-1/2} (\log \lambda)^{\frac{1}{2}}$$

and

$$(21) \quad \int_\beta |\varphi_{\lambda_j}| ds \geq \|\varphi_{\lambda_j}\|_\infty^{-1} \|\varphi_{\lambda_j}\|_{L^2(\beta)}^2 \geq C \lambda_j^{-1/2} \log \lambda_j.$$

The first inequality is generic while the second uses the QER property. The inequalities are inconsistent if $\varphi_{\lambda_j} \geq 0$ on β , and that shows that eigenfunctions have many sign changes along the geodesic. The estimate also the well-known sup norm bound

$$(22) \quad \|\varphi_\lambda\|_\infty \leq \lambda^{1/4} / \log \lambda$$

for eigenfunctions on negatively curved surfaces [Be].

The same argument shows that the number of singular points of odd eigenfunctions tends to infinity and one can adapt it to prove that the number of critical points of even eigenfunctions (on the geodesic) tend to infinity.

2.6. Complexification of M and Grauert tubes. The next series of results concerns ‘complex nodal sets’, i.e. complex zeros of analytic continuations of eigenfunctions to the complexification of M . It is difficult to draw conclusions about real nodal sets from knowledge of their complexifications. But complex nodal sets are simpler to study than real nodal sets and the results are stronger, just as complex algebraic varieties behave in simpler ways than real algebraic varieties.

A real analytic manifold M always possesses a unique complexification $M_{\mathbb{C}}$ generalizing the complexification of \mathbb{R}^m as \mathbb{C}^m . The complexification is an open complex manifold in which M embeds $\iota : M \rightarrow M_{\mathbb{C}}$ as a totally real submanifold (Bruhat-Whitney)

The Riemannian metric determines a special kind of distance function on $M_{\mathbb{C}}$ known as a Grauert tube function. In fact, it is observed in [GS1] that the Grauert tube function is obtained from the distance function by setting $\sqrt{\rho}(\zeta) = i\sqrt{r^2(\zeta, \bar{\zeta})}$ where $r^2(x, y)$ is the squared distance function in a neighborhood of the diagonal in $M \times M$.

One defines the Grauert tubes $M_{\tau} = \{\zeta \in M_{\mathbb{C}} : \sqrt{\rho}(\zeta) \leq \tau\}$. There exists a maximal τ_0 for which $\sqrt{\rho}$ is well defined, known as the Grauert tube radius. For $\tau \leq \tau_0$, M_{τ} is a strictly pseudo-convex domain in $M_{\mathbb{C}}$. Since (M, g) is real analytic, the exponential map $\exp_x t\xi$ admits an analytic continuation in t and the imaginary time exponential map

$$(23) \quad E : B_{\epsilon}^*M \rightarrow M_{\mathbb{C}}, \quad E(x, \xi) = \exp_x i\xi$$

is, for small enough ϵ , a diffeomorphism from the ball bundle B_{ϵ}^*M of radius ϵ in T^*M to the Grauert tube M_{ϵ} in $M_{\mathbb{C}}$. We have $E^*\omega = \omega_{T^*M}$ where $\omega = i\partial\bar{\partial}\rho$ and where ω_{T^*M} is the canonical symplectic form; and also $E^*\sqrt{\rho} = |\xi|$ [GS1, LS1]. It follows that E^* conjugates the geodesic flow on B^*M to the Hamiltonian flow $\exp tH_{\sqrt{\rho}}$ of $\sqrt{\rho}$ with respect to ω , i.e.

$$(24) \quad E(g^t(x, \xi)) = \exp t\Xi_{\sqrt{\rho}}(\exp_x i\xi).$$

In general E only extends as a diffeomorphism to a certain maximal radius ϵ_{\max} . We assume throughout that $\epsilon < \epsilon_{\max}$.

2.7. Equidistribution of nodal sets in the complex domain. One may also consider the complex nodal sets

$$(25) \quad \mathcal{N}_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\epsilon} : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

and the complex critical point sets

$$(26) \quad \mathcal{C}_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\epsilon} : \partial\varphi_j^{\mathbb{C}}(\zeta) = 0\}.$$

The following is proved in [Ze5]:

THEOREM 5. *Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Then for all but a sparse subsequence of λ_j ,*

$$\frac{1}{\lambda_j} \int_{\mathcal{N}_{\varphi_{\lambda_j}^{\mathbb{C}}}} f \omega_g^{n-1} \rightarrow \frac{i}{\pi} \int_{M_{\epsilon}} f \bar{\partial}\partial\sqrt{\rho} \wedge \omega_g^{n-1}.$$

The proof is based on quantum ergodicity of analytic continuation of eigenfunctions to Grauert tubes and the growth estimates ergodic eigenfunctions satisfy.

We will say that a sequence $\{\varphi_{j_k}\}$ of L^2 -normalized eigenfunctions is *quantum ergodic* if

$$(27) \quad \langle A\varphi_{j_k}, \varphi_{j_k} \rangle \rightarrow \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu, \quad \forall A \in \Psi^0(M).$$

Here, $\Psi^s(M)$ denotes the space of pseudodifferential operators of order s , and $d\mu$ denotes Liouville measure on the unit cosphere bundle S^*M of (M, g) . More generally, we denote by

$d\mu_r$ the (surface) Liouville measure on ∂B_r^*M , defined by

$$(28) \quad d\mu_r = \frac{\omega^m}{d|\xi|_g} \text{ on } \partial B_r^*M.$$

We also denote by α the canonical action 1-form of T^*M .

2.8. Intersection of nodal sets and real analytic curves on surfaces. In recent work, intersections of nodal sets and curves on surfaces M^2 have been used in a variety of articles to obtain upper and lower bounds on nodal points and domains. The work often is based on restriction theorems for eigenfunctions. Some of the recent articles on restriction theorems and/or nodal intersections are [TZ, TZ2, GRS, JJ, JJ2, Mar, Yo, Po].

First we consider a basic upper bound on the number of intersection points:

THEOREM 6. *Let $\Omega \subset \mathbb{R}^2$ be a piecewise analytic domain and let $n_{\partial\Omega}(\lambda_j)$ be the number of components of the nodal set of the j th Neumann or Dirichlet eigenfunction which intersect $\partial\Omega$. Then there exists C_Ω such that $n_{\partial\Omega}(\lambda_j) \leq C_\Omega \lambda_j$.*

In the Dirichlet case, we delete the boundary when considering components of the nodal set.

The method of proof of Theorem 6 generalizes from $\partial\Omega$ to a rather large class of real analytic curves $C \subset \Omega$, even when $\partial\Omega$ is not real analytic. Let us call a real analytic curve C a *good curve* if there exists a constant $a > 0$ so that for all λ_j sufficiently large,

$$(29) \quad \frac{\|\varphi_{\lambda_j}\|_{L^2(\partial\Omega)}}{\|\varphi_{\lambda_j}\|_{L^2(C)}} \leq e^{a\lambda_j}.$$

Here, the L^2 norms refer to the restrictions of the eigenfunction to C and to $\partial\Omega$. The following result deals with the case where $C \subset \Omega$ is an *interior* real-analytic curve. The real curve C may then be holomorphically continued to a complex curve $C_{\mathbb{C}} \subset \mathbb{C}^2$ obtained by analytically continuing a real analytic parametrization of C .

THEOREM 7. *Suppose that $\Omega \subset \mathbb{R}^2$ is a C^∞ plane domain, and let $C \subset \Omega$ be a good interior real analytic curve in the sense of (29). Let $n(\lambda_j, C) = \#Z_{\varphi_{\lambda_j}} \cap C$ be the number of intersection points of the nodal set of the j -th Neumann (or Dirichlet) eigenfunction with C . Then there exists $A_{C,\Omega} > 0$ depending only on C, Ω such that $n(\lambda_j, C) \leq A_{C,\Omega} \lambda_j$.*

The proof of Theorem 7 is somewhat simpler than that of Theorem 6, i.e. good interior analytic curves are somewhat simpler than the boundary itself. On the other hand, it is clear that the boundary is good and it is hard to prove that other curves are good. A recent paper of J. Jung shows that many natural curves in the hyperbolic plane are ‘good’ [JJ]. See also [ElHajT] for general results on good curves.

The upper bounds of Theorem 6 - 7 are proved by analytically continuing the restricted eigenfunction to the analytic continuation of the curve. We then give a similar upper bound on complex zeros. Since real zeros are also complex zeros, we then get an upper bound on complex zeros. An obvious question is whether the order of magnitude estimate is sharp. Simple examples in the unit disc show that there are no non-trivial lower bounds on numbers of intersection points. But when the dynamics is ergodic we expect to prove an equi-distribution theorem for nodal intersection points (in progress). Ergodicity once again

implies that eigenfunctions oscillate as much as possible and therefore saturate bounds on zeros.

Let $\gamma \subset M^2$ be a generic geodesic arc on a real analytic Riemannian surface. For small ϵ , the parametrization of γ may be analytically continued to a strip,

$$\gamma_{\mathbb{C}} : S_{\tau} := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \rightarrow M_{\tau}.$$

Then the eigenfunction restricted to γ is

$$\gamma_{\mathbb{C}}^* \varphi_j^{\mathbb{C}}(t + i\tau) = \varphi_j(\gamma_{\mathbb{C}}(t + i\tau)) \text{ on } S_{\tau}.$$

Let

$$(30) \quad \mathcal{N}_{\lambda_j}^{\gamma} := \{(t + i\tau) : \gamma_{\mathbb{C}}^* \varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau) = 0\}$$

be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points.

Then as a current of integration,

$$(31) \quad [\mathcal{N}_{\lambda_j}^{\gamma}] = i\partial\bar{\partial}_{t+i\tau} \log \left| \gamma_{\mathbb{C}}^* \varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau) \right|^2.$$

The following result is proved in [Ze6]:

THEOREM 8. *Let (M, g) be real analytic with ergodic geodesic flow. Then for generic γ there exists a subsequence of eigenvalues λ_{j_k} of density one such that*

$$\frac{i}{\pi\lambda_{j_k}} \partial\bar{\partial}_{t+i\tau} \log \left| \gamma_{\mathbb{C}}^* \varphi_{\lambda_{j_k}}^{\mathbb{C}}(t + i\tau) \right|^2 \rightarrow \delta_{\tau=0} ds.$$

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain— and are distributed by arc-length measure on the real geodesic.

The key point is that

$$\frac{1}{\lambda_{j_k}} \log |\varphi_{\lambda_{j_k}}^{\mathbb{C}}(\gamma(t + i\tau))|^2 \rightarrow |\tau|.$$

Thus, the maximal growth occurs along individual (generic) geodesics.

2.9. L^p norms of eigenfunctions. In §2.1 we mentioned that lower bounds on $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_{\lambda}})$ are related to lower bounds on $\|\varphi_{\lambda}\|_{L^1}$ and to upper bounds on $\|\varphi_{\lambda}\|_{L^p}$ for certain p . Such L^p bounds are interesting for all p and depend on the shapes of the eigenfunctions.

In (5) we stated the Weyl law on the number of eigenvalues. There also exists a pointwise local Weyl law which is relevant to the pointwise behavior of eigenfunctions. The pointwise spectral function along the diagonal is defined by

$$(32) \quad E(\lambda, x, x) = N(\lambda, x) := \sum_{\lambda_j \leq \lambda} |\varphi_j(x)|^2.$$

The pointwise Weyl law asserts tht

$$(33) \quad N(\lambda, x) = \frac{1}{(2\pi)^n} |B^n| \lambda^n + R(\lambda, x),$$

where $R(\lambda, x) = O(\lambda^{n-1})$ uniformly in x . These results are proved by studying the cosine transform

$$(34) \quad E(t, x, x) = \sum_{\lambda_j \leq \lambda} \cos t\lambda_j |\varphi_j(x)|^2,$$

which is the fundamental (even) solution of the wave equation restricted to the diagonal. Background on the wave equation is given in §14.

We note that the Weyl asymptote $\frac{1}{(2\pi)^n} |B^n| \lambda^n$ is continuous, while the spectral function (32) is piecewise constant with jumps at the eigenvalues λ_j . Hence the remainder must jump at an eigenvalue λ , i.e.

$$(35) \quad R(\lambda, x) - R(\lambda - 0, x) = \sum_{j: \lambda_j = \lambda} |\varphi_j(x)|^2 = O(\lambda^{n-1}).$$

on any compact Riemannian manifold. It follows immediately that

$$(36) \quad \sup_M |\varphi_j| \lesssim \lambda_j^{\frac{n-1}{2}}.$$

There exist (M, g) for which this estimate is sharp, such as the standard spheres. However, as (22) the sup norms are smaller on manifolds of negative curvature. In fact, (36) is very rarely sharp and the actual size of the sup-norms and other L^p norms of eigenfunctions is another interesting problem in global harmonic analysis. In [SoZ] it is proved that if the bound (36) is achieved by some sequence of eigenfunctions, then there must exist a “partial blow-down point” or self-focal point p where a positive measure of directions $\omega \in S_p^*M$ so that the geodesic with initial value (p, ω) returns to p at some time $T(p, \omega)$. Recently the authors have improved the result in the real analytic case, and we sketch the new result in §12.

To state it, we need some further notation and terminology. We only consider real analytic metrics for the sake of simplicity. We call a point p a *self-focal point* or *ablow-down point* if there exists a time $T(p)$ so that $\exp_p T(p)\omega = p$ for all $\omega \in S_p^*M$. Such a point is self-conjugate in a very strong sense. In terms of symplectic geometry, the flowout manifold

$$(37) \quad \Lambda_p = \bigcup_{0 \leq t \leq T(p)} G^t S_p^*M$$

is an embedded Lagrangian submanifold of S^*M whose projection

$$\pi : \Lambda_p \rightarrow M$$

has a “blow-down singularity” at $t = 0, t = T(p)$ (see[STZ]). Focal points come in two basic kinds, depending on the first return map

$$(38) \quad \Phi_p : S_p^*M \rightarrow S_p^*M, \quad \Phi_p(\xi) := \gamma'_{p,\xi}(T(p)),$$

where $\gamma_{p,\xi}$ is the geodesic defined by the initial data $(p, \xi) \in S_x^*M$. We say that p is a *pole* if

$$\Phi_p = Id : S_p^*M \rightarrow S_p^*M.$$

On the other hand, it is possible that $\Phi_p = Id$ only on a codimension one set in S_p^*M . We call such a Φ_p *twisted*.

Examples of poles are the poles of a surface of revolution (in which case all geodesic loops at x_0 are smoothly closed). Examples of self-focal points with fully twisted return map are

the four umbilic points of two-dimensional tri-axial ellipsoids, from which all geodesics loop back at time 2π but are almost never smoothly closed. The only smoothly closed directions are the geodesic (and its time reversal) defined by the middle length ‘equator’.

At a self-focal point we have a kind of analogue of (18) but not on S^*M but just on S_p^*M . We define the Perron-Frobenius operator at a self-focal point by

$$(39) \quad U_x : L^2(S_x^*M, d\mu_x) \rightarrow L^2(S_x^*M, d\mu_x), \quad U_x f(\xi) := f(\Phi_x(\xi)) \sqrt{|J_x(\xi)|}.$$

Here, J_x is the Jacobian of the map Φ_x , i.e. $\Phi_x^*|d\xi| = J_x(\xi)|d\xi|$.

The new result of C.D. Sogge and the author is the following:

THEOREM 9. *If (M, g) is real analytic and has maximal eigenfunction growth, then it possesses a self-focal point whose first return map Φ_p has an invariant L^2 function in $L^2(S_p^*M)$. Equivalently, it has an L^1 invariant measure in the class of the Euclidean volume density μ_p on S_p^*M .*

For instance, the twisted first return map at an umbilic point of an ellipsoid has no such finite invariant measure. Rather it has two fixed points, one of which is a source and one a sink, and the only finite invariant measures are delta-functions at the fixed points. It also has an infinite invariant measure on the complement of the fixed points, similar to $\frac{dx}{x}$ on \mathbb{R}_+ .

The results of [SoZ, STZ, SoZ2] are stated for the L^∞ norm but the same results are true for L^p norms above a critical index p_m depending on the dimension (§12). The analogous problem for lower L^p norms is of equal interest, but the geometry of the extremals changes from analogues of zonal harmonics to analogues of Gaussian beams or highest weight harmonics. For the lower L^p norms there are also several new developments which are discussed in §12.

2.10. Quasi-modes. A significant generalization of eigenfunctions are *quasi-modes*, which are special kinds of approximate solutions of the eigenvalue problem. The two basic types are:

- (i) Lagrangian distributions given by oscillatory integrals which are approximate solutions of the eigenvalue problem, i.e. which satisfy $\|(\Delta + \mu_k^2)u_k\| = O(\mu_k^{-p})$ for some $p \geq 0$.
- (ii) Any sequence $\{u_k\}_{k=1}^\infty$ of L^2 normalized solutions of an approximate eigenvalue problem. In [STZ] we worked with a more general class of “admissible” quasi-modes. A sequence $\{\psi_\lambda\}$, $\lambda = \lambda_j$, $j = 1, 2, \dots$ is a sequence of admissible quasimodes if $\|\psi_\lambda\|_2 = 1$ and

$$(40) \quad \|(\Delta + \lambda^2)\psi_\lambda\|_2 + \|S_{2\lambda}^\perp \psi_\lambda\|_\infty = o(\lambda).$$

Here, S_μ^\perp denotes the projection onto the $[\mu, \infty)$ part of the spectrum of $\sqrt{-\Delta}$, and in what follows $S_\mu = I - S_\mu^\perp$, i.e., $S_\mu f = \sum_{\lambda_j < \mu} e_j(f)$, where $e_j(f)$ is the projection of f onto the eigenspace with eigenvalue λ_j .

The more special type (i) are studied in [Arn, BB, CV2, K, R1, R2] and are analogous to Hadamard parametrices for the wave kernel. However, none may exist on a given (M, g) . For instance, Gaussian beams [BB, R1, R2] are special quasi-modes which concentrate along a closed geodesic. On the standard S^n they exist along any closed geodesic and are eigenfunctions. But one needs a stable elliptic closed geodesic to construct a Gaussian beam and

such closed geodesics do not exist when g has negative curvature. They do however exist in many cases, but the associated Gaussian beams are rarely exact eigenfunctions, and are typically just approximate eigenfunctions. More general quasi-modes are constructed from geodesic flow-invariant closed immersed Lagrangian submanifolds $\Lambda \subset S_g^*M$. But such invariant Lagrangian submanifolds are also rare. They do arise in many interesting cases and in particular the unit co-sphere bundle S_g^*M is foliated by flow-invariant Lagrangian tori when the geodesic flow is integrable. This is essentially the definition of complete integrability.

For $p = 1$, the more general type (ii) always exist: it suffices to define u_k as superpositions of eigenfunctions with frequencies λ_j drawn from sufficiently narrow windows $[\lambda, \lambda + O(\lambda^{-p})]$. A model case admissible quasi-modes would be a sequence of L^2 -normalized functions $\{\psi_{\lambda_j}\}$ whose $\sqrt{-\Delta}$ spectrum lies in intervals of the form $[\lambda_j - o(1), \lambda_j + o(1)]$ as $\lambda_j \rightarrow \infty$. If $p = 1$ then for generic metrics g , it follows from the Weyl law with remainder that there exist $\simeq C_g \lambda^{m-1}$ eigenvalues in the window and one can construct many quasi-modes. For higher values of p existence of refined quasi-modes depends on the number of eigenvalues in very short intervals, and in general one only knows they exist by constructions of type (i).

An important example of such a quasi-mode is a sequence of “shrinking spectral projections”, i.e. the L^2 -normalized projection kernels

$$\Phi_j^z(x) = \frac{\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(x, z)}{\sqrt{\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(z, z)}}$$

with second point frozen at a point $z \in M$ and with width $\epsilon_j \rightarrow 0$. Here, $\chi_{[\lambda_j, \lambda_j + \epsilon_j]}(x, z)$ is the orthogonal projection onto the sum of the eigenspaces V_λ with $\lambda \in [\lambda_j, \lambda_j + \epsilon_j]$. The zonal eigenfunctions of a surface of revolution are examples of such shrinking spectral projections for a sufficiently small ϵ_j , and when z is a partial focus such $\Phi_j^z(x)$ are generalizations of zonal eigenfunctions. On a general Zoll manifold, shrinking spectral projections of widths $\epsilon_j = O(\lambda_j^{-1})$ are the direct analogues of zonal spherical harmonics, and they would satisfy the analog of (40) where $o(\lambda)$ is replaced by the much stronger $O(\lambda^{-1})$.

There are several motivations to consider quasi-modes as well as actual modes (eigenfunctions). First, many results about eigenfunctions automatically hold for quasi-modes as well. Indeed, it is difficult to distinguish modes and quasi-modes when using a wave kernel construction, and many of the methods apply equally to modes and quasi-modes. Second, quasi-modes are often geometrically beautiful. They are not stationary states but retain their shape under propagation by the wave group $U(t) = \exp it\sqrt{\Delta}$ for a “very long time” (essentially the Ehrenfest time) before breaking up (see [Ze2] for background). They are natural extremals for L^p norms when they exist. Third, they are often the objects needed to close the gap between necessary and sufficient conditions on eigenfunction growth. It is usually interesting to know whether a theorem about eigenfunctions extends to quasi-modes as well.

2.11. Format of these lectures and references to the literature. In keeping with the format of the Park City summer school, we concentrate on the topics of the five lectures rather than give a systematic exposition of the subject. The more detailed account will appear in [Ze0]. Various details of the proof are given as Exercises for the reader. The “details” are intended to be stimulating and fundamental, rather than the tedious and routine aspects of proofs often left to readers in textbooks. As a result, the exercises vary widely in difficulty

and amount of background assumed. Problems labelled **Problems** are not exercises; they are problems whose solutions are not currently known.

The technical backbone of the semi-classical analysis of eigenfunctions consists of wave equation methods combined with the machinery of Fourier integral operators and Pseudo-differential operators. We do not have time to review this theory. The main results we need are the construction of parametrices for the ‘propagator’ $E(t) = \cos t\sqrt{\Delta}$ and the Poisson kernel $\exp -\tau\sqrt{\Delta}$. We also need Fourier analysis to construct approximate spectral projections $\rho(\sqrt{\Delta} - \lambda)$ and to prove Tauberian theorems relating smooth expansions and cutoffs.

The books [GSj, DSj, D2, GS2, GSt2, Sogb, Sogb2, Zw] give textbook treatments of the semi-classical methods with applications to spectral asymptotics. Somewhat more classical background on the wave equation with many explicit formulae in model cases can be found in [TI, TII]. General spectral theory and the relevant functional analysis can also be found in [RS]. The series [HoI, HoII, HoIII, HoIV] gives a systematic presentation of Fourier integral operator theory: stationary phase and Tauberian theorems can be found in [HoI], Weyl’s law and spectral asymptotics can be found in [HoIII, HoIV].

In [Ze0] the author gives a more systematic presentation of results on nodal sets, L^p norms and other aspects of eigenfunctions. Earlier surveys [Ze, Ze2, Ze3] survey related material. Other monographs on Δ -eigenfunctions can be found in [HL] and [Sogb2]. The methods of [HL] mainly involve the local harmonic analysis of eigenfunctions and rely more on classical elliptic estimates, on frequency functions and of one-variable complex analysis. The exposition in [Sogb2] is close to the one given here but does not extend to the recent results that we highlight in these lectures and in [Ze0].

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3. FOUNDATIONAL RESULTS ON NODAL SETS

As mentioned above, the nodal domains of an eigenfunction are the connected components of $M \setminus \mathcal{N}_{\varphi_\lambda}$. In the case of a domain with boundary and Dirichlet boundary conditions, the nodal set is defined by taking the closure of the zero set in $M \setminus \partial M$.

The eigenfunction is either positive or negative in each nodal domain and changes sign as the nodal set is crossed from one domain to an adjacent domain. Thus the set of nodal domains can be given the structure of a bi-partite graph [H]. Since the eigenfunction has one sign in each nodal domain, it is the ground state eigenfunction with Dirichlet boundary conditions in each nodal domain.

In the case of domains $\Omega \subset \mathbb{R}^n$ (with the Euclidean metric), the Faber-Krahn inequality states that the lowest eigenvalue (ground state eigenvalue, bass note) $\lambda_1(\Omega)$ for the Dirichlet problem has the lower bound,

$$(41) \quad \lambda_1(\Omega) \geq |\Omega|^{-\frac{2}{n}} C_n^{\frac{2}{n}} j_{\frac{n-2}{2}},$$

where $|\Omega|$ is the Euclidean volume of Ω , $C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in \mathbb{R}^n and where $j_{m,1}$ is the first positive zero of the Bessel function J_m . That is, among all domains of a fixed volume the unit ball has the lowest bass note.

3.1. Vanishing order and scaling near zeros. By the vanishing order $\nu(u, a)$ of u at a is meant the largest positive integer such that $D^\alpha u(a) = 0$ for all $|\alpha| \leq \nu$. A unique continuation theorem shows that the vanishing order of an eigenfunction at each zero is finite. The following estimate is a quantitative version of this fact.

THEOREM 3.1. (see [DF]; [Lin] *Proposition 1.2 and Corollary 1.4*; and [H] *Theorem 2.1.8*.) *Suppose that M is compact and of dimension n . Then there exist constants $C(n), C_2(n)$ depending only on the dimension such that the the vanishing order $\nu(u, a)$ of u at $a \in M$ satisfies $\nu(u, a) \leq C(n) N(0, 1) + C_2(n)$ for all $a \in B_{1/4}(0)$. In the case of a global eigenfunction, $\nu(\varphi_\lambda, a) \leq C(M, g)\lambda$.*

Highest weight spherical harmonics $C_n(x_1 + ix_2)^N$ on S^2 are examples which vanish at the maximal order of vanishing at the poles $x_1 = x_2 = 0, x_3 = \pm 1$.

The following Bers scaling rule extracts the leading term in the Taylor expansion of the eigenfunction around a zero:

PROPOSITION 3.2. [Bers, HW2] *Assume that φ_λ vanishes to order k at x_0 . Let $\varphi_\lambda(x) = \varphi_k^{x_0}(x) + \varphi_{k+1}^{x_0} + \dots$ denote the C^∞ Taylor expansion of φ_λ into homogeneous terms in normal coordinates x centered at x_0 . Then $\varphi_k^{x_0}(x)$ is a Euclidean harmonic homogeneous polynomial of degree k .*

To prove this, one substitutes the homogeneous expansion into the equation $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$ and rescales $x \rightarrow \lambda x$, i.e. one applies the dilation operator

$$(42) \quad D_\lambda^{x_0} \varphi_\lambda(u) = \varphi(x_0 + \frac{u}{\lambda}).$$

The rescaled eigenfunction is an eigenfunction of the locally rescaled Laplacian

$$(43) \quad \Delta_\lambda^{x_0} := \lambda^{-2} D_\lambda^{x_0} \Delta_g (D_\lambda^{x_0})^{-1} = \sum_{j=1}^n \frac{\partial^2}{\partial u_j^2} + \dots$$

in Riemannian normal coordinates u at x_0 but now with eigenvalue 1,

$$(44) \quad \begin{aligned} D_\lambda^{x_0} \Delta_g (D_\lambda^{x_0})^{-1} \varphi(x_0 + \frac{u}{\lambda}) &= \lambda^2 \varphi(x_0 + \frac{u}{\lambda}) \\ \implies \Delta_\lambda^{x_0} \varphi(x_0 + \frac{u}{\lambda}) &= \varphi(x_0 + \frac{u}{\lambda}). \end{aligned}$$

Since $\varphi(x_0 + \frac{u}{\lambda})$ is, modulo lower order terms, an eigenfunction of a standard flat Laplacian on \mathbb{R}^n , it behaves near a zero as a sum of homogeneous Euclidean harmonic polynomials.

In dimension 2, a homogeneous harmonic polynomial of degree N is the real or imaginary part of the unique holomorphic homogeneous polynomial z^N of this degree, i.e. $p_N(r, \theta) = r^N \sin N\theta$. As observed in [Ch], there exists a C^1 local diffeomorphism χ in a disc around a zero x_0 so that $\chi(x_0) = 0$ and so that $\varphi_N^{x_0} \circ \chi = p_N$. It follows that the restriction of φ_λ to a curve H is C^1 equivalent around a zero to p_N restricted to $\chi(H)$. The nodal set of p_N around 0 consists of N rays, $\{r(\cos \theta, \sin \theta) : r > 0, p_N|_{S^1}(v) = 0\}$. It follows that the local structure of the nodal set in a small disc around a singular point p is C^1 equivalent to N

equi-angular rays emanating from p . We refer to [HW, HW2, Ch, Ch1, Ch2, Bes] for further background and results.

Question Is there any useful scaling behavior of φ_λ around its critical points?

3.2. Proof of Proposition 1. The proofs are based on rescaling the eigenvalue problem in small balls.

Proof. Fix x_0, r and consider $B(x_0, r)$. If φ_λ has no zeros in $B(x_0, r)$, then $B(x_0, r) \subset D_{j;\lambda}$ must be contained in the interior of a nodal domain $D_{j;\lambda}$ of φ_λ . Now $\lambda^2 = \lambda_1^2(D_{j;\lambda})$ where $\lambda_1^2(D_{j;\lambda})$ is the smallest Dirichlet eigenvalue for the nodal domain. By domain monotonicity of the lowest Dirichlet eigenvalue (i.e. $\lambda_1(\Omega)$ decreases as Ω increases), $\lambda^2 \leq \lambda_1^2(D_{j;\lambda}) \leq \lambda_1^2(B(x_0, r))$. To complete the proof we show that $\lambda_1^2(B(x_0, r)) \leq \frac{C}{r^2}$ where C depends only on the metric. This is proved by comparing $\lambda_1^2(B(x_0, r))$ for the metric g with the lowest Dirichlet Eigenvalue $\lambda_1^2(B(x_0, cr); g_0)$ for the Euclidean ball $B(x_0, cr; g_0)$ centered at x_0 of radius cr with Euclidean metric g_0 equal to g with coefficients frozen at x_0 ; c is chosen so that $B(x_0, cr; g_0) \subset B(x_0, r, g)$. Again by domain monotonicity, $\lambda_1^2(B(x_0, r, g)) \leq \lambda_1^2(B(x_0, cr; g_0))$ for $c < 1$. By comparing Rayleigh quotients $\frac{\int_\Omega |df|^2 dV_g}{\int_\Omega f^2 dV_g}$ one easily sees that $\lambda_1^2(B(x_0, cr; g)) \leq C\lambda_1^2(B(x_0, cr; g_0))$ for some C depending only on the metric. But by explicit calculation with Bessel functions, $\lambda_1^2(B(x_0, cr; g_0)) \leq \frac{C}{r^2}$. Thus, $\lambda^2 \leq \frac{C}{r^2}$. □

For background we refer to [Ch].

3.3. A second proof. Another proof is given in [HL]: Let u_r denote the ground state Dirichlet eigenfunction for $B(x_0, r)$. Then $u_r > 0$ on the interior of $B(x_0, r)$. If $B(x_0, r) \subset D_{j;\lambda}$ then also $\varphi_\lambda > 0$ in $B(x_0, r)$. Hence the ratio $\frac{u_r}{\varphi_\lambda}$ is smooth and non-negative, vanishes only on $\partial B(x_0, r)$, and must have its maximum at a point y in the interior of $B(x_0, r)$. At this point (recalling that our Δ is minus the sum of squares),

$$\nabla \left(\frac{u_r}{\varphi_\lambda} \right) (y) = 0, \quad -\Delta \left(\frac{u_r}{\varphi_\lambda} \right) (y) \leq 0,$$

so at y ,

$$0 \geq -\Delta \left(\frac{u_r}{\varphi_\lambda} \right) = -\frac{\varphi_\lambda \Delta u_r - u_r \Delta \varphi_\lambda}{\varphi_\lambda^2} = -\frac{(\lambda_1^2(B(x_0, r)) - \lambda^2) \varphi_\lambda u_r}{\varphi_\lambda^2}.$$

Since $\frac{\varphi_\lambda u_r}{\varphi_\lambda^2} > 0$, this is possible only if $\lambda_1(B(x_0, r)) \geq \lambda$.

To complete the proof we note that if $r = \frac{A}{\lambda}$ then the metric is essentially Euclidean. We rescale the ball by $x \rightarrow \lambda x$ (with coordinates centered at x_0) and then obtain an essentially Euclidean ball of radius r . Then $\lambda_1(B(x_0, \frac{r}{\lambda})) = \lambda \lambda_1 B_{g_0}(x_0, r)$. Therefore we only need to choose r so that $\lambda_1 B_{g_0}(x_0, r) = 1$.

Problem Are the above results true as well for quasi-modes of order zero (§2.10, §15.3)?

3.4. Rectifiability of the nodal set. We recall that the nodal set of an eigenfunction φ_λ is its zero set. When zero is a regular value of φ_λ the nodal set is a smooth hypersurface. This is a generic property of eigenfunctions [U]. It is pointed out in [Bae] that eigenfunctions can always be locally represented in the form

$$\varphi_\lambda(x) = v(x) \left(x_1^k + \sum_{j=0}^{k-1} x_1^j u_j(x') \right),$$

in suitable coordinates (x_1, x') near p , where φ_λ vanishes to order k at p , where $u_j(x')$ vanishes to order $k - j$ at $x' = 0$, and where $v(x) \neq 0$ in a ball around p . It follows that the nodal set is always countably $n - 1$ rectifiable when $\dim M = n$.

4. LOWER BOUNDS FOR $\mathcal{H}^{m-1}(\mathcal{N}_\lambda)$ FOR C^∞ METRICS

In this section we review the lower bounds on $\mathcal{H}^{n-1}(Z_{\varphi_\lambda})$ from [CM, SoZ, SoZa, HS, HW]. Here

$$\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) = \int_{Z_{\varphi_\lambda}} dS$$

is the Riemannian surface measure, where dS denotes the Riemannian volume element on the nodal set, i.e. the insert $iota_n dV_g$ of the unit normal into the volume form of (M, g) . The main result is:

THEOREM 4.1. *Let (M, g) be a C^∞ Riemannian manifold. Then there exists a constant C independent of λ such that*

$$C\lambda^{1-\frac{n-1}{2}} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}).$$

Remark: In a recent article [BlSo], M. Blair and C. Sogge improve this result on manifolds of non-positive curvature by showing that the right side divided by the left side tends to infinity. There exists a related proof using a comparison of diffusion processes in [Ste]. The result generalizes in a not completely straightforward way to Schrödinger operators $-\frac{\hbar^2}{2}\Delta_g + V$ for certain potentials V [ZZh] (see also [Jin] for generalizations of [DF] to Schrödinger operators). The new issue is the separation of the domain into classically allowed and forbidden regions. In [HZZ] the density of zeros in both regions is studied for random Hermite functions.

We sketch the proof of Theorem 4.1 from [SoZ, SoZa]. The starting point is an identity from [SoZ] (inspired by an identity in [Dong]):

PROPOSITION 4.2. *For any $f \in C^2(M)$,*

$$(45) \quad \int_M |\varphi_\lambda| (\Delta_g + \lambda^2) f dV_g = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla_g \varphi_\lambda| f dS,$$

When $f \equiv 1$ we obtain

COROLLARY 4.3.

$$(46) \quad \lambda^2 \int_M |\varphi_\lambda| dV_g = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla_g \varphi_\lambda| f dS,$$

EXERCISE 2. *Prove this identity by decomposing M into a union of nodal domains.*

Hint: The nodal domains form a partition of M , i.e.

$$M = \bigcup_{j=1}^{N_+(\lambda)} D_j^+ \cup \bigcup_{k=1}^{N_-(\lambda)} D_k^- \cup \mathcal{N}_\lambda,$$

where the D_j^+ and D_k^- are the positive and negative nodal domains of φ_λ , i.e., the connected components of the sets $\{\varphi_\lambda > 0\}$ and $\{\varphi_\lambda < 0\}$.

Let us assume for the moment that 0 is a regular value for φ_λ , i.e., $\Sigma = \emptyset$. Then each D_j^+ has smooth boundary ∂D_j^+ , and so if ∂_ν is the Riemann outward normal derivative on this set, by the Gauss-Green formula we have

$$\begin{aligned} \int_{D_j^+} ((\Delta + \lambda^2)f) |\varphi_\lambda| dV &= \int_{D_j^+} ((\Delta + \lambda^2)f) \varphi_\lambda dV \\ (47) \qquad \qquad \qquad &= \int_{D_j^+} f (\Delta + \lambda^2)\varphi_\lambda dV - \int_{\partial D_j^+} f \partial_\nu \varphi_\lambda dS \\ &= \int_{\partial D_j^+} f |\nabla \varphi_\lambda| dS. \end{aligned}$$

We use that $-\partial_\nu \varphi_\lambda = |\nabla \varphi_\lambda|$ since $\varphi_\lambda = 0$ on ∂D_j^+ and φ_λ decreases as it crosses ∂D_j^+ from D_j^+ . A similar argument shows that

$$\begin{aligned} \int_{D_k^-} ((\Delta + \lambda^2)f) |\varphi_\lambda| dV &= - \int_{D_k^-} ((\Delta + \lambda^2)f) \varphi_\lambda dV \\ (48) \qquad \qquad \qquad &= \int_{\partial D_k^-} f \partial_\nu \varphi_\lambda dS = \int_{\partial D_k^-} f |\nabla \varphi_\lambda| dS, \end{aligned}$$

using in the last step that φ_λ increases as it crosses ∂D_k^- from D_k^- .

If we sum these two identities over j and k , we get

$$\begin{aligned} \int_M ((\Delta + \lambda^2)f) |\varphi_\lambda| dV &= \sum_j \int_{D_j^+} ((\Delta + \lambda^2)f) |\varphi_\lambda| dV \\ &\quad + \sum_k \int_{D_k^-} ((\Delta + \lambda^2)f) |\varphi_\lambda| dV \\ &= \sum_j \int_{\partial D_j^+} f |\nabla \varphi_\lambda| dS \\ &\quad + \sum_k \int_{\partial D_k^-} f |\nabla \varphi_\lambda| dS = 2 \int_{\mathcal{N}_\lambda} f |\nabla \varphi_\lambda| dS, \end{aligned}$$

using the fact that \mathcal{N}_λ is the disjoint union of the ∂D_j^+ and the disjoint union of the ∂D_k^- .

The lower bound of Theorem 4.1 follows from the identity in Corollary 4.3 and the following lemma:

LEMMA 4.4. *If $\lambda > 0$ then*

$$(49) \qquad \qquad \qquad \|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}$$

Here, $A(\lambda) \lesssim B(\lambda)$ means that there exists a constant independent of λ so that $A(\lambda) \leq CB(\lambda)$.

By Lemma 4.4 and Corollary 4.3, we have

$$(50) \quad \begin{aligned} \lambda^2 \int_M |\varphi_\lambda| dV &= 2 \int_{\mathcal{N}_\lambda} |\nabla_g \varphi_\lambda|_g dS \leq 2 |\mathcal{N}_\lambda| \|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \\ &\lesssim 2 |\mathcal{N}_\lambda| \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}. \end{aligned}$$

Thus Theorem 4.1 follows from the somewhat curious cancellation of $\|\varphi_\lambda\|_{L^1}$ from the two sides of the inequality.

Problem Show that Corollary 4.3 and Lemma 4.4 are true modulo $O(1)$ for quasi-modes of order zero (§2.10, §15.3).

4.1. Proof of Lemma 4.4.

Proof. The main point is to construct a designer reproducing kernel K_λ for φ_λ : Let $\hat{\rho} \in C_0^\infty(\mathbb{R})$ satisfy $\rho(0) = \int \hat{\rho} dt = 1$. Define the operator

$$(51) \quad \rho(\lambda - \sqrt{\Delta}) : L^2(M) \rightarrow L^2(M)$$

by

$$(52) \quad \rho(\lambda - \sqrt{\Delta})f = \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} e^{-it\sqrt{-\Delta}} f dt.$$

Then (51) is a function of Δ and has φ_λ as an eigenfunction with eigenvalue $\rho(\lambda - \lambda) = \rho(0) = 1$. Hence,

$$\rho(\lambda - \sqrt{\Delta})\varphi_\lambda = \varphi_\lambda.$$

EXERCISE 3. Check that (52) has the spectral expansion,

$$(53) \quad \rho(\lambda - \sqrt{\Delta})f = \sum_{j=0}^{\infty} \rho(\lambda - \lambda_j) E_j f,$$

where $E_j f$ is the projection of f onto the λ_j -eigenspace of $\sqrt{-\Delta_g}$. Conclude that (52) reproduces φ_λ if $\rho(0) = 1$.

We may choose ρ further so that $\hat{\rho}(t) = 0$ for $t \notin [\epsilon/2, \epsilon]$.

CLAIM If $\text{supp } \hat{\rho} \subset [\epsilon/2, \epsilon]$, then the kernel $K_\lambda(x, y)$ of $\rho(\lambda - \sqrt{\Delta})$ for ϵ sufficiently small satisfies

$$(54) \quad |\nabla_g K_\lambda(x, y)| \leq C \lambda^{1+\frac{n-1}{2}}.$$

The Claim proves the Lemma, because

$$\begin{aligned}
\nabla_x \varphi_\lambda(x) &= \nabla_x \rho(\lambda - \sqrt{\Delta}) \varphi_\lambda(x) \\
&= \int_M \nabla_x K_\lambda(x, y) \varphi_\lambda(y) dV(y) \\
&\leq C \sup_{x, y} |\nabla_x K_\lambda(x, y)| \int_M |\varphi_\lambda| dV \\
&\leq \lambda^{1 + \frac{n-1}{2}} \|\varphi_\lambda\|_{L^1}
\end{aligned}$$

which implies the lemma.

The gradient estimate on $K_\lambda(x, y)$ is based on the following “parametrix” for the designer reproducing kernel:

PROPOSITION 4.5.

$$(55) \quad K_\lambda(x, y) = \lambda^{\frac{n-1}{2}} a_\lambda(x, y) e^{i\lambda r(x, y)},$$

where $a_\lambda(x, y)$ is bounded with bounded derivatives in (x, y) and where $r(x, y)$ is the Riemannian distance between points.

Proof. Let $U(t) = e^{-it\sqrt{\Delta}}$. We may write

$$(56) \quad \rho(\lambda - \sqrt{\Delta}) = \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} U(t, x, y) dt.$$

As reviewed in §14.2, for small t and x, y near the diagonal one may construct the Hadamard parametrix,

$$U(t, x, y) = \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} A t, (x, y, \theta) d\theta$$

modulo a smooth remainder (which may be neglected).

EXERCISE 4. *Explain why the remainder may be neglected. How many of the terms in the parametrix construction does one need in the proof of Proposition 4.5? (Hint: if one truncates the amplitude after a finite number of terms in the Hadamard parametrix, the remainder lies in C^k and then the contribution to (56) decays as $\lambda \rightarrow \infty$.)*

Thus,

$$K_\lambda(x, y) = \int_{\mathbb{R}} \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} e^{it\lambda} \hat{\rho}(t) A t, (x, y, \theta) d\theta dt.$$

We change variables $\theta \rightarrow \lambda\theta$ to obtain

$$K_\lambda(x, y) = \lambda \int_{\mathbb{R}} \int_0^\infty e^{i\lambda[\theta(r^2(x, y) - t^2) + t]} \hat{\rho}(t) A t, (x, y, \lambda\theta) d\theta dt.$$

We then apply stationary phase. The phase is

$$\theta(r^2(x, y) - t^2) + t$$

and the critical point equations are

$$r^2(x, y) = t^2, \quad 2t\theta = 1, \quad (t \in (\epsilon, 2\epsilon)).$$

The power of θ in the amplitude is $\theta^{\frac{n-1}{2}}$. The change of variables thus puts in $\lambda^{\frac{n+1}{2}}$. But we get λ^{-1} from stationary phase with two variables (t, θ) .

The value of the phase at the critical point is $e^{it\lambda} = e^{i\lambda r(x,y)}$. The Hessian in (t, θ) is $2t$ and it is invertible. Hence,

$$K_\lambda(x, y) \simeq \lambda^{\frac{n-1}{2}} e^{i\lambda r(x,y)} a(\lambda, x, y),$$

where

$$a \sim a_0 + \lambda^{-1} a_{-1} + \dots$$

and

$$a_0 = A_0(r(x, y), (x, y, \frac{2}{r(x, y)})).$$

□

Proposition 4.5 implies that $|\nabla_g K_\lambda(x, y)| \leq C \lambda^{1+\frac{n-1}{2}}$ by directly differentiating the expression. The extra power of λ comes from the “phase factor” $e^{i\lambda r(x,y)}$.

This concludes the proof of Lemma 4.4. □

Remark: There are many ‘reproducing kernels’ if one only requires them to reproduce one eigenfunction. A very common choice is the spectral projections operator

$$\Pi_{[\lambda, \lambda+1]}(x, y) = \sum_{j: \lambda_j \in [\lambda, \lambda+1]} \varphi_j(x) \varphi_j(y)$$

for the interval $[\lambda, \lambda + 1]$. It reproduces *all* eigenfunction φ_k with $\lambda_k \in [\lambda, \lambda + 1]$. This reproducing kernel cannot be used in our application because $\Pi_\lambda(x, x) \simeq \lambda^{n-1}$, as follows from the local Weyl law. Similarly, $\sup_{x,y} |\nabla_x \Pi_\lambda(x, y)| \simeq \lambda^n$. The reader may check these statements on the spectral projections kernel for the standard sphere (§16).

4.2. Modifications. Hezari-Sogge modified the proof Proposition 4.2 in [HS] to prove

THEOREM 4.6. *For any C^∞ compact Riemannian manifold, the L^2 -normalized eigenfunctions satisfy*

$$\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \geq C \lambda \|\varphi_\lambda\|_{L^1}^2.$$

They first apply the Schwarz inequality to get

$$(57) \quad \lambda^2 \int_M |\varphi_\lambda| dV_g \leq 2(\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}))^{1/2} \left(\int_{Z_{\varphi_\lambda}} |\nabla_g \varphi_\lambda|^2 dS \right)^{1/2}.$$

They then use the test function

$$(58) \quad f = (1 + \lambda^2 \varphi_\lambda^2 + |\nabla_g \varphi_\lambda|_g^2)^{\frac{1}{2}}$$

in Proposition 4.2 to show that

$$(59) \quad \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla_g \varphi_\lambda|^2 dS \leq \lambda^3.$$

See also [Ar] for the generalization to the nodal bounds to Dirichlet and Neumann eigenfunctions of bounded domains.

Theorem 4.6 shows that Yau's conjectured lower bound would follow for a sequence of eigenfunctions satisfying $\|\varphi_\lambda\|_{L^1} \geq C > 0$ for some positive constant C .

4.3. Lower bounds on L^1 norms of eigenfunctions. The following universal lower bound is optimal as (M, g) ranges over all compact Riemannian manifolds.

PROPOSITION 10. *For any (M, g) and any L^2 -normalized eigenfunction, $\|\varphi_\lambda\|_{L^1} \geq C_g \lambda^{-\frac{n-1}{4}}$.*

Remark: There are few results on L^1 norms of eigenfunctions. The reason is probably that $|\varphi_\lambda|^2 dV$ is the natural probability measure associated to eigenfunctions. It is straightforward to show that the expected L^1 norm of random L^2 -normalized spherical harmonics of degree N and their generalizations to any (M, g) is a positive constant C_N with a uniform positive lower bound. One expects eigenfunctions in the ergodic case to have the same behavior.

PROBLEM 1. *A difficult but interesting problem would be to show that $\|\varphi_\lambda\|_{L^1} \geq C_0 > 0$ on a compact hyperbolic manifold. A partial result in this direction would be useful.*

4.4. Dong's upper bound. Let (M, g) be a compact C^∞ Riemannian manifold of dimension n , let φ_λ be an L^2 -normalized eigenfunction of the Laplacian,

$$\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda,$$

Let

$$(60) \quad q = |\nabla \varphi|^2 + \lambda^2 \varphi^2.$$

In Theorem 2.2 of [D], R. T. Dong proves the bound (for M of any dimension n),

$$(61) \quad \mathcal{H}^{n-1}(\mathcal{N} \cap \Omega) \leq \frac{1}{2} \int_{\Omega} |\nabla \log q| + \sqrt{n} \text{vol}(\Omega) \lambda + \text{vol}(\partial\Omega).$$

He also proves (Theorem 3.3) that on a surface,

$$(62) \quad \Delta \log q \geq -\lambda + 2 \min(K, 0) + 4\pi \sum_i (k_i - 1) \delta_{p_i},$$

where $\{p_i\}$ are the singular points and k_i is the order of p_i . In Dong's notation, $\lambda > 0$. Using a weak Harnack inequality together with (62), Dong proves ([D], (25)) that in dimension two,

$$(63) \quad \int_{B_R} |\nabla \log q| \leq C_g R \lambda + C'_g \lambda^2 R^3.$$

Combining with (61) produces the upper bound $\mathcal{H}^1(\mathcal{N} \cap \Omega) \leq \lambda^{3/2}$ in dimension 2.

PROBLEM 2. *To what extent can one generalize these estimates to higher dimensions?*

4.5. Other level sets. Although nodal sets are special, it is of interest to bound the Hausdorff surface measure of any level set $\mathcal{N}_{\varphi_\lambda}^c := \{\varphi_\lambda = c\}$. Let $\text{sgn}(x) = \frac{x}{|x|}$.

PROPOSITION 4.7. *For any C^∞ Riemannian manifold, and any $f \in C(M)$ we have,*

$$(64) \quad \int_M f(\Delta + \lambda^2) |\varphi_\lambda - c| dV + \lambda^2 c \int f \text{sgn}(\varphi_\lambda - c) dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}^c} f |\nabla \varphi_\lambda| dS.$$

This identity has similar implications for $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^c)$ and for the equidistribution of level sets.

COROLLARY 4.8. *For $c \in \mathbb{R}$*

$$\lambda^2 \int_{\varphi_\lambda \geq c} \varphi_\lambda dV = \int_{\mathcal{N}_{\varphi_\lambda}^c} |\nabla \varphi_\lambda| dS.$$

One can obtain lower bounds on $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^c)$ as in the case of nodal sets. However the integrals of $|\varphi_\lambda|$ no longer cancel out. The numerator is smaller since one only integrates over $\{\varphi_\lambda \geq c\}$. Indeed, $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^c)$ must tend to zero as c tends to the maximum possible threshold $\lambda^{\frac{n-1}{2}}$ for $\sup_M |\varphi_\lambda|$.

The Corollary follows by integrating Δ by parts, and by using the identity,

$$(65) \quad \begin{aligned} \int_M |\varphi_\lambda - c| + c \text{sgn}(\varphi_\lambda - c) dV &= \int_{\varphi_\lambda > c} \varphi_\lambda dV - \int_{\varphi_\lambda < c} \varphi_\lambda dV \\ &= 2 \int_{\varphi_\lambda > c} \varphi_\lambda dV, \end{aligned}$$

since $0 = \int_M \varphi_\lambda dV = \int_{\varphi_\lambda > c} \varphi_\lambda dV + \int_{\varphi_\lambda < c} \varphi_\lambda dV$.

PROBLEM 3. *A difficult problem would be to study $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}^c)$ as a function of (c, λ) and try to find thresholds where the behavior changes. For random spherical harmonics, $\sup_M |\varphi_\lambda| \simeq \sqrt{\log \lambda}$ and one would expect the level set volumes to be very small above this height except in special cases.*

4.6. Examples. The lower bound of Theorem 4.1 is far from the lower bound conjectured by Yau, which by Theorem 4 is correct at least in the real analytic case. In this section we go over the model examples to understand why the methods are not always getting sharp results.

4.6.1. Flat tori. We have, $|\nabla \sin \langle k, x \rangle|^2 = \cos^2 \langle k, x \rangle |k|^2$. Since $\cos \langle k, x \rangle = 1$ when $\sin \langle k, x \rangle = 0$ the integral is simply $|k|$ times the surface volume of the nodal set, which is known to be of size $|k|$. Also, we have $\int_{\mathbf{T}} |\sin \langle k, x \rangle| dx \geq C$. Thus, our method gives the sharp lower bound $\mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \geq C\lambda^1$ in this example.

So the upper bound is achieved in this example. Also, we have $\int_{\mathbf{T}} |\sin \langle k, x \rangle| dx \geq C$. Thus, our method gives the sharp lower bound $\mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \geq C\lambda^1$ in this example. Since $\cos \langle k, x \rangle = 1$ when $\sin \langle k, x \rangle = 0$ the integral is simply $|k|$ times the surface volume of the nodal set, which is known to be of size $|k|$.

4.6.2. *Spherical harmonics on S^2 .* For background on spherical harmonics we refer to §16.

The L^1 of Y_0^N norm can be derived from the asymptotics of Legendre polynomials

$$P_N(\cos \theta) = \sqrt{2}(\pi N \sin \theta)^{-\frac{1}{2}} \cos \left(\left(N + \frac{1}{2}\right)\theta - \frac{\pi}{4} \right) + O(N^{-3/2})$$

where the remainder is uniform on any interval $\epsilon < \theta < \pi - \epsilon$. We have

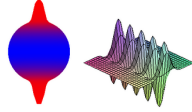
$$\|Y_0^N\|_{L^1} = 4\pi \sqrt{\frac{(2N+1)}{2\pi}} \int_0^{\pi/2} |P_N(\cos r)| dv(r) \sim C_0 > 0,$$

i.e. the L^1 norm is asymptotically a positive constant. Hence $\int_{Z_{Y_0^N}} |\nabla Y_0^N| ds \simeq C_0 N^2$. In

this example $|\nabla Y_0^N|_{L^\infty} = N^{\frac{3}{2}}$ saturates the sup norm bound. The length of the nodal line of Y_0^N is of order λ , as one sees from the rotational invariance and by the fact that P_N has N zeros. The defect in the argument is that the bound $|\nabla Y_0^N|_{L^\infty} = N^{\frac{3}{2}}$ is only obtained on the nodal components near the poles, where each component has length $\simeq \frac{1}{N}$.

EXERCISE 5. Calculate the L^1 norms of (L^2 -normalized) zonal spherical harmonics and Gaussian beams.

The left image is a zonal spherical harmonic of degree N on S^2 : it has high peaks of height \sqrt{N} at the north and south poles. The right image is a Gaussian beam: its height along the equator is $N^{1/4}$ and then it has Gaussian decay transverse to the equator.



Gaussian beams

Gaussian beams are Gaussian shaped lumps which are concentrated on $\lambda^{-\frac{1}{2}}$ tubes $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ around closed geodesics and have height $\lambda^{\frac{n-1}{4}}$. We note that their L^1 norms decrease like $\lambda^{-\frac{(n-1)}{4}}$, i.e. they saturate the L^p bounds of [Sog] for small p . In such cases we have $\int_{Z_{\varphi_\lambda}} |\nabla \varphi_\lambda| dS \simeq \lambda^2 \|\varphi_\lambda\|_{L^1} \simeq \lambda^{2-\frac{n-1}{4}}$. It is likely that Gaussian beams are minimizers of the L^1 norm among L^2 -normalized eigenfunctions of Riemannian manifolds. Also, the gradient bound $\|\nabla \varphi_\lambda\|_{L^\infty} = O(\lambda^{\frac{n+1}{2}})$ is far off for Gaussian beams, the correct upper bound being $\lambda^{1+\frac{n-1}{4}}$. If we use these estimates on $\|\varphi_\lambda\|_{L^1}$ and $\|\nabla \varphi_\lambda\|_{L^\infty}$, our method gives $\mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \geq C\lambda^{1-\frac{n-1}{2}}$, while λ is the correct lower bound for Gaussian beams in the case of surfaces of revolution (or any real analytic case). The defect is again that the gradient estimate is achieved only very close to the closed geodesic of the Gaussian beam. Outside of the tube $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ of radius $\lambda^{-\frac{1}{2}}$ around the geodesic, the Gaussian beam and all of its derivatives decay like $e^{-\lambda d^2}$ where d is the distance to the geodesic. Hence $\int_{Z_{\varphi_\lambda}} |\nabla \varphi_\lambda| dS \simeq \int_{Z_{\varphi_\lambda} \cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |\nabla \varphi_\lambda| dS$. Applying the gradient bound for Gaussian beams to the latter integral gives $\mathcal{H}^{n-1}(Z_{\varphi_\lambda} \cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)) \geq C\lambda^{1-\frac{n-1}{2}}$, which is sharp since the intersection $Z_{\varphi_\lambda} \cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ cuts across γ in $\simeq \lambda$ equally spaced points (as one sees from the Gaussian beam approximation).

5. QUANTUM ERGODIC RESTRICTION THEOREM FOR DIRICHLET OR NEUMANN DATA

QER (quantum ergodic restriction) theorems for Dirichlet data assert the quantum ergodicity of restrictions $\varphi_j|_H$ of eigenfunctions or their normal derivatives to hypersurfaces $H \subset M$. In this section we briefly review the QER theorem for hypersurfaces of [TZ2, CTZ]. For lack of space, we must assume the reader’s familiarity with quantum ergodicity on the global manifold M . We refer to [Ze2, Ze3, Ze6, Zw] for recent expositions.

5.1. Quantum ergodic restriction theorems for Dirichlet data. Roughly speaking, the QER theorem for Dirichlet data says that restrictions of eigenfunctions to hypersurfaces $H \subset M$ for (M, g) with ergodic geodesic flow are quantum ergodic along H as long as H is *asymmetric* for the geodesic flow. Here we note that a tangent vector ξ to H of length ≤ 1 is the projection to TH of two unit tangent vectors ξ_{\pm} to M . The $\xi_{\pm} = \xi + r\nu$ where ν is the unit normal to H and $|\xi|^2 + r^2 = 1$. There are two possible signs of r corresponding to the two choices of “inward” resp. “outward” normal. Asymmetry of H with respect to the geodesic flow G^t means that the two orbits $G^t(\xi_{\pm})$ almost never return at the same time to the same place on H . A generic hypersurface is asymmetric [TZ2]. We refer to [TZ2] (Definition 1) for the precise definition of “positive measure of microlocal reflection symmetry” of H . By asymmetry we mean that this measure is zero.

We write $h_j = \lambda_j^{-\frac{1}{2}}$ and employ the calculus of semi-classical pseudo-differential operators [Zw] where the pseudo-differential operators on H are denoted by $a^w(y, hD_y)$ or $Op_{h_j}(a)$. The unit co-ball bundle of H is denoted by B^*H .

THEOREM 5.1. *Let (M, g) be a compact surface with ergodic geodesic flow, and let $H \subset M$ be a closed curve which is asymmetric with respect to the geodesic flow. Then there exists a density-one subset S of \mathbb{N} such that for $a \in S^{0,0}(T^*H \times [0, h_0])$,*

$$\lim_{j \rightarrow \infty; j \in S} \langle Op_{h_j}(a)\varphi_{h_j}|_H, \varphi_{h_j}|_H \rangle_{L^2(H)} = \omega(a),$$

where

$$\omega(a) = \frac{4}{\text{vol}(S^*M)} \int_{B^*H} a_0(s, \sigma) (1 - |\sigma|^2)^{-\frac{1}{2}} dsd\sigma.$$

In particular this holds for multiplication operators f .

There is a similar result for normalized Neumann data. The normalized Neumann data of an eigenfunction along H is denoted by

$$(66) \quad \lambda_j^{-\frac{1}{2}} D_{\nu} \varphi_j|_H.$$

Here, $D_{\nu} = \frac{1}{i} \partial_{\nu}$ is a fixed choice of unit normal derivative.

We define the microlocal lifts of the Neumann data as the linear functionals on semi-classical symbols $a \in S_{sc}^0(H)$ given by

$$\mu_h^N(a) := \int_{B^*H} a d\Phi_h^N := \langle Op_H(a)hD_{\nu}\varphi_h|_H, hD_{\nu}\varphi_h|_H \rangle_{L^2(H)}.$$

THEOREM 5.2. *Let (M, g) be a compact surface with ergodic geodesic flow, and let $H \subset M$ be a closed curve which is asymmetric with respect to the geodesic flow. Then there exists a*

density-one subset S of \mathbb{N} such that for $a \in S^{0,0}(T^*H \times [0, h_0])$,

$$\lim_{h_j \rightarrow 0^+; j \in S} \mu_h^N(a) \rightarrow \omega(a),$$

where

$$\omega(a) = \frac{4}{\text{vol}(S^*M)} \int_{B^*H} a_0(s, \sigma) (1 - |\sigma|^2)^{\frac{1}{2}} ds d\sigma.$$

In particular this holds for multiplication operators f .

5.2. Quantum ergodic restriction theorems for Cauchy data. Our application is to the hypersurface $H = \text{Fix}(\sigma)$ (17) given by the fixed point set of the isometric involution σ . Such a hypersurface (i.e. curve) fails to be asymmetric. However there is a quantum ergodic restriction theorem for Cauchy data in [CTZ] which does apply and shows that the even eigenfunctions are quantum ergodic along H , hence along each component γ .

The normalized Cauchy data of an eigenfunction along γ is denoted by

$$(67) \quad CD(\varphi_h) := \{(\varphi_h|_\gamma, hD_\nu \varphi_h|_\gamma)\}.$$

Here, D_ν is a fixed choice of unit normal derivative. The first component of the Cauchy data is called the Dirichlet data and the second is called the Neumann data.

THEOREM 5.3. *Assume that (M, g) has an orientation reversing isometric involution with separating fixed point set H . Let γ be a component of H . Let φ_h be the sequence of even ergodic eigenfunctions. Then,*

$$\begin{aligned} & \langle Op_\gamma(a) \varphi_h|_\gamma, \varphi_h|_\gamma \rangle_{L^2(\gamma)} \\ & \rightarrow_{h \rightarrow 0^+} \frac{4}{2\pi \text{Area}(M)} \int_{B^*\gamma} a_0(s, \sigma) (1 - |\sigma|^2)^{-1/2} ds d\sigma. \end{aligned}$$

In particular, this holds when $Op_\gamma(a)$ is multiplication by a smooth function f .

Here we use the semi-classical notation $h_j = \lambda_\varphi^{-\frac{1}{4}}$. ergodic along γ , but we do not use this result here. We refer to [TZ2, CTZ, Zw] for background and for notation concerning pseudo-differential operators.

We further define the microlocal lifts of the Neumann data as the linear functionals on semi-classical symbols $a \in S_{sc}^0(\gamma)$ given by

$$\mu_h^N(a) := \int_{B^*\gamma} a d\Phi_h^N := \langle Op_\gamma(a) hD_\nu \varphi_h|_\gamma, hD_\nu \varphi_h|_\gamma \rangle_{L^2(\gamma)}.$$

We also define the *renormalized microlocal lifts* of the Dirichlet data by

$$\mu_h^D(a) := \int_{B^*\gamma} a d\Phi_h^{RD} := \langle Op_\gamma(a) (1 + h^2 \Delta_\gamma) \varphi_h|_\gamma, \varphi_h|_\gamma \rangle_{L^2(\gamma)}.$$

Here, $h^2 \Delta_\gamma$ denotes the negative tangential Laplacian $-h^2 \frac{d^2}{ds^2}$ for the induced metric on γ , so that the symbol $1 - |\sigma|^2$ of the operator $(1 + h^2 \Delta_\gamma)$ vanishes on the tangent directions $S^*\gamma$ of γ . Finally, we define the microlocal lift $d\Phi_h^{CD}$ of the Cauchy data to be the sum

$$(68) \quad d\Phi_h^{CD} := d\Phi_h^N + d\Phi_h^{RD}.$$

Let $B^*\gamma$ denote the unit ‘‘ball-bundle’’ of γ (which is the interval $\sigma \in (-1, 1)$ at each point s), s denotes arc-length along γ and σ is the dual symplectic coordinate. The first result of

[CTZ] relates QE (quantum ergodicity) on M to quantum ergodicity on a hypersurface γ . A sequence of eigenfunctions is QE globally on M if

$$\langle A\varphi_{j_k}, \varphi_{j_k} \rangle \rightarrow \frac{1}{\mu(S^*M)} \int_{S^*M} \sigma_A d\mu,$$

where $d\mu_L$ is Liouville measure, i.e. the measure induced on the co-sphere bundle by the symplectic volume measure and the Hamiltonian $H(x, \xi) = |\xi|_g$.

THEOREM 5.4. *Assume that $\{\varphi_h\}$ is a quantum ergodic sequence of eigenfunctions on M . Then the sequence $\{d\Phi_h^{CD}\}$ (68) of microlocal lifts of the Cauchy data of φ_h is quantum ergodic on γ in the sense that for any $a \in S_{sc}^0(\gamma)$,*

$$\begin{aligned} & \langle Op_H(a)hD_\nu\varphi_h|_\gamma, hD_\nu\varphi_h|_\gamma \rangle_{L^2(\gamma)} + \langle Op_\gamma(a)(1 + h^2\Delta_\gamma)\varphi_h|_\gamma, \varphi_h|_\gamma \rangle_{L^2(\gamma)} \\ & \rightarrow_{h \rightarrow 0^+} \frac{4}{\mu(S^*M)} \int_{B^*\gamma} a_0(s, \sigma)(1 - |\sigma|^2)^{1/2} ds d\sigma \end{aligned}$$

where a_0 is the principal symbol of $Op_\gamma(a)$.

When applied to even eigenfunctions under an orientation-reversing isometric involution with separating fixed point set, the Neumann data vanishes, and we obtain

COROLLARY 5.1. *Let (M, g) have an orientation-reversing isometric involution with separating fixed point set H and let γ be one of its components. Then for any sequence of even quantum ergodic eigenfunctions of (M, g) ,*

$$\begin{aligned} & \langle Op_\gamma(a)(1 + h^2\Delta_\gamma)\varphi_h|_\gamma, \varphi_h|_\gamma \rangle_{L^2(\gamma)} \\ & \rightarrow_{h \rightarrow 0^+} \frac{4}{\mu(S^*M)} \int_{B^*\gamma} a_0(s, \sigma)(1 - |\sigma|^2)^{1/2} ds d\sigma \end{aligned}$$

For applications to zeros along γ , we need a limit formula for the integrals $\int_\gamma f\varphi_h^2 ds$, i.e. a quantum ergodicity result for Dirichlet data. We invert the operator $(1 + h^2\Delta_\gamma)$ and obtain

THEOREM 5.5. *Assume that $\{\varphi_h\}$ is a quantum ergodic sequence on M . Then, there exists a sub-sequence of density one as $h \rightarrow 0^+$ such that for all $a \in S_{sc}^0(\gamma)$,*

$$\begin{aligned} & \langle (1 + h^2\Delta_\gamma + i0)^{-1}Op_\gamma(a)hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_\gamma \rangle_{L^2(\gamma)} + \langle Op_\gamma(a)\varphi_h|_\gamma, \varphi_h|_\gamma \rangle_{L^2(\gamma)} \\ & \rightarrow_{h \rightarrow 0^+} \frac{4}{2\pi \text{Area}(M)} \int_{B^*\gamma} a_0(s, \sigma)(1 - |\sigma|^2)^{-1/2} ds d\sigma. \end{aligned}$$

Theorem 5.3 follows from Theorem 5.5 since the Neumann term drops out (as before) under the hypothesis of Corollary 5.1.

6. COUNTING INTERSECTIONS OF NODAL SETS AND GEODESICS

As discussed in the introduction §2.5, the QER results can be used to obtain results on intersections of nodal sets with geodesics in dimension two. In general, we do not know how to use intersection results to obtain lower bounds on numbers of nodal domains unless we assume a symmetry condition on the surface. But begin with general results on intersection that do not assume any symmetries.

THEOREM 6.1. *Let (M, g) be a C^∞ compact negatively curved surface, and let H be a closed curve which is asymmetric with respect to the geodesic flow. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of Δ -eigenfunctions of (M, g) , there exists a density 1 subset A of \mathbb{N} such that*

$$\begin{cases} \lim_{j \in A, j \rightarrow \infty} \# \mathcal{N}_{\varphi_j} \cap H = \infty \\ \lim_{j \in A, j \rightarrow \infty} \# \{x \in H : \partial_\nu \varphi_j(x) = 0\} = \infty. \end{cases}$$

Furthermore, there are an infinite number of zeros where $\varphi_j|_H$ (resp. $\partial_\nu \varphi_j|_H$) changes sign.

We now add the assumption of a symmetry as discussed in the introduction in §2.3.

THEOREM 6.2. *Let (M, g) be a compact negatively curved C^∞ surface with an orientation-reversing isometric involution $\sigma : M \rightarrow M$ with $\text{Fix}(\sigma)$ separating. Let $\gamma \subset \text{Fix}(\sigma)$. Then for any orthonormal eigenbasis $\{\varphi_j\}$ of $L^2_{\text{even}}(M)$, resp. $\{\psi_j\}$ of $L^2_{\text{odd}}(M)$, one can find a density 1 subset A of \mathbb{N} such that*

$$\begin{cases} \lim_{j \in A, j \rightarrow \infty} \# \mathcal{N}_{\varphi_j} \cap \gamma = \infty \\ \lim_{j \in A, j \rightarrow \infty} \# \Sigma_{\psi_j} \cap \gamma = \infty. \end{cases}$$

Furthermore, there are an infinite number of zeros where $\varphi_j|_H$ (resp. $\partial_\nu \psi_j|_H$) changes sign.

We now sketch the proof.

6.1. Kuznecov sum formula on surfaces. The first step is to use an old result [Ze9] on the asymptotics of the ‘periods’ $\int_\gamma f \varphi_j ds$ of eigenfunctions over closed geodesics when f is a smooth function.

THEOREM 6.3. [Ze9] (Corollary 3.3) *Let $f \in C^\infty(\gamma)$. Then there exists a constant $c > 0$ such that,*

$$\sum_{\lambda_j < \lambda} \left| \int_\gamma f \varphi_j ds \right|^2 = c \left| \int_\gamma f ds \right|^2 \sqrt{\lambda} + O_f(1).$$

There is a similar result for the normal derivative ∂_ν of eigenfunctions along γ .

THEOREM 6.4. *Let $f \in C^\infty(\gamma)$. Then there exists a constant $c > 0$ such that,*

$$\sum_{\lambda_j < \lambda} \left| \lambda_j^{-1/2} \int_\gamma f \partial_\nu \varphi_j ds \right|^2 = c \left| \int_\gamma f ds \right|^2 \sqrt{\lambda} + O_f(1).$$

These ‘Kuznecov sum formulae’ do not imply individual results about asymptotic periods of the full sequence of eigenfunction. However, because the terms are positive, there must exist a subsequence of eigenfunctions φ_j of natural density one so that, for all $f \in C^\infty(\gamma)$,

$$(69) \quad \begin{cases} \left| \int_\gamma f \varphi_j ds \right| \\ \lambda_j^{-\frac{1}{2}} \left| \int_\gamma f \partial_\nu \varphi_j ds \right| \end{cases} = O_f(\lambda_j^{-1/4} (\log \lambda_j)^{1/2})$$

Indeed, this follows by Chebychev's inequality. Denote by $N(\lambda)$ the number of eigenfunctions in $\{j \mid \lambda < \lambda_j < 2\lambda\}$. Then for each f ,

$$\frac{1}{N(\lambda)} |\{j \mid \lambda < \lambda_j < 2\lambda, \left| \int_{\gamma_i} f \varphi_j ds \right|^2 \geq \lambda_j^{-1/2} \log \lambda_j\}| = O_f\left(\frac{1}{\log \lambda}\right).$$

It follows that the upper density of exceptions to (69) tends to zero. We then choose a countable dense set $\{f_n\}$ and apply the diagonalization argument of [Ze4] (Lemma 3) or [Zw] Theorem 15.5 step (2)) to conclude that there exists a density one subsequence for which (69) holds for all $f \in C^\infty(\gamma)$. The same holds for the normal derivative.

We then use the argument sketched in §2.5 of the introduction. If we combine (69) with the QER result (21), we conclude that a full density subsequence of eigenfunctions of a negatively curved surface must have an unbounded number of sign changing zeros along γ . In the asymmetric case we use Theorem 5.1 while in the symmetric case we use Theorem 5.5 in the QER step.

Problem The QER step and the Kuznecov extend to manifolds with boundary (although the Kuznecov sum formula of [Ze9] has so far only been proved in the boundaryless case). But the main obstacle to generalizing the results on intersections and nodal domains is that the logarithmic improvement in (22) has not been generalized to the boundary problems with chaotic billiards (to the author's knowledge).

7. COUNTING NODAL DOMAINS

We now sketch the proof of Theorem 2.1. Granted that there are many zeros of even eigenfunctions along each component γ of $\text{Fix}(\sigma)$ and many zeros of normal derivatives of odd eigenfunctions, the remaining point is to relate such zeros to nodal domains. The nodal sets cross the geodesic transversally but may link up in complicated ways far from the separating set. In dimension two, use the Euler inequality for graphs to relate numbers of intersection points and numbers of nodal domains.

Problem: The QER and Kuznecov sum formula results are valid in all dimensions. It is only the topological step which is simpler in dimension two. Can one find any generalizations of this argument to higher dimensions?

In dimension two we can construct an embedded graph from the nodal set $\mathcal{N}_{\varphi_\lambda}$ as follows.

- (1) For each embedded circle which does not intersect γ , we add a vertex.
- (2) Each singular point is a vertex.
- (3) If $\gamma \not\subset \mathcal{N}_{\varphi_\lambda}$, then each intersection point in $\gamma \cap \mathcal{N}_{\varphi_\lambda}$ is a vertex.
- (4) Edges are the arcs of $\mathcal{N}_{\varphi_\lambda}$ ($\mathcal{N}_{\varphi_\lambda} \cup \gamma$, when φ_λ is even) which join the vertices listed above.

We thus obtain a graph embedded into the surface M . The *faces* f of G are the connected components of $M \setminus V(G) \cup \bigcup_{e \in E(G)} e$. The set of faces is denoted $F(G)$. An edge $e \in E(G)$ is *incident* to f if the boundary of f contains an interior point of e . Every edge is incident to at least one and to at most two faces; if e is incident to f then $e \subset \partial f$. The faces are not assumed to be cells and the sets $V(G), E(G), F(G)$ are not assumed to form a CW complex.

Now let $v(\varphi_\lambda)$ be the number of vertices, $e(\varphi_\lambda)$ be the number of edges, $f(\varphi_\lambda)$ be the number of faces, and $m(\varphi_\lambda)$ be the number of connected components of the graph. Then by Euler's formula (Appendix F, [G]),

$$(70) \quad v(\varphi_\lambda) - e(\varphi_\lambda) + f(\varphi_\lambda) - m(\varphi_\lambda) \geq 1 - 2g_M$$

where g_M is the genus of the surface. We use this inequality to give a lower bound for the number of nodal domains for even and odd eigenfunctions.

In the odd case, we get a lower bound using the large number of singular points of the odd eigenfunctions. Note that since an odd eigenfunction vanishes on $\text{Fix}(\sigma)$, the points where $\partial_\nu \psi_j = 0$ belong to the singular set.

We claim that for an odd eigenfunction ψ_j ,

$$N(\psi_j) \geq \#(\Sigma_{\psi_j} \cap \gamma) + 2 - 2g_M,$$

Proof: for an odd eigenfunction ψ_j , $\gamma \subset \mathcal{N}_{\psi_j}$. Therefore $f(\psi_j) = N(\psi_j)$. Let $n(\psi_j) = \#\Sigma_{\psi_j} \cap \gamma$ be the number of singular points on γ . These points correspond to vertices having degree at least 4 on the graph, hence

$$(71) \quad \begin{aligned} 0 &= \sum_{x:\text{vertices}} \deg(x) - 2e(\psi_j) \\ &\geq 2(v(\psi_j) - n(\psi_j)) + 4n(\psi_j) - 2e(\psi_j). \end{aligned}$$

Therefore

$$e(\psi_j) - v(\psi_j) \geq n(\psi_j),$$

and plugging into (70) with $m(\psi_j) \geq 1$, we obtain

$$N(\psi_j) \geq n(\psi_j) + 2 - 2g_M.$$

Now consider an even eigenfunction φ_j . In this case (following the idea of [GRS]), we relate the number of intersection points of the nodal set with γ to the number of 'inert nodal domains', i.e. nodal domains which are invariant under the isometric involution, $\sigma U = U$. We claim that,

$$N(\varphi_j) \geq \frac{1}{2} \#(\mathcal{N}_{\varphi_j} \cap \gamma) + 1 - g_M.$$

Proof: For an even eigenfunction φ_j , let $N_{in}(\varphi_j)$ be the number of 'inert' nodal domain U . Let $N_{sp}(\varphi_j)$ be the number of the rest (split nodal domains). From the assumption that $\text{Fix}(\sigma)$ is separating, inert domains intersect γ in a finite non-empty union of relative open intervals, and $\text{Fix}(\sigma)$ divides each inert nodal domain into two connected components. Hence each inert nodal domain may correspond to two faces on the graph, depending on whether the nodal domain intersects γ or not. Therefore $f(\varphi_j) \leq 2N_{in}(\varphi_j) + N_{sp}(\varphi_j)$.

Each point in $\mathcal{N}_{\varphi_j} \cap \gamma$ corresponds to a vertex having degree at least 4 on the graph. The nodal intersections are non-singular points. Hence by the same reasoning as the odd case, we have

$$N(\varphi_j) \geq N_{in} + \frac{1}{2} N_{sp}(\varphi_j) \geq \frac{f(\varphi_j)}{2} \geq \frac{n(\varphi_j)}{2} + 1 - g_M$$

where $n(\varphi_j) = \#\mathcal{N}_{\varphi_j} \cap \gamma$.

This concludes the proof of Theorem 2.1.

8. ANALYTIC CONTINUATION OF EIGENFUNCTIONS TO THE COMPLEX DOMAIN

We next discuss three results that use analytic continuation of eigenfunctions to the complex domain. First is the Donnelly-Fefferman volume bound Theorem 4. We sketch a somewhat simplified proof which will appear in more detail in [Ze0]. Second we discuss the equidistribution theory of nodal sets in the complex domain in the ergodic case [Ze5] and in the completely integrable case [Ze8]. Third, we discuss nodal intersection bounds. This includes bounds on the number of nodal lines intersecting the boundary in [TZ] for the Dirichlet or Neuman problem in a plane domain, the number (and equi-distribution) of nodal intersections with geodesics in the complex domain [Ze6] and results on nodal intersections and nodal domains for the modular surface

8.1. **Grauert tubes.** . As examples, we have:

- $M = \mathbb{R}^m/\mathbb{Z}^m$ is $M_{\mathbb{C}} = \mathbb{C}^m/\mathbb{Z}^m$.
- The unit sphere S^n defined by $x_1^2 + \dots + x_{n+1}^2 = 1$ in \mathbb{R}^{n+1} is complexified as the complex quadric $S_{\mathbb{C}}^2 = \{(z_1, \dots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \dots + z_n^2 = 1\}$.
- The hyperboloid model of hyperbolic space is the hypersurface in \mathbb{R}^{n+1} defined by

$$\mathbb{H}^n = \{x_1^2 + \dots + x_n^2 - x_{n+1}^2 = -1, \ x_n > 0\}.$$

Then,

$$H_{\mathbb{C}}^n = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \dots + z_n^2 - z_{n+1}^2 = -1\}.$$

- Any real algebraic subvariety of \mathbb{R}^m has a similar complexification.
- Any Lie group G (or symmetric space) admits a complexification $G_{\mathbb{C}}$.

Let us consider examples of holomorphic continuations of eigenfunctions:

- On the flat torus $\mathbb{R}^m/\mathbb{Z}^m$, the real eigenfunctions are $\cos\langle k, x \rangle, \sin\langle k, x \rangle$ with $k \in 2\pi\mathbb{Z}^m$. The complexified torus is $\mathbb{C}^m/\mathbb{Z}^m$ and the complexified eigenfunctions are $\cos\langle k, \zeta \rangle, \sin\langle k, \zeta \rangle$ with $\zeta = x + i\xi$.
- On the unit sphere S^m , eigenfunctions are restrictions of homogeneous harmonic functions on \mathbb{R}^{m+1} . The latter extend holomorphically to holomorphic harmonic polynomials on \mathbb{C}^{m+1} and restrict to holomorphic function on $S_{\mathbb{C}}^m$.
- On \mathbf{H}^m , one may use the hyperbolic plane waves $e^{(i\lambda+1)\langle z, b \rangle}$, where $\langle z, b \rangle$ is the (signed) hyperbolic distance of the horocycle passing through z and b to 0. They may be holomorphically extended to the maximal tube of radius $\pi/4$.
- On compact hyperbolic quotients \mathbf{H}^m/Γ , eigenfunctions can be then represented by Helgason's generalized Poisson integral formula [H],

$$\varphi_{\lambda}(z) = \int_B e^{(i\lambda+1)\langle z, b \rangle} dT_{\lambda}(b).$$

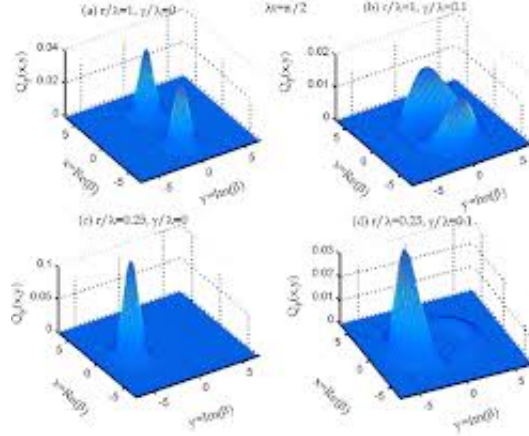
Here, $z \in D$ (the unit disc), $B = \partial D$, and $dT_{\lambda} \in \mathcal{D}'(B)$ is the boundary value of φ_{λ} , taken in a weak sense along circles centered at the origin 0. To analytically continue φ_{λ} it suffices to analytically continue $\langle z, b \rangle$. Writing the latter as $\langle \zeta, b \rangle$, we have:

$$(72) \quad \varphi_{\lambda}^{\mathbb{C}}(\zeta) = \int_B e^{(i\lambda+1)\langle \zeta, b \rangle} dT_{\lambda}(b).$$

The (L^2 -normalizations of the) modulus squares

$$(73) \quad |\varphi_j^{\mathbb{C}}(\zeta)|^2 : M_\epsilon \rightarrow \mathbb{R}_+$$

are sometimes known as Husimi functions. They are holomorphic extensions of L^2 -normalized functions but are not themselves L^2 normalized on M_ϵ . However, as will be discussed below, their L^2 norms may on the Grauert tubes (and their boundaries) can be determined. One can then ask how the mass of the normalized Husimi function is distributed in phase space, or how the L^p norms behave.



8.2. Weak * limit problem for Husimi measures in the complex domain. One of the general problems of quantum dynamics is to determine all of the weak* limits of the sequence,

$$\left\{ \frac{|\varphi_j^{\mathbb{C}}(z)|^2}{\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\epsilon)}^2} d\mu_\epsilon \right\}_{j=1}^\infty.$$

Here, $d\mu_\epsilon$ is the natural measure on ∂M_ϵ corresponding to the contact volume form on S_ϵ^*M . Recall that a sequence μ_n of probability measures on a compact space X is said to converge weak* to a measure μ if $\int_X f d\mu_n \rightarrow \int_X f d\mu$ for all $f \in C(X)$. We refer to Theorem 10.3 for the ergodic case. In the integrable case one has localization results, which are not presented here.

8.3. Background on currents and PSH functions. We next consider logarithms of Husimi functions, which are PSH = (pluri-subharmonic) functions on M_ϵ . A function f on a domain in a complex manifold is PSH if $i\partial\bar{\partial}f$ is a *positive (1,1) current*. That is, $i\partial\bar{\partial}f$ is a singular form of type $\sum_{i\bar{j}} a_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ with $(a_{j\bar{k}})$ positive definite Hermitian. If f is a local holomorphic function, then $\log|f(z)|$ is PSH and $i\partial\bar{\partial}\log|f(z)| = [Z_f]$. General references are [GH, HoC].

In Theorem 5, we regard the zero set $[Z_f]$ as a *current of integration*, i.e. as a linear functional on $(m-1, m-1)$ forms ψ

$$\langle [Z_{\varphi_j}], \psi \rangle = \int_{Z_{\varphi_j}} \psi.$$

Recall that a *current* is a linear functional (distribution) on smooth forms. We refer to [GH] for background. On a complex manifold one has (p, q) forms with $p dz_j$ and $q d\bar{z}_k$'s. In (96)

we use the Kähler hypersurface volume form ω_g^{m-1} (where $\omega_g = i\partial\bar{\partial}\rho$) to make Z_{φ_j} into a measure:

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f \omega_g^{m-1}, \quad (f \in C(M)).$$

A sequence of $(1, 1)$ currents E_k converges weak* to a current E if $\langle E_k, \psi \rangle \rightarrow \langle E, \psi \rangle$ for all smooth $(m-1, m-1)$ forms. Thus, for all f

$$[Z_{\varphi_j}] \rightarrow i\partial\bar{\partial}\sqrt{\rho} \iff \int_{Z_{\varphi_j}} f \omega^{m-1} \rightarrow i \int_{M_\epsilon} f \partial\bar{\partial}\sqrt{\rho} \wedge \omega^{m-1, m-1}.$$

8.4. Poincaré-Lelong formula. One of the two key reasons for the gain in simplicity is that there exists a simple analytical formula for the delta-function on the nodal set. The *Poincaré-Lelong formula* gives an exact formula for the delta-function on the zero set of φ_j

$$(74) \quad i\partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}(z)|^2 = [\mathcal{N}_{\varphi_j^{\mathbb{C}}}]$$

Thus, if ψ is an $(n-1, n-1)$ form,

$$\int_{\mathcal{N}_{\varphi_j^{\mathbb{C}}}} \psi = \int_{M_\epsilon} \psi \wedge i\partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}(z)|^2.$$

8.5. Pluri-subharmonic functions and compactness. In the real domain, we have emphasized the problem of finding weak* limits of the probability measures (14) and of their microlocal lifts or Wigner measures in phase space. The same problem exists in the complex domain for the sequence of Husimi functions (73). However, there also exists a new problem involving the sequence of normalized logarithms

$$(75) \quad \{u_j := \frac{1}{\lambda_j} \log |\varphi_j^{\mathbb{C}}(z)|^2\}_{j=1}^\infty.$$

A key fact is that this sequence is pre-compact in $L^p(M_\epsilon)$ for all $p < \infty$ and even that

$$(76) \quad \left\{ \frac{1}{\lambda_j} \nabla \log |\varphi_j^{\mathbb{C}}(z)|^2 \right\}_{j=1}^\infty$$

is pre-compact in $L^1(M_\epsilon)$.

LEMMA 8.1. (*Hartog's Lemma; (see [Hol, Theorem 4.1.9]):*) *Let $\{v_j\}$ be a sequence of subharmonic functions in an open set $X \subset \mathbb{R}^m$ which have a uniform upper bound on any compact set. Then either $v_j \rightarrow -\infty$ uniformly on every compact set, or else there exists a subsequence v_{j_k} which is convergent to some $u \in L^1_{loc}(X)$. Further, $\limsup_n u_n(x) \leq u(x)$ with equality almost everywhere. For every compact subset $K \subset X$ and every continuous function f ,*

$$\limsup_{n \rightarrow \infty} \sup_K (u_n - f) \leq \sup_K (u - f).$$

In particular, if $f \geq u$ and $\epsilon > 0$, then $u_n \leq f + \epsilon$ on K for n large enough.

8.6. **A general weak* limit problem for logarithms of Husimi functions.** The study of exponential growth rates gives rise to a new kind new weak* limit problem for complexified eigenfunctions.

PROBLEM 8.2. *Find the weak* limits G on M_ϵ of sequences*

$$\frac{1}{\lambda_{j_k}} \log |\varphi_{j_k}^{\mathbb{C}}(z)|^2 \rightarrow G??$$

(*The limits are actually in L^1 and not just weak.*)

See Theorems 10.4 , 10.5 and 10.7 for the solution to this problem (modulo sparse subsequences) in the ergodic case.

Here is a general Heuristic principle to pin down the possible G : If $\frac{1}{\lambda_{j_k}} \log |\varphi_{j_k}^{\mathbb{C}}(z)|^2 \rightarrow G(z)$ then

$$|\varphi_{j_k}^{\mathbb{C}}(z)|^2 \simeq e^{\lambda_j G(z)} (1 + \text{SOMETHING SMALLER}) (\lambda_j \rightarrow \infty).$$

But $\Delta_{\mathbb{C}} |\varphi_{j_k}^{\mathbb{C}}(z)|^2 = \lambda_{j_k}^2 |\varphi_{j_k}^{\mathbb{C}}(z)|^2$, so we should have

CONJECTURE 8.3. *Any limit G as above solves the Hamilton-Jacobi equation,*

$$(\nabla_{\mathbb{C}} G)^2 = 1.$$

(Note: The weak* limits of $\frac{|\varphi_j^{\mathbb{C}}(z)|^2}{\|\varphi_j^{\mathbb{C}}\|_{L^2(\partial M_\epsilon)}} d\mu_\epsilon$ must be supported in $\{G = G_{\max}\}$ (i.e. in the set of maximum values).

9. POISSON OPERATOR AND SZEGÖ OPERATORS ON GRAUERT TUBES

9.1. **Poisson operator and analytic Continuation of eigenfunctions.** The half-wave group of (M, g) is the unitary group $U(t) = e^{it\sqrt{\Delta}}$ generated by the square root of the positive Laplacian. Its Schwartz kernel is a distribution on $\mathbb{R} \times M \times M$ with the eigenfunction expansion

$$(77) \quad U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_j(x) \varphi_j(y).$$

By the Poisson operator we mean the analytic continuation of $U(t)$ to positive imaginary time,

$$(78) \quad e^{-\tau\sqrt{\Delta}} = U(i\tau).$$

The eigenfunction expansion then converges absolutely to a real analytic function on $\mathbb{R}_+ \times M \times M$.

Let $A(\tau)$ denote the operator of analytic continuation of a function on M to the Grauert tube M_τ . Since

$$(79) \quad U_{\mathbb{C}}(i\tau)\varphi_\lambda = e^{-\tau\lambda}\varphi_\lambda^{\mathbb{C}},$$

it is simple to see that

$$(80) \quad A(\tau) = U_{\mathbb{C}}(i\tau)e^{\tau\sqrt{\Delta}}$$

where $U_{\mathbb{C}}(i\tau, \zeta, y)$ is the analytic continuation of the Poisson kernel in x to M_{τ} . In terms of the eigenfunction expansion, one has

$$(81) \quad U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \varphi_j(y), \quad (\zeta, y) \in M_{\epsilon} \times M.$$

This is a very useful observation because $U_{\mathbb{C}}(i\tau)e^{\tau\sqrt{\Delta}}$ is a Fourier integral operator with complex phase and can be related to the geodesic flow. The analytic continuability of the Poisson operator to M_{τ} implies that every eigenfunction analytically continues to the same Grauert tube.

9.2. Analytic continuation of the Poisson wave group. The analytic continuation of the Poisson-wave kernel to M_{τ} in the x variable is discussed in detail in [Ze8] and ultimately derives from the analysis by Hadamard of his parametrix construction. We only briefly discuss it here and refer to [Ze8] for further details. In the case of Euclidean \mathbb{R}^n and its wave kernel $U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} d\xi$ which analytically continues to $t + i\tau, \zeta = x + ip \in \mathbb{C}_+ \times \mathbb{C}^n$ as the integral

$$U_{\mathbb{C}}(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y \rangle} d\xi.$$

The integral clearly converges absolutely for $|p| < \tau$.

Exact formulae of this kind exist for S^m and \mathbf{H}^m . For a general real analytic Riemannian manifold, there exists an oscillatory integral expression for the wave kernel of the form,

$$(82) \quad U(t, x, y) = \int_{T_y^* M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x) \rangle} A(t, x, y, \xi) d\xi$$

where $A(t, x, y, \xi)$ is a polyhomogeneous amplitude of order 0. The holomorphic extension of (82) to the Grauert tube $|\zeta| < \tau$ in x at time $t = i\tau$ then has the form

$$(83) \quad U_{\mathbb{C}}(i\tau, \zeta, y) = \int_{T_y^* M} e^{-\tau|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(\zeta) \rangle} A(t, \zeta, y, \xi) d\xi \quad (\zeta = x + ip).$$

9.3. Complexified spectral projections. The next step is to holomorphically extend the spectral projectors $d\Pi_{[0, \lambda]}(x, y) = \sum_j \delta(\lambda - \lambda_j) \varphi_j(x) \varphi_j(y)$ of $\sqrt{\Delta}$. The complexified diagonal spectral projections measure is defined by

$$(84) \quad d_{\lambda} \Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) |\varphi_j^{\mathbb{C}}(\zeta)|^2.$$

Henceforth, we generally omit the superscript and write the kernel as $\Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta})$. This kernel is not a tempered distribution due to the exponential growth of $|\varphi_j^{\mathbb{C}}(\zeta)|^2$. Since many asymptotic techniques assume spectral functions are of polynomial growth, we simultaneously consider the damped spectral projections measure

$$(85) \quad d_{\lambda} P_{[0, \lambda]}^{\tau}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

which is a temperate distribution as long as $\sqrt{\rho}(\zeta) \leq \tau$. When we set $\tau = \sqrt{\rho}(\zeta)$ we omit the τ and put

$$(86) \quad d_\lambda P_{[0,\lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2.$$

The integral of the spectral measure over an interval I gives

$$\Pi_I(x, y) = \sum_{j:\lambda_j \in I} \varphi_j(x) \varphi_j(y).$$

Its complexification gives the spectral projections kernel along the anti-diagonal,

$$(87) \quad \Pi_I(\zeta, \bar{\zeta}) = \sum_{j:\lambda_j \in I} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

and the integral of (85) gives its temperate version

$$(88) \quad P_I^\tau(\zeta, \bar{\zeta}) = \sum_{j:\lambda_j \in I} e^{-2\tau\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

or in the crucial case of $\tau = \sqrt{\rho}(\zeta)$,

$$(89) \quad P_I(\zeta, \bar{\zeta}) = \sum_{j:\lambda_j \in I} e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2,$$

9.4. Poisson operator as a complex Fourier integral operator. The damped spectral projection measure $d_\lambda P_{[0,\lambda]}^\tau(\zeta, \bar{\zeta})$ (85) is dual under the real Fourier transform in the t variable to the restriction

$$(90) \quad U(t + 2i\tau, \zeta, \bar{\zeta}) = \sum_j e^{(-2\tau+it)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2$$

to the anti-diagonal of the mixed Poisson-wave group. The adjoint of the Poisson kernel $U(i\tau, x, y)$ also admits an anti-holomorphic extension in the y variable. The sum (90) are the diagonal values of the complexified wave kernel

$$(91) \quad \begin{aligned} U(t + 2i\tau, \zeta, \bar{\zeta}') &= \int_M U(t + i\tau, \zeta, y) E(i\tau, y, \bar{\zeta}') dV_g(x) \\ &= \sum_j e^{(-2\tau+it)\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \overline{\varphi_j^{\mathbb{C}}(\zeta')}. \end{aligned}$$

We obtain (91) by orthogonality of the real eigenfunctions on M .

Since $U(t + 2i\tau, \zeta, y)$ takes its values in the CR holomorphic functions on ∂M_τ , we consider the Sobolev spaces $\mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\tau)$ of CR holomorphic functions on the boundaries of the strictly pseudo-convex domains M_ϵ , i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau) = W^{s+\frac{m-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau),$$

where W_s is the s th Sobolev space and where $\mathcal{O}(\partial M_\epsilon)$ is the space of boundary values of holomorphic functions. The inner product on $\mathcal{O}^0(\partial M_\tau)$ is with respect to the Liouville measure

$$(92) \quad d\mu_\tau = (i\partial\bar{\partial}\sqrt{\rho})^{m-1} \wedge d^c\sqrt{\rho}.$$

We then regard $U(t + i\tau, \zeta, y)$ as the kernel of an operator from $L^2(M) \rightarrow \mathcal{O}^0(\partial M_\tau)$. It equals its composition $\Pi_\tau \circ U(t + i\tau)$ with the Szegő projector

$$\Pi_\tau : L^2(\partial M_\tau) \rightarrow \mathcal{O}^0(\partial M_\tau)$$

for the tube M_τ , i.e. the orthogonal projection onto boundary values of holomorphic functions in the tube.

This is a useful expression for the complexified wave kernel, because $\tilde{\Pi}_\tau$ is a complex Fourier integral operator with a small wave front relation. More precisely, the real points of its canonical relation form the graph Δ_Σ of the identity map on the symplectic one $\Sigma_\tau \subset T^*\partial M_\tau$ spanned by the real one-form $d^c\rho$, i.e.

$$(93) \quad \Sigma_\tau = \{(\zeta; rd^c\rho(\zeta)), \quad \zeta \in \partial M_\tau, r > 0\} \subset T^*(\partial M_\tau).$$

We note that for each τ , there exists a symplectic equivalence $\Sigma_\tau \simeq T^*M$ by the map $(\zeta, rd^c\rho(\zeta)) \rightarrow (E_{\mathbb{C}}^{-1}(\zeta), r\alpha)$, where $\alpha = \xi \cdot dx$ is the action form (cf. [GS2]).

The following result was first stated by Boutet de Monvel[Bou] and has been proved in detail in [Ze8, L, Ste].

THEOREM 9.1. $\Pi_\epsilon \circ U(i\epsilon) : L^2(M) \rightarrow \mathcal{O}(\partial M_\epsilon)$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ associated to the canonical relation

$$\Gamma = \{(y, \eta, \iota_\epsilon(y, \eta))\} \subset T^*M \times \Sigma_\epsilon.$$

Moreover, for any s ,

$$\Pi_\epsilon \circ U(i\epsilon) : W^s(M) \rightarrow \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\epsilon)$$

is a continuous isomorphism.

In [Ze8] we give the following sharpening of the sup norm estimates of [Bou]:

PROPOSITION 9.2. *Suppose (M, g) is real analytic. Then*

$$\sup_{\zeta \in M_\tau} |\varphi_\lambda^{\mathbb{C}}(\zeta)| \leq C\lambda^{\frac{m+1}{2}} e^{\tau\lambda}, \quad \sup_{\zeta \in M_\tau} \left| \frac{\partial \varphi_\lambda^{\mathbb{C}}(\zeta)}{\partial \zeta_j} \right| \leq C\lambda^{\frac{m+3}{2}} e^{\tau\lambda}$$

The proof follows easily from the fact that the complexified Poisson kernel is a complex Fourier integral operator of finite order. The estimates can be improved further.

9.5. Toeplitz dynamical construction of the wave group. There exists an alternative to the parametrix constructions of Hadamard-Riesz, Lax, Hörmander and others which are reviewed in §14. It is useful for constructing the wave group $U(t)$ for large t , when it is awkward to use the group property $U(t/N)^N = U(t)$. As in Theorem 9.1 we denote by $U(i\epsilon)$ the operator with kernel $U(i\epsilon, \zeta, y)$ with $\zeta \in \partial M_\epsilon, y \in M$. We also denote by $U^*(i\epsilon) : \mathcal{O}(\partial M_\epsilon) \rightarrow L^2(M)$ the adjoint operator. Further, let

$$T_{g^t} : L^2(\partial M_\epsilon, d\mu_\epsilon) \rightarrow L^2(\partial M_\epsilon, d\mu_\epsilon)$$

be the unitary translation operator

$$T_{g^t} f(\zeta) = f(g^t(\zeta))$$

where $d\mu_\epsilon$ is the contact volume form on ∂M_ϵ and g^t is the Hamiltonian flow of $\sqrt{\rho}$ on M_ϵ .

PROPOSITION 9.3. *There exists a symbol $\sigma_{\epsilon,t}$ such that*

$$U(t) = U^*(i\epsilon)\sigma_{\epsilon,t}T_{g^t}U(i\epsilon).$$

The proof of this Proposition is to verify that the right side is a Fourier integral operator with canonical relation the graph of the geodesic flow. One then constructs $\sigma_{\epsilon,t}$ so that the symbols match. The proof is given in [Ze6]. Related constructions are given in [G1, BoGu].

10. EQUIDISTRIBUTION OF COMPLEX NODAL SETS OF REAL ERGODIC EIGENFUNCTIONS

We now consider global results when hypotheses are made on the dynamics of the geodesic flow. The main purpose of this section is to sketch the proof of Theorem 5. Use of the global wave operator brings into play the relation between the geodesic flow and the complexified eigenfunctions, and this allows one to prove global results on nodal hypersurfaces that reflect the dynamics of the geodesic flow. In some cases, one can determine not just the volume, but the limit distribution of complex nodal hypersurfaces. Since we have discussed this result elsewhere [Ze6] we only briefly review it here.

The complex nodal hypersurface of an eigenfunction is defined by

$$(94) \quad Z_{\varphi_\lambda^{\mathbb{C}}} = \{\zeta \in M_{\epsilon_0} : \varphi_\lambda^{\mathbb{C}}(\zeta) = 0\}.$$

As discussed in §8.3, there exists a natural current of integration over the nodal hypersurface in any Grauert tube M_ϵ with $\epsilon < \epsilon_0$, given by

$$(95) \quad \langle [Z_{\varphi_\lambda^{\mathbb{C}}}], \varphi \rangle = \frac{i}{2\pi} \int_{M_\epsilon} \partial\bar{\partial} \log |\varphi_\lambda^{\mathbb{C}}|^2 \wedge \varphi = \int_{Z_{\varphi_\lambda^{\mathbb{C}}}} \varphi, \quad \varphi \in \mathcal{D}^{(m-1, m-1)}(M_\epsilon).$$

In the second equality we used the Poincaré-Lelong formula (§8.4). We recall that $\mathcal{D}^{(m-1, m-1)}(M_\epsilon)$ stands for smooth $(m-1, m-1)$ -forms with support in $M_{*\epsilon}$.

The nodal hypersurface $Z_{\varphi_\lambda^{\mathbb{C}}}$ also carries a natural volume form $|Z_{\varphi_\lambda^{\mathbb{C}}}|$ as a complex hypersurface in a Kähler manifold. By Wirtinger's formula, it equals the restriction of $\frac{\omega_g^{m-1}}{(m-1)!}$ to $Z_{\varphi_\lambda^{\mathbb{C}}}$. Hence, one can regard $Z_{\varphi_\lambda^{\mathbb{C}}}$ as defining the measure

$$(96) \quad \langle |Z_{\varphi_\lambda^{\mathbb{C}}}|, \varphi \rangle = \int_{Z_{\varphi_\lambda^{\mathbb{C}}}} \varphi \frac{\omega_g^{m-1}}{(m-1)!}, \quad \varphi \in C(B_\epsilon^* M).$$

We prefer to state results in terms of the current $[Z_{\varphi_\lambda^{\mathbb{C}}}]$ since it carries more information.

We re-state Theorem 5 as follows:

THEOREM 10.1. *Let (M, g) be real analytic, and let $\{\varphi_{j_k}\}$ denote a quantum ergodic sequence of eigenfunctions of its Laplacian Δ . Let M_{ϵ_0} be the maximal Grauert tube around M . Let $\epsilon < \epsilon_0$. Then:*

$$\frac{1}{\lambda_{j_k}} [Z_{\varphi_{j_k}^{\mathbb{C}}}] \rightarrow \frac{i}{\pi} \partial\bar{\partial} \sqrt{\rho} \text{ weakly in } \mathcal{D}'^{(1,1)}(M_\epsilon),$$

in the sense that, for any continuous test form $\psi \in \mathcal{D}^{(m-1, m-1)}(M_\epsilon)$, we have

$$\frac{1}{\lambda_{j_k}} \int_{Z_{\varphi_{j_k}^{\mathbb{C}}}} \psi \rightarrow \frac{i}{\pi} \int_{M_\epsilon} \psi \wedge \partial\bar{\partial} \sqrt{\rho}.$$

Equivalently, for any $\varphi \in C(M_\epsilon)$,

$$\frac{1}{\lambda_{j_k}} \int_{Z_{\varphi_{j_k}^{\mathbb{C}}}} \varphi \frac{\omega_g^{m-1}}{(m-1)!} \rightarrow \frac{i}{\pi} \int_{M_\epsilon} \varphi \partial\bar{\partial} \sqrt{\rho} \wedge \frac{\omega_g^{m-1}}{(m-1)!}.$$

10.1. Sketch of the proof. The first step is to find a nice way to express $\varphi_j^{\mathbb{C}}$ on $M_{\mathbb{C}}$. Very often, when we analytically continue a function, we lose control over its behavior. The trick is to observe that the complexified wave group analytically continues the eigenfunctions. Recall that $U(t) = \exp it\sqrt{\Delta}$. Define the Poisson operator as

$$U(i\tau) = e^{-\tau\sqrt{\Delta}}.$$

Note that

$$U(i\tau)\varphi_j = e^{-\tau\lambda_j}\varphi_j.$$

The next step is to analytically continue $U(i\tau, x, y)$ in $x \rightarrow z \in M_{\tau}$.

The complexified Poisson kernel is define by

$$U(i\tau, \zeta, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \varphi_j(y).$$

It is holomorphic in $\zeta \in M_{\tau}$, i.e. when $\sqrt{\rho}(\zeta) < \tau$. But the main point is that it remains a Fourier integral operator after analytic continuation:

THEOREM 10.2. (Hadamard, Boutet de Monvel, Z, M. Stenzel, G. Lebeau) $U(i\epsilon, z, y) : L^2(M) \rightarrow H^2(\partial M_{\epsilon})$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ quantizing the complexified exponential map $\exp_y i\epsilon \frac{\eta}{|\eta|} : S_{\epsilon}^* \rightarrow \partial M_{\epsilon}$.

We first observe that

$$U(i\tau)\varphi_{\lambda_j} = e^{-\tau\lambda_j}\varphi_{\lambda_j}^{\mathbb{C}}.$$

This follows immediately by integrating

$$U(i\tau, \zeta, y) = \sum_{k=0}^{\infty} e^{-\tau\lambda_k} \varphi_k^{\mathbb{C}}(\zeta) \varphi_k(y)$$

against φ_j and using orthogonality.

But we know that $U(i\tau)\varphi_{\lambda_j}$ is a Fourier integral operator. It is a fact that such an operator can only change L^2 norms by powers of λ_j . So

$$\|U(i\tau)\varphi_{\lambda_j}\|_{L^2(\partial M_{\epsilon})}^2$$

has polynomial growth in λ_j and therefore we have,

$$\|\varphi_{\lambda_j}\|_{L^2(\partial M_{\epsilon})}^2 = \lambda^{\text{somepower}} e^{\tau\lambda_j}.$$

The power is relevant because we are taking the normalized logarithm.

The first step is to prove quantum ergodicity of the complexified eigenfunctions:

THEOREM 10.3. Assume the geodesic flow of (M, g) is ergodic. Then

$$\frac{|\varphi_{j_k}^{\epsilon}(z)|^2}{\|\varphi_{j_k}^{\epsilon}\|_{L^2(\partial M_{\epsilon})}^2} \rightarrow 1, \text{ weak } * \text{ on } C(\partial M_{\epsilon}),$$

along a density one subsequence of λ_j . I.e. for any continuous V ,

$$\int_{\partial M_{\epsilon}} V \frac{|\varphi_{j_k}^{\epsilon}(z)|^2}{\|\varphi_{j_k}^{\epsilon}\|_{L^2(\partial M_{\epsilon})}^2} d\text{vol} \rightarrow \int_{\partial M_{\epsilon}} V d\text{vol}.$$

Thus, Husimi measures tend to 1 weakly as measures. We then apply Hartogs' Lemma 8.1 to obtain,

LEMMA 10.4. *We have: For all but a sparse subsequence of eigenvalues,*

$$\frac{1}{\lambda_{j_k}} \log \frac{|\varphi_{j_k}^\epsilon(z)|^2}{\|\varphi_{j_k}^\epsilon\|_{L^2(\partial M_\epsilon)}^2} \rightarrow 0, \text{ in } L^1(M_\epsilon).$$

This is almost obvious from the QE theorem. The limit is ≤ 0 and it were < 0 on a set of positive measure it would contradict

$$\frac{|\varphi_{j_k}^\epsilon(z)|^2}{\|\varphi_{j_k}^\epsilon\|_{L^2(\partial M_\epsilon)}^2} \rightarrow 1.$$

Combine Lemma 10.4 with Poincare- Lelong:

$$\frac{1}{\lambda_{j_k}} [Z_{j_k}] = i\partial\bar{\partial} \log |\varphi_{j_k}^C|^2.$$

We get

$$\frac{1}{\lambda_{j_k}} \partial\bar{\partial} \log |\varphi_{j_k}^C|^2 \sim \frac{1}{\lambda_{j_k}} \partial\bar{\partial} \log \|\varphi_{j_k}^C\|_{L^2(\partial M_\epsilon)}^2 \text{ weak } * \text{ on } M_\epsilon.$$

To complete proof we need to prove:

$$(97) \quad \frac{1}{\lambda_j} \log \|\varphi_j^C\|_{\partial M_\epsilon}^2 \rightarrow 2\epsilon.$$

But $U(i\epsilon) = e^{-\epsilon\lambda_j} \varphi_j^C$, hence $\|\varphi_\lambda^C\|_{L^2(\partial M_\epsilon)}^2$ equals $e^{2\epsilon\lambda_j}$ times

$$\langle U(i\epsilon)\varphi_\lambda, U(i\epsilon)\varphi_\lambda \rangle = \langle U(i\epsilon)^*U(i\epsilon)\varphi_\lambda, \varphi_\lambda \rangle.$$

But $U(i\epsilon)^*U(i\epsilon)$ is a pseudodifferential operator of order $\frac{n-1}{2}$. Its symbol $|\xi|^{-\frac{n-1}{2}}$ doesn't contribute to the logarithm.

We now provide more details on each step.

10.2. Growth properties of complexified eigenfunctions. In this section we prove Lemma 10.4 in more detail. We state it in combination with (97).

THEOREM 10.5. *If the geodesic flow is ergodic, then for all but a sparse subsequence of λ_j ,*

$$\frac{1}{\lambda_{j_k}} \log |\varphi_{j_k}^C(z)|^2 \rightarrow \sqrt{\rho} \text{ in } L^1(M_\epsilon).$$

The Grauert tube function is a maximal PSH function with bound $\leq \epsilon$ on M_ϵ . Hence Theorem 10.5 says that ergodic eigenfunctions have the maximum exponential growth rate possible for any eigenfunctions.

A key object in the proof is the sequence of functions $U_\lambda(x, \xi) \in C^\infty(M_\epsilon)$ defined by

$$(98) \quad \begin{cases} U_\lambda(x, \xi) := \frac{\varphi_\lambda^C(x, \xi)}{\rho_\lambda(x, \xi)}, & (x, \xi) \in M_\epsilon, \text{ where} \\ \rho_\lambda(x, \xi) := \|\varphi_\lambda^C\|_{\partial M_{|\xi|g}} \|L^2(\partial M_{|\xi|g}) \end{cases}$$

Thus, $\rho_\lambda(x, \xi)$ is the L^2 -norm of the restriction of $\varphi_\lambda^{\mathbb{C}}$ to the sphere bundle $\{\partial M_\epsilon\}$ where $\epsilon = |\xi|_g$. U_λ is of course not holomorphic, but its restriction to each sphere bundle is CR holomorphic there, i.e.

$$(99) \quad u_\lambda^\epsilon = U_\lambda|_{\partial M_\epsilon} \in \mathcal{O}^0(\partial M_\epsilon).$$

Our first result gives an ergodicity property of holomorphic continuations of ergodic eigenfunctions.

LEMMA 10.6. *Assume that $\{\varphi_{j_k}\}$ is a quantum ergodic sequence of Δ -eigenfunctions on M in the sense of (27). Then for each $0 < \epsilon < \epsilon_0$,*

$$|U_{j_k}|^2 \rightarrow \frac{1}{\mu_1(S^*M)} \sqrt{\rho}^{-m+1}, \quad \text{weakly in } L^1(M_\epsilon, \omega^m).$$

We note that $\omega^m = r^{m-1} dr d\omega d\text{vol}(x)$ in polar coordinates, so the right side indeed lies in L^1 . The actual limit function is otherwise irrelevant. The next step is to use a compactness argument to obtain strong convergence of the normalized logarithms of the sequence $\{|U_\lambda|^2\}$. The first statement of the following lemma immediately implies the second.

LEMMA 10.7. *Assume that $|U_{j_k}|^2 \rightarrow \frac{1}{\mu_\epsilon(\partial M_\epsilon)} \sqrt{\rho}^{-m+1}$, weakly in $L^1(M_\epsilon, \omega^m)$. Then:*

- (1) $\frac{1}{\lambda_{j_k}} \log |U_{j_k}|^2 \rightarrow 0$ strongly in $L^1(M_\epsilon)$.
- (2) $\frac{1}{\lambda_j} \partial \bar{\partial} \log |U_{j_k}|^2 \rightarrow 0$, weakly in $\mathcal{D}'^{(1,1)}(M_\epsilon)$.

Separating out the numerator and denominator of $|U_j|^2$, we obtain that

$$(100) \quad \frac{1}{\lambda_{j_k}} \partial \bar{\partial} \log |\varphi_{j_k}^{\mathbb{C}}|^2 - \frac{2}{\lambda_{j_k}} \partial \bar{\partial} \log \rho_{\lambda_{j_k}} \rightarrow 0, \quad (\lambda_{j_k} \rightarrow \infty).$$

The next lemma shows that the second term has a weak limit:

LEMMA 10.8. *For $0 < \epsilon < \epsilon_0$,*

$$\frac{1}{\lambda_{j_k}} \log \rho_{\lambda_{j_k}}(x, \xi) \rightarrow \sqrt{\rho}, \quad \text{in } L^1(M_\epsilon) \text{ as } \lambda_{j_k} \rightarrow \infty.$$

Hence,

$$\frac{1}{\lambda_{j_k}} \partial \bar{\partial} \log \rho_{\lambda_{j_k}} \rightarrow \partial \bar{\partial} \sqrt{\rho} \quad (\lambda_j \rightarrow \infty) \text{ weakly in } \mathcal{D}'(M_\epsilon).$$

It follows that the left side of (100) has the same limit, and that will complete the proof of Theorem 10.1.

10.3. Proof of Lemma 10.6 and Theorem 10.3. We begin by proving a weak limit formula for the CR holomorphic functions u_λ^ϵ defined in (99) for fixed ϵ .

LEMMA 10.9. *Assume that $\{\varphi_{j_k}\}$ is a quantum ergodic sequence. Then for each $0 < \epsilon < \epsilon_0$,*

$$|u_{j_k}^\epsilon|^2 \rightarrow \frac{1}{\mu_\epsilon(\partial M_\epsilon)}, \quad \text{weakly in } L^1(\partial M_\epsilon, d\mu_\epsilon).$$

That is, for any $a \in C(\partial M_\epsilon)$,

$$\int_{\partial M_\epsilon} a(x, \xi) |u_{j_k}^\epsilon((x, \xi))|^2 d\mu_\epsilon \rightarrow \frac{1}{\mu_\epsilon(\partial M_\epsilon)} \int_{\partial M_\epsilon} a(x, \xi) d\mu_\epsilon.$$

Proof. It suffices to consider $a \in C^\infty(\partial M_\epsilon)$. We then consider the Toeplitz operator $\Pi_\epsilon a \Pi_\epsilon$ on $\mathcal{O}^0(\partial M_\epsilon)$. We have,

$$\begin{aligned} \langle \Pi_\epsilon a \Pi_\epsilon u_j^\epsilon, u_j^\epsilon \rangle &= e^{2\epsilon\lambda_j} \|\varphi_\lambda^\mathbb{C}\|_{L^2(\partial M_\epsilon)}^{-2} \langle \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, U(i\epsilon)\varphi_j \rangle_{L^2(\partial M_\epsilon)} \\ (101) \qquad \qquad \qquad &= e^{2\epsilon\lambda_j} \|\varphi_\lambda^\mathbb{C}\|_{L^2(\partial M_\epsilon)}^{-2} \langle U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}. \end{aligned}$$

It is not hard to see that $U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)$ is a pseudodifferential operator on M of order $-\frac{m-1}{2}$ with principal symbol $\tilde{a}|\xi|_g^{-\frac{m-1}{2}}$, where \tilde{a} is the (degree 0) homogeneous extension of a to $T^*M - 0$. The normalizing factor $e^{2\epsilon\lambda_j} \|\varphi_\lambda^\mathbb{C}\|_{L^2(\partial M_\epsilon)}^{-2}$ has the same form with $a = 1$. Hence, the expression on the right side of (101) may be written as

$$(102) \qquad \frac{\langle U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}{\langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_j, \varphi_j \rangle_{L^2(M)}}.$$

By the standard quantum ergodicity result on compact Riemannian manifolds with ergodic geodesic flow (see [Shn, Ze4, CV] for proofs and references) we have

$$(103) \qquad \frac{\langle U(i\epsilon)^* \Pi_\epsilon a \Pi_\epsilon U(i\epsilon)\varphi_{j_k}, \varphi_{j_k} \rangle_{L^2(M)}}{\langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_{j_k}, \varphi_{j_k} \rangle_{L^2(M)}} \rightarrow \frac{1}{\mu_\epsilon(\partial M_\epsilon)} \int_{\partial M_\epsilon} a d\mu_\epsilon.$$

More precisely, the numerator is asymptotic to the right side times $\lambda^{-\frac{m-1}{2}}$, while the denominator has the same asymptotics when a is replaced by 1. We also use that $\frac{1}{\mu_\epsilon(\partial M_\epsilon)} \int_{\partial M_\epsilon} a d\mu_\epsilon$ equals the analogous average of \tilde{a} over ∂M_ϵ . Taking the ratio produces (103).

Combining (101), (103) and the fact that

$$\langle \Pi_\epsilon a \Pi_\epsilon u_j^\epsilon, u_j^\epsilon \rangle = \int_{\partial B_\epsilon^* M} a |u_j^\epsilon|^2 d\mu_\epsilon$$

completes the proof of the lemma. □

We now complete the proof of Lemma 10.6, i.e. we prove that

$$(104) \qquad \int_{M_\epsilon} a |U_{j_k}|^2 \omega^m \rightarrow \frac{1}{\mu_\epsilon(\partial M_\epsilon)} \int_{M_\epsilon} a \sqrt{\rho}^{-m+1} \omega^m$$

for any $a \in C(M_\epsilon)$. It is only necessary to relate the surface Liouville measures $d\mu_r$ (28) to the Kähler volume measure. One may write $d\mu_r = \frac{d}{dt}|_{t=r} \chi_t \omega^m$, where χ_t is the characteristic function of $M_t = \{\sqrt{\rho} \leq t\}$. By homogeneity of $|\xi|_g$, $\mu_r(\partial M_r) = r^{m-1} \mu_\epsilon(\partial M_\epsilon)$. If $a \in C(M_\epsilon)$, then $\int_{M_\epsilon} a \omega^m = \int_0^\epsilon \{\int_{\partial M_r} a d\mu_r\} dr$. By Lemma 10.9, we have

$$\begin{aligned} (105) \qquad \int_{M_\epsilon} a |U_{j_k}|^2 \omega^m &= \int_0^\epsilon \{\int_{\partial M_r} a |u_{j_k}^r|^2 d\mu_r\} dr \rightarrow \int_0^\epsilon \left\{ \frac{1}{\mu_r(\partial B_r^*)} \int_{\partial M_r} a d\mu_r \right\} dr \\ &= \frac{1}{\mu_\epsilon(\partial M_\epsilon)} \int_{M_\epsilon} a r^{-m+1} \omega^m, \\ &\implies w^* - \lim_{\lambda \rightarrow \infty} |U_{j_k}|^2 = \frac{1}{\mu_1(\partial M_\epsilon)} \sqrt{\rho}^{-m+1}. \end{aligned}$$

10.4. Proof of Lemma 10.8. In fact, one has

$$\frac{1}{\lambda} \log \rho_\lambda(x, \xi) \rightarrow \sqrt{\rho}, \quad \text{uniformly in } M_\epsilon \text{ as } \lambda \rightarrow \infty.$$

Proof. Again using $U(i\epsilon)\varphi_\lambda = e^{-\lambda\epsilon}\varphi_\lambda^{\mathbb{C}}$, we have:

$$\begin{aligned} \rho_\lambda^2(x, \xi) &= \langle \Pi_\epsilon \varphi_\lambda^{\mathbb{C}}, \Pi_\epsilon \varphi_\lambda^{\mathbb{C}} \rangle_{L^2(\partial B_\epsilon^* M)} \quad (\epsilon = |\xi|_{g_x}) \\ (106) \qquad &= e^{2\lambda\epsilon} \langle \Pi_\epsilon U(i\epsilon)\varphi_\lambda, \Pi_\epsilon U(i\epsilon)\varphi_\lambda \rangle_{L^2(\partial B_\epsilon^* M)} \\ &= e^{2\lambda\epsilon} \langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_\lambda, \varphi_\lambda \rangle_{L^2(M)}. \end{aligned}$$

Hence,

$$(107) \qquad \frac{2}{\lambda} \log \rho_\lambda(x, \xi) = 2|\xi|_{g_x} + \frac{1}{\lambda} \log \langle U(i\epsilon)^* \Pi_\epsilon U(i\epsilon)\varphi_\lambda, \varphi_\lambda \rangle.$$

The second term on the right side is the matrix element of a pseudo-differential operator, hence is bounded by some power of λ . Taking the logarithm gives a remainder of order $\frac{\log \lambda}{\lambda}$. \square

10.5. Proof of Lemma 10.7.

Proof. We wish to prove that

$$\psi_j := \frac{1}{\lambda_j} \log |U_j|^2 \rightarrow 0 \quad \text{in } L^1(M_\epsilon).$$

As we have said, this is almost obvious from Lemmas 10.6 and 10.9. If the conclusion is not true, then there exists a subsequence ψ_{j_k} satisfying $\|\psi_{j_k}\|_{L^1(B_\epsilon^* M)} \geq \delta > 0$. To obtain a contradiction, we use Lemma 8.1.

To see that the hypotheses are satisfied in our example, it suffices to prove these statements on each surface ∂M_ϵ with uniform constants independent of ϵ . On the surface ∂M_ϵ , $U_j = u_j^\epsilon$. By the Sobolev inequality in $\mathcal{O}^{\frac{m-1}{4}}(\partial M_\epsilon)$, we have

$$\begin{aligned} \sup_{(x, \xi) \in \partial M_\epsilon} |u_j^\epsilon(x, \xi)| &\leq \lambda_j^m \|u_j^\epsilon(x, \xi)\|_{L^2(\partial M_\epsilon)} \\ &\leq \lambda_j^m. \end{aligned}$$

Taking the logarithm, dividing by λ_j , and combining with the limit formula of Lemma 10.8 proves (i) - (ii).

We now settle the dichotomy above by proving that the sequence $\{\psi_j\}$ does not tend uniformly to $-\infty$ on compact sets. That would imply that $\psi_j \rightarrow -\infty$ uniformly on the spheres ∂M_ϵ for each $\epsilon < \epsilon_0$. Hence, for each ϵ , there would exist $K > 0$ such that for $k \geq K$,

$$(108) \qquad \frac{1}{\lambda_{j_k}} \log |u_{j_k}^\epsilon(z)| \leq -1.$$

However, (108) implies that

$$|u_{j_k}(z)| \leq e^{-2\lambda_{j_k}} \quad \forall z \in \partial M_\epsilon,$$

which is inconsistent with the hypothesis that $|u_{j_k}^\epsilon(z)| \rightarrow 1$ in $\mathcal{D}'(\partial M_\epsilon)$.

Therefore, there must exist a subsequence, which we continue to denote by $\{\psi_{j_k}\}$, which converges in $L^1(M_{\epsilon_0})$ to some $\psi \in L^1(M_{\epsilon_0})$. Then,

$$\psi(z) = \limsup_{k \rightarrow \infty} \psi_{j_k} \leq 2|\xi|_g \quad (\text{a.e.}).$$

Now let

$$\psi^*(z) := \limsup_{w \rightarrow z} \psi(w) \leq 0$$

be the upper-semicontinuous regularization of ψ . Then ψ^* is plurisubharmonic on M_ϵ and $\psi^* = \psi$ almost everywhere.

If $\psi^* \leq 2|\xi|_g - \delta$ on a set U_δ of positive measure, then $\psi_{j_k}(\zeta) \leq -\delta/2$ for $\zeta \in U_\delta$, $k \geq K$; i.e.,

$$(109) \quad |\psi_{j_k}(\zeta)| \leq e^{-\delta\lambda_{j_k}}, \quad \zeta \in U_\delta, \quad k \geq K.$$

This contradicts the weak convergence to 1 and concludes the proof. □

11. INTERSECTIONS OF NODAL SETS AND ANALYTIC CURVES ON REAL ANALYTIC SURFACES

It is often possible to obtain more refined results on nodal sets by studying their intersections with some fixed (and often special) hypersurface. This has been most successful in dimension two. In §11.1 we discuss upper bounds on the number of intersection points of the nodal set with the boundary of a real analytic plane domain and more general ‘good’ analytic curves. To obtain lower bounds or asymptotics, we need to add some dynamical hypotheses. In case of ergodic geodesic flow, we can obtain equidistribution theorems for intersections of nodal sets and geodesics on surfaces. The dimensional restriction is due to the fact that the results are partly based on the quantum ergodic restriction theorems of [TZ, TZ2], which concern restrictions of eigenfunctions to hypersurfaces. Nodal sets and geodesics have complementary dimensions and intersect in points, and therefore it makes sense to count the number of intersections. But we do not yet have a mechanism for studying restrictions to geodesics when $\dim M \geq 3$.

11.1. Counting nodal lines which touch the boundary in analytic plane domains.

In this section, we review the results of [TZ] giving upper bounds on the number of intersections of the nodal set with the boundary of an analytic (or more generally piecewise analytic) plane domain. One may expect that the results of this section can also be generalized to higher dimensions by measuring codimension two nodal hypersurface volumes within the boundary.

Thus we would like to count the number of nodal lines (i.e. components of the nodal set) which touch the boundary. Here we assume that 0 is a regular value so that components of the nodal set are either loops in the interior (closed nodal loops) or curves which touch the boundary in two points (open nodal lines). It is known that for generic piecewise analytic plane domains, zero is a regular value of all the eigenfunctions φ_{λ_j} , i.e. $\nabla \varphi_{\lambda_j} \neq 0$ on $\mathcal{N}_{\varphi_{\lambda_j}}[U]$; we then call the nodal set regular. Since the boundary lies in the nodal set for Dirichlet boundary conditions, we remove it from the nodal set before counting components.

Henceforth, the number of components of the nodal set in the Dirichlet case means the number of components of $\mathcal{N}_{\varphi_{\lambda_j}} \setminus \partial\Omega$.

We now sketch the proof of Theorems 6 in the case of Neumann boundary conditions. By a piecewise analytic domain $\Omega^2 \subset \mathbb{R}^2$, we mean a compact domain with piecewise analytic boundary, i.e. $\partial\Omega$ is a union of a finite number of piecewise analytic curves which intersect only at their common endpoints. Such domains are often studied as archtypes of domains with ergodic billiards and quantum chaotic eigenfunctions, in particular the Bunimovich stadium or Sinai billiard.

For the Neumann problem, the boundary nodal points are the same as the zeros of the boundary values $\varphi_{\lambda_j}|_{\partial\Omega}$ of the eigenfunctions. The number of boundary nodal points is thus twice the number of open nodal lines. Hence in the Neumann case, the Theorem follows from:

THEOREM 11.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain. Then the number $n(\lambda_j) = \#\mathcal{N}_{\varphi_{\lambda_j}} \cap \partial\Omega$ of zeros of the boundary values $\varphi_{\lambda_j}|_{\partial\Omega}$ of the j th Neumann eigenfunction satisfies $n(\lambda_j) \leq C_\Omega \lambda_j$, for some $C_\Omega > 0$.*

This is a more precise version of Theorem 6 since it does not assume that 0 is a regular value. We prove Theorem 11.1 by analytically continuing the boundary values of the eigenfunctions and counting *complex zeros and critical points* of analytic continuations of Cauchy data of eigenfunctions. When $\partial\Omega \in C^\omega$, the eigenfunctions can be holomorphically continued to an open tube domain in \mathbb{C}^2 projecting over an open neighborhood W in \mathbb{R}^2 of Ω which is independent of the eigenvalue. We denote by $\Omega_{\mathbb{C}} \subset \mathbb{C}^2$ the points $\zeta = x + i\xi \in \mathbb{C}^2$ with $x \in \Omega$. Then $\varphi_{\lambda_j}(x)$ extends to a holomorphic function $\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)$ where $x \in W$ and where $|\xi| \leq \epsilon_0$ for some $\epsilon_0 > 0$.

Assuming $\partial\Omega$ real analytic, we define the (interior) complex nodal set by

$$\mathcal{N}_{\varphi_{\lambda_j}}^{\mathbb{C}} = \{\zeta \in \Omega_{\mathbb{C}} : \varphi_{\lambda_j}^{\mathbb{C}}(\zeta) = 0\}.$$

THEOREM 11.2. *Suppose that $\Omega \subset \mathbb{R}^2$ is a piecewise real analytic plane domain, and denote by $(\partial\Omega)_{\mathbb{C}}$ the union of the complexifications of its real analytic boundary components.*

(1) *Let $n(\lambda_j, \partial\Omega_{\mathbb{C}}) = \#Z_{\varphi_{\lambda_j}}^{\partial\Omega_{\mathbb{C}}}$ be the number of complex zeros on the complex boundary.*

Then there exists a constant $C_\Omega > 0$ independent of the radius of $(\partial\Omega)_{\mathbb{C}}$ such that $n(\lambda_j, \partial\Omega_{\mathbb{C}}) \leq C_\Omega \lambda_j$.

The theorems on real nodal lines and critical points follow from the fact that real zeros and critical points are also complex zeros and critical points, hence

$$(110) \quad n(\lambda_j) \leq n(\lambda_j, \partial\Omega_{\mathbb{C}}).$$

All of the results are sharp, and are already obtained for certain sequences of eigenfunctions on a disc (see §4.6).

To prove 11.2, we represent the analytic continuations of the boundary values of the eigenfunctions in terms of layer potentials. Let $G(\lambda_j, x_1, x_2)$ be any ‘Green’s function’ for the Helmholtz equation on Ω , i.e. a solution of $(-\Delta - \lambda_j^2)G(\lambda_j, x_1, x_2) = \delta_{x_1}(x_2)$ with $x_1, x_2 \in \bar{\Omega}$. By Green’s formula,

$$(111) \quad \varphi_{\lambda_j}(x, y) = \int_{\partial\Omega} (\partial_\nu G(\lambda_j, q, (x, y))\varphi_{\lambda_j}(q) - G(\lambda_j, q, (x, y))\partial_\nu \varphi_{\lambda_j}(q)) d\sigma(q),$$

where $(x, y) \in \mathbb{R}^2$, where $d\sigma$ is arc-length measure on $\partial\Omega$ and where ∂_ν is the normal derivative by the interior unit normal. Our aim is to analytically continue this formula.

In the case of Neumann eigenfunctions φ_λ in Ω ,

$$(112) \quad \varphi_{\lambda_j}(x, y) = \int_{\partial\Omega} \frac{\partial}{\partial\nu_q} G(\lambda_j, q, (x, y)) u_{\lambda_j}(q) d\sigma(q), \quad (x, y) \in \Omega^\circ \quad (\text{Neumann}).$$

To obtain concrete representations we need to choose G . We choose the real ambient Euclidean Green's function S

$$(113) \quad S(\lambda_j, \xi, \eta; x, y) = -Y_0(\lambda_j r((x, y); (\xi, \eta))),$$

where $r = \sqrt{zz^*}$ is the distance function (the square root of r^2 above) and where Y_0 is the Bessel function of order zero of the second kind. The Euclidean Green's function has the form

$$(114) \quad S(\lambda_j, \xi, \eta; x, y) = A(\lambda_j, \xi, \eta; x, y) \log \frac{1}{r} + B(\lambda_j, \xi, \eta; x, y),$$

where A and B are entire functions of r^2 . The coefficient $A = J_0(\lambda_j r)$ is known as the Riemann function.

By the 'jumps' formulae, the double layer potential $\frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, (x, y))$ on $\partial\Omega \times \bar{\Omega}$ restricts to $\partial\Omega \times \partial\Omega$ as $\frac{1}{2}\delta_{\tilde{q}}(\tilde{q}) + \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q)$ (see e.g. [TI, TII]). Hence in the Neumann case the boundary values u_{λ_j} satisfy,

$$(115) \quad u_{\lambda_j}(q) = 2 \int_{\partial\Omega} \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q) u_{\lambda_j}(\tilde{q}) d\sigma(\tilde{q}) \quad (\text{Neumann}).$$

We have,

$$(116) \quad \frac{\partial}{\partial\nu_{\tilde{q}}} S(\lambda_j, \tilde{q}, q) = -\lambda_j Y_1(\lambda_j r) \cos \angle(q - \tilde{q}, \nu_{\tilde{q}}).$$

It is equivalent, and sometimes more convenient, to use the (complex valued) Euclidean outgoing Green's function $\text{Ha}_0^{(1)}(kz)$, where $\text{Ha}_0^{(1)} = J_0 + iY_0$ is the Hankel function of order zero. It has the same form as (114) and only differs by the addition of the even entire function J_0 to the B term. If we use the Hankel free outgoing Green's function, then in place of (116) we have the kernel

$$(117) \quad \begin{aligned} N(\lambda_j, q(s), q(s')) &= \frac{i}{2} \partial_{\nu_y} \text{Ha}_0^{(1)}(\lambda_j |q(s) - y|)|_{y=q(s')} \\ &= -\frac{i}{2} \lambda_j \text{Ha}_1^{(1)}(\lambda_j |q(s) - q(s')|) \cos \angle(q(s') - q(s), \nu_{q(s')}), \end{aligned}$$

and in place of (115) we have the formula

$$(118) \quad u_{\lambda_j}(q(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q(t)) u_{\lambda_j}(q(s)) ds.$$

The next step is to analytically continue the layer potential representations (115) and (118). The main point is to express the analytic continuations of Cauchy data of Neumann and Dirichlet eigenfunctions in terms of the real Cauchy data. For brevity, we only consider (115) but essentially the same arguments apply to the free outgoing representation (118).

As mentioned above, both $A(\lambda_j, \xi, \eta, x, y)$ and $B(\lambda_j, \xi, \eta, x, y)$ admit analytic continuations. In the case of A , we use a traditional notation $R(\zeta, \zeta^*, z, z^*)$ for the analytic continuation and for simplicity of notation we omit the dependence on λ_j .

The details of the analytic continuation are complicated when the curve is the boundary, and they simplify when the curve is interior. So we only continue the sketch of the proof in the interior case.

As above, the arc-length parametrization of C is denoted by $q_C : [0, 2\pi] \rightarrow C$ and the corresponding arc-length parametrization of the boundary, $\partial\Omega$, by $q : [0, 2\pi] \rightarrow \partial\Omega$. Since the boundary and C do not intersect, the logarithm $\log r^2(q(s); q_C^C(t))$ is well defined and the holomorphic continuation of equation (118) is given by:

$$(119) \quad \varphi_{\lambda_j}^C(q_C^C(t)) = \int_0^{2\pi} N(\lambda_j, q(s), q_C^C(t)) u_{\lambda_j}(q(s)) d\sigma(s),$$

From the basic formula (117) for $N(\lambda_j, q, q_C)$ and the standard integral formula for the Hankel function $\text{Ha}_1^{(1)}(z)$, one easily gets an asymptotic expansion in λ_j of the form:

$$(120) \quad N(\lambda_j, q(s), q_C^C(t)) = e^{i\lambda_j r(q(s); q_C^C(t))} \sum_{m=0}^k a_m(q(s), q_C^C(t)) \lambda_j^{1/2-m} + O(e^{i\lambda_j r(q(s); q_C^C(t))} \lambda_j^{1/2-k-1}).$$

Note that the expansion in (120) is valid since for interior curves,

$$C_0 := \min_{(q_C(t), q(s)) \in C \times \partial\Omega} |q_C(t) - q(s)|^2 > 0.$$

Then, $\text{Re } r^2(q(s); q_C^C(t)) > 0$ as long as

$$(121) \quad |\text{Im } q_C^C(t)|^2 < C_0.$$

So, the principal square root of r^2 has a well-defined holomorphic extension to the tube (121) containing C . We have denoted this square root by r in (120).

Substituting (120) in the analytically continued single layer potential integral formula (119) proves that for $t \in A(\epsilon)$ and $\lambda_j > 0$ sufficiently large,

$$(122) \quad \varphi_{\lambda_j}^C(q_C^C(t)) = 2\pi \lambda_j^{1/2} \int_0^{2\pi} e^{i\lambda_j r(q(s); q_C^C(t))} a_0(q(s), q_C^C(t)) (1 + O(\lambda_j^{-1})) u_{\lambda_j}(q(s)) d\sigma(s).$$

Taking absolute values of the integral on the RHS in (122) and applying the Cauchy-Schwartz inequality proves

LEMMA 11.3. *For $t \in [0, 2\pi] + i[-\epsilon, \epsilon]$ and $\lambda_j > 0$ sufficiently large*

$$|\varphi_{\lambda_j}^C(q_C^C(t))| \leq C_1 \lambda_j^{1/2} \exp \lambda_j (\max_{q(s) \in \partial\Omega} \text{Re } ir(q(s); q_C^C(t))) \cdot \|u_{\lambda_j}\|_{L^2(\partial\Omega)}.$$

From the pointwise upper bounds in Lemma 11.3, it is immediate that

$$(123) \quad \log \max_{q_C^C(t) \in Q_C^C(A(\epsilon))} |\varphi_{\lambda_j}^C(q_C^C(t))| \leq C_{\max} \lambda_j + C_2 \log \lambda_j + \log \|u_{\lambda_j}\|_{L^2(\partial\Omega)},$$

where,

$$C_{\max} = \max_{(q(s), q_C^C(t)) \in \partial\Omega \times Q_C^C(A(\epsilon))} \text{Re } ir(q(s); q_C^C(t)).$$

Finally, we use that $\log \|u_{\lambda_j}\|_{L^2(\partial\Omega)} = O(\lambda_j)$ by the assumption that C is a good curve and apply Proposition 11.4 to get that $n(\lambda_j, C) = O(\lambda_j)$.

The following estimate, suggested by Lemma 6.1 of Donnelly-Fefferman [DF], gives an upper bound on the number of zeros in terms of the growth of the family:

PROPOSITION 11.4. *Suppose that C is a good real analytic curve in the sense of (29). Normalize u_{λ_j} so that $\|u_{\lambda_j}\|_{L^2(C)} = 1$. Then, there exists a constant $C(\epsilon) > 0$ such that for any $\epsilon > 0$,*

$$n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2))) \leq C(\epsilon) \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} \log |u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))|.$$

Proof. Let G_ϵ denote the Dirichlet Green's function of the 'annulus' $Q_C^{\mathbb{C}}(A(\epsilon))$. Also, let $\{a_k\}_{k=1}^{n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2)))}$ denote the zeros of $u_{\lambda_j}^{\mathbb{C}}$ in the sub-annulus $Q_C^{\mathbb{C}}(A(\epsilon/2))$. Let $U_{\lambda_j} = \frac{u_{\lambda_j}^{\mathbb{C}}}{\|u_{\lambda_j}^{\mathbb{C}}\|_{Q_C^{\mathbb{C}}(A(\epsilon))}}$ where $\|u\|_{Q_C^{\mathbb{C}}(A(\epsilon))} = \max_{\zeta \in Q_C^{\mathbb{C}}(A(\epsilon))} |u(\zeta)|$. Then,

$$\begin{aligned} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| &= \int_{Q_C^{\mathbb{C}}(A(\epsilon/2))} G_\epsilon(q_C^{\mathbb{C}}(t), w) \partial \bar{\partial} \log |u_{\lambda_j}^{\mathbb{C}}(w)| + H_{\lambda_j}(q_C^{\mathbb{C}}(t)) \\ &= \sum_{a_k \in Q_C^{\mathbb{C}}(A(\epsilon/2)): u_{\lambda_j}^{\mathbb{C}}(a_k)=0} G_\epsilon(q_C^{\mathbb{C}}(t), a_k) + H_{\lambda_j}(q_C^{\mathbb{C}}(t)), \end{aligned}$$

since $\partial \bar{\partial} \log |u_{\lambda_j}^{\mathbb{C}}(w)| = \sum_{a_k \in C_{\mathbb{C}}: u_{\lambda_j}^{\mathbb{C}}(a_k)=0} \delta_{a_k}$. Moreover, the function H_{λ_j} is sub-harmonic on $Q_C^{\mathbb{C}}(A(\epsilon))$ since

$$\begin{aligned} \partial \bar{\partial} H_{\lambda_j} &= \partial \bar{\partial} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| - \sum_{a_k \in Q_C^{\mathbb{C}}(A(\epsilon/2)): u_{\lambda_j}^{\mathbb{C}}(a_k)=0} \partial \bar{\partial} G_\epsilon(q_C^{\mathbb{C}}(t), a_k) \\ &= \sum_{a_k \in Q_C^{\mathbb{C}}(A(\epsilon)) \setminus Q_C^{\mathbb{C}}(A(\epsilon/2))} \delta_{a_k} > 0. \end{aligned}$$

So, by the maximum principle for subharmonic functions,

$$\max_{Q_C^{\mathbb{C}}(A(\epsilon))} H_{\lambda_j}(q_C^{\mathbb{C}}(t)) \leq \max_{\partial Q_C^{\mathbb{C}}(A(\epsilon))} H_{\lambda_j}(q_C^{\mathbb{C}}(t)) = \max_{\partial Q_C^{\mathbb{C}}(A(\epsilon))} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| = 0.$$

It follows that

$$(124) \quad \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| \leq \sum_{a_k \in Q_C^{\mathbb{C}}(A(\epsilon/2)): u_{\lambda_j}^{\mathbb{C}}(a_k)=0} G_\epsilon(q_C^{\mathbb{C}}(t), a_k),$$

hence that

$$(125) \quad \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon/2))} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| \leq \left(\max_{z, w \in Q_C^{\mathbb{C}}(A(\epsilon/2))} G_\epsilon(z, w) \right) n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2))).$$

Now $G_\epsilon(z, w) \leq \max_{w \in Q_C^{\mathbb{C}}(\partial A(\epsilon))} G_\epsilon(z, w) = 0$ and $G_\epsilon(z, w) < 0$ for $z, w \in Q_C^{\mathbb{C}}(A(\epsilon/2))$. It follows that there exists a constant $\nu(\epsilon) < 0$ so that $\max_{z, w \in Q_C^{\mathbb{C}}(A(\epsilon/2))} G_\epsilon(z, w) \leq \nu(\epsilon)$. Hence,

$$(126) \quad \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon/2))} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| \leq \nu(\epsilon) n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2))).$$

Since both sides are negative, we obtain

$$\begin{aligned}
 n(\lambda_j, Q_C^{\mathbb{C}}(A(\epsilon/2))) &\leq \frac{1}{|\nu(\epsilon)|} \left| \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon/2))} \log |U_{\lambda_j}(q_C^{\mathbb{C}}(t))| \right| \\
 (127) \quad &\leq \frac{1}{|\nu(\epsilon)|} \left(\max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} \log |u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| - \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon/2))} \log |u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| \right) \\
 &\leq \frac{1}{|\nu(\epsilon)|} \max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon))} \log |u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))|,
 \end{aligned}$$

where in the last step we use that $\max_{q_C^{\mathbb{C}}(t) \in Q_C^{\mathbb{C}}(A(\epsilon/2))} \log |u_{\lambda_j}^{\mathbb{C}}(q_C^{\mathbb{C}}(t))| \geq 0$, which holds since $|u_{\lambda_j}^{\mathbb{C}}| \geq 1$ at some point in $Q_C^{\mathbb{C}}(A(\epsilon/2))$. Indeed, by our normalization, $\|u_{\lambda_j}\|_{L^2(C)} = 1$, and so there must already exist points on the real curve C with $|u_{\lambda_j}| \geq 1$. Putting $C(\epsilon) = \frac{1}{|\nu(\epsilon)|}$ finishes the proof. \square

This completes the proof of Theorem 7.

11.2. Application to Pleijel's conjecture. I. Polterovich [Po] observed that Theorem 6 can be used to prove an old conjecture of A. Pleijel regarding Courant's nodal domain theorem, which says that the number n_k of nodal domains (components of $\Omega \setminus Z_{\varphi_{\lambda_k}}$) of the k th eigenfunction satisfies $n_k \leq k$. Pleijel improved this result for Dirichlet eigenfunctions of plane domains: For any plane domain with Dirichlet boundary conditions, $\limsup_{k \rightarrow \infty} \frac{n_k}{k} \leq \frac{4}{j_1^2} \simeq 0.691\dots$, where j_1 is the first zero of the J_0 Bessel function. He conjectured that the same result should be true for a free membrane, i.e. for Neumann boundary conditions. This was recently proved in the real analytic case by I. Polterovich [Po]. His argument is roughly the following: Pleijel's original argument applies to all nodal domains which do not touch the boundary, since the eigenfunction is a Dirichlet eigenfunction in such a nodal domain. The argument does not apply to nodal domains which touch the boundary, but by the Theorem above the number of such domains is negligible for the Pleijel bound.

11.3. Equidistribution of intersections of nodal lines and geodesics on surfaces.

We fix $(x, \xi) \in S^*M$ and let

$$(128) \quad \gamma_{x,\xi} : \mathbb{R} \rightarrow M, \quad \gamma_{x,\xi}(0) = x, \quad \gamma'_{x,\xi}(0) = \xi \in T_x M$$

denote the corresponding parametrized geodesic. Our goal is to determine the asymptotic distribution of intersection points of $\gamma_{x,\xi}$ with the nodal set of a highly eigenfunction. As usual, we cannot cope with this problem in the real domain and therefore analytically continue it to the complex domain. Thus, we consider the intersections

$$\mathcal{N}_{\lambda_j}^{\gamma_{x,\xi}^{\mathbb{C}}} = Z_{\varphi_j^{\mathbb{C}}} \cap \gamma_{x,\xi}^{\mathbb{C}}$$

of the complex nodal set with the (image of the) complexification of a generic geodesic If

$$(129) \quad S_\epsilon = \{(t + i\tau) \in \mathbb{C} : |\tau| \leq \epsilon\}$$

then $\gamma_{x,\xi}$ admits an analytic continuation

$$(130) \quad \gamma_{x,\xi}^{\mathbb{C}} : S_\epsilon \rightarrow M_\epsilon.$$

In other words, we consider the zeros of the pullback,

$$\{\gamma_{x,\xi}^* \varphi_{\lambda_j}^{\mathbb{C}} = 0\} \subset S_\epsilon.$$

We encode the discrete set by the measure

$$(131) \quad [\mathcal{N}_{\lambda_j}^{\gamma_{x,\xi}^c}] = \sum_{(t+i\tau): \varphi_j^c(\gamma_{x,\xi}^c(t+i\tau))=0} \delta_{t+i\tau}.$$

We would like to show that for generic geodesics, the complex zeros on the complexified geodesic condense on the real points and become uniformly distributed with respect to arclength. This does not always occur: as in our discussion of QER theorems, if $\gamma_{x,\xi}$ is the fixed point set of an isometric involution, then ‘‘odd’’ eigenfunctions under the involution will vanish on the geodesic. The additional hypothesis is that QER holds for $\gamma_{x,\xi}$. The following is proved ([Ze3]):

THEOREM 11.5. *Let (M^2, g) be a real analytic Riemannian surface with ergodic geodesic flow. Let $\gamma_{x,\xi}$ satisfy the QER hypothesis. Then there exists a subsequence of eigenvalues λ_{j_k} of density one such that for any $f \in C_c(S_\epsilon)$,*

$$\lim_{k \rightarrow \infty} \sum_{(t+i\tau): \varphi_j^c(\gamma_{x,\xi}^c(t+i\tau))=0} f(t+i\tau) = \int_{\mathbb{R}} f(t) dt.$$

In other words,

$$\text{weak}^* \lim_{k \rightarrow \infty} \frac{i}{\pi \lambda_{j_k}} [\mathcal{N}_{\lambda_{j_k}}^{\gamma_{x,\xi}^c}] = \delta_{\tau=0},$$

in the sense of weak* convergence on $C_c(S_\epsilon)$. Thus, the complex nodal set intersects the (parametrized) complexified geodesic in a discrete set which is asymptotically (as $\lambda \rightarrow \infty$) concentrated along the real geodesic with respect to its arclength.

This concentration- equidistribution result is a ‘restricted’ version of the result of §10. As noted there, the limit distribution of complex nodal sets in the ergodic case is a singular current $dd^c \sqrt{\rho}$. The motivation for restricting to geodesics is that restriction magnifies the singularity of this current. In the case of a geodesic, the singularity is magnified to a delta-function; for other curves there is additionally a smooth background measure.

The assumption of ergodicity is crucial. For instance, in the case of a flat torus, say \mathbb{R}^2/L where $L \subset \mathbb{R}^2$ is a generic lattice, the real eigenfunctions are $\cos \langle \lambda, x \rangle, \sin \langle \lambda, x \rangle$ where $\lambda \in L^*$, the dual lattice, with eigenvalue $-|\lambda|^2$. Consider a geodesic $\gamma_{x,\xi}(t) = x + t\xi$. Due to the flatness, the restriction $\sin \langle \lambda, x_0 + t\xi_0 \rangle$ of the eigenfunction to a geodesic is an eigenfunction of the Laplacian $-\frac{d^2}{dt^2}$ of submanifold metric along the geodesic with eigenvalue $-\langle \lambda, \xi_0 \rangle^2$. The complexification of the restricted eigenfunction is $\sin \langle \lambda, x_0 + (t+i\tau)\xi_0 \rangle$ and its exponent of its growth is $\tau |\langle \frac{\lambda}{|\lambda|}, \xi_0 \rangle|$, which can have a wide range of values as the eigenvalue moves along different rays in L^* . The limit current is $i\partial\bar{\partial}$ applied to the limit and thus also has many limits

The proof involves several new principles which played no role in the global result of §10 and which are specific to geodesics. However, the first steps in the proof are the same as in the global case. By the Poincaré-Lelong formula, we may express the current of summation over the intersection points in (131) in the form,

$$(132) \quad [\mathcal{N}_{\lambda_j}^{\gamma_{x,\xi}^c}] = i\partial\bar{\partial}_{t+i\tau} \log \left| \gamma_{x,\xi}^* \varphi_{\lambda_j}^c(t+i\tau) \right|^2.$$

Thus, the main point of the proof is to determine the asymptotics of $\frac{1}{\lambda_j} \log \left| \gamma_{x,\xi}^* \varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau) \right|^2$. When we freeze τ we put

$$(133) \quad \gamma_{x,\xi}^\tau(t) = \gamma_{x,\xi}^{\mathbb{C}}(t + i\tau).$$

PROPOSITION 11.6. (*Growth saturation*) *If $\{\varphi_{j_k}\}$ satisfies QER along any arcs of $\gamma_{x,\xi}$, then in $L^1_{loc}(S_\tau)$, we have*

$$\lim_{k \rightarrow \infty} \frac{1}{\lambda_{j_k}} \log \left| \gamma_{x,\xi}^{\tau*} \varphi_{\lambda_{j_k}}^{\mathbb{C}}(t + i\tau) \right|^2 = |\tau|.$$

Proposition 11.6 immediately implies Theorem 11.5 since we can apply $\partial\bar{\partial}$ to the L^1 convergent sequence $\frac{1}{\lambda_{j_k}} \log \left| \gamma_{x,\xi}^* \varphi_{\lambda_{j_k}}^{\mathbb{C}}(t + i\tau) \right|^2$ to obtain $\partial\bar{\partial}|\tau|$.

The upper bound in Proposition 11.6 follows immediately from the known global estimate

$$\lim_{k \rightarrow \infty} \frac{1}{\lambda_j} \log |\varphi_{j_k}(\gamma_{x,\xi}^{\mathbb{C}}(\zeta))| \leq |\tau|$$

on all of ∂M_τ . Hence the difficult point is to prove that this growth rate is actually obtained upon restriction to $\gamma_{x,\xi}^{\mathbb{C}}$. This requires new kinds of arguments related to the QER theorem.

- Complexifications of restrictions of eigenfunctions to geodesics have incommensurate Fourier modes, i.e. higher modes are exponentially larger than lower modes.
- The quantum ergodic restriction theorem in the real domain shows that the Fourier coefficients of the top allowed modes are ‘large’ (i.e. as large as the lower modes). Consequently, the L^2 norms of the complexified eigenfunctions along arcs of $\gamma_{x,\xi}^{\mathbb{C}}$ achieve the lower bound of Proposition 11.6.
- Invariance of Wigner measures along the geodesic flow implies that the Wigner measures of restrictions of complexified eigenfunctions to complexified geodesics should tend to constant multiples of Lebesgue measures dt for each $\tau > 0$. Hence the eigenfunctions everywhere on $\gamma_{x,\xi}^{\mathbb{C}}$ achieve the growth rate of the L^2 norms.

These principles are most easily understood in the case of periodic geodesics. We let $\gamma_{x,\xi} : S^1 \rightarrow M$ parametrize the geodesic with arc-length (where $S^1 = \mathbb{R}/L\mathbb{Z}$ where L is the length of $\gamma_{x,\xi}$).

LEMMA 11.7. *Assume that $\{\varphi_j\}$ satisfies QER along the periodic geodesic $\gamma_{x,\xi}$. Let $\|\gamma_{x,\xi}^{\tau*} \varphi_j^{\mathbb{C}}\|_{L^2(S^1)}^2$ be the L^2 -norm of the complexified restriction of φ_j along $\gamma_{x,\xi}^\tau$. Then,*

$$\lim_{\lambda_j \rightarrow \infty} \frac{1}{\lambda_j} \log \|\gamma_{x,\xi}^{\tau*} \varphi_j^{\mathbb{C}}\|_{L^2(S^1)}^2 = |\tau|.$$

To prove Lemma 11.7, we study the orbital Fourier series of $\gamma_{x,\xi}^{\tau*} \varphi_j$ and of its complexification. The orbital Fourier coefficients are

$$\nu_{\lambda_j}^{x,\xi}(n) = \frac{1}{L_\gamma} \int_0^{L_\gamma} \varphi_{\lambda_j}(\gamma_{x,\xi}(t)) e^{-\frac{2\pi i n t}{L_\gamma}} dt,$$

and the orbital Fourier series is

$$(134) \quad \varphi_{\lambda_j}(\gamma_{x,\xi}(t)) = \sum_{n \in \mathbb{Z}} \nu_{\lambda_j}^{x,\xi}(n) e^{\frac{2\pi i n t}{L_\gamma}}.$$

Hence the analytic continuation of $\gamma_{x,\xi}^{\tau*} \varphi_j$ is given by

$$(135) \quad \varphi_{\lambda_j}^{\mathbb{C}}(\gamma_{x,\xi}(t + i\tau)) = \sum_{n \in \mathbb{Z}} \nu_{\lambda_j}^{x,\xi}(n) e^{\frac{2\pi i n(t+i\tau)}{L_\gamma}}.$$

By the Paley-Wiener theorem for Fourier series, the series converges absolutely and uniformly for $|\tau| \leq \epsilon_0$. By “energy localization” only the modes with $|n| \leq \lambda_j$ contribute substantially to the L^2 norm. We then observe that the Fourier modes decouple, since they have different exponential growth rates. We use the QER hypothesis in the following way:

LEMMA 11.8. *Suppose that $\{\varphi_{\lambda_j}\}$ is QER along the periodic geodesic $\gamma_{x,\xi}$. Then for all $\epsilon > 0$, there exists $C_\epsilon > 0$ so that*

$$\sum_{n: |n| \geq (1-\epsilon)\lambda_j} |\nu_{\lambda_j}^{x,\xi}(n)|^2 \geq C_\epsilon.$$

Lemma 11.8 implies Lemma 11.7 since it implies that for any $\epsilon > 0$,

$$\sum_{n: |n| \geq (1-\epsilon)\lambda_j} |\nu_{\lambda_j}^{x,\xi}(n)|^2 e^{-2n\tau} \geq C_\epsilon e^{2\tau(1-\epsilon)\lambda_j}.$$

To go from asymptotics of L^2 norms of restrictions to Proposition 11.6 we then use the third principle:

PROPOSITION 11.9. *(Lebesgue limits) If $\gamma_{x,\xi}^* \varphi_j \neq 0$ (identically), then for all $\tau > 0$ the sequence*

$$U_j^{x,\xi,\tau} = \frac{\gamma_{x,\xi}^{\tau*} \varphi_j^{\mathbb{C}}}{\|\gamma_{x,\xi}^{\tau*} \varphi_j^{\mathbb{C}}\|_{L^2(S^1)}}$$

is QUE with limit measure given by normalized Lebesgue measure on S^1 .

The proof of Proposition 11.6 is completed by combining Lemma 11.7 and Proposition 11.9. Theorem 11.5 follows easily from Proposition 11.6.

The proof for non-periodic geodesics is considerably more involved, since one cannot use Fourier analysis in quite the same way.

11.4. Real zeros and complex analysis.

PROBLEM 4. *An important but apparently rather intractable problem is, how to obtain information on the real zeros from knowledge of the complex nodal distribution? There are several possible approaches:*

- *Try to intersect the nodal current with the current of integration over the real points $M \subset M_\epsilon$. I.e. try to slice the complex nodal set with the real domain.*
- *Thicken the real slice slightly by studying the behavior of the nodal set in M_ϵ as $\epsilon \rightarrow 0$. The sharpest version is to try to re-scale the nodal set by a factor of λ^{-1} to zoom in on the zeros which are within λ^{-1} of the real domain. They may not be real but at least one can control such “almost real” zeros. Try to understand (at least in real dimension 2) how the complex nodal set ‘sprouts’ from the real nodal set. How do the connected components of the real nodal set fit together in the complex nodal set?*

- *Intersect the nodal set with geodesics. This magnifies the singularity along the real domain and converts nodal sets to isolated points.*

12. L^p NORMS OF EIGENFUNCTIONS

In §4.3 we pointed out that lower bounds on $\|\varphi_\lambda\|_{L^1}$ lead to improved lower bounds on Hausdorff measures of nodal sets. In this section we consider general L^p -norm problems for eigenfunctions.

12.1. Generic upper bounds on L^p norms. We have already explained that the pointwise Weyl law (33) and remainder jump estimate (35) leads to the general sup norm bound for L^2 -normalized eigenfunctions,

$$(136) \quad \|\varphi_\lambda\|_{L^\infty} \leq C_g \lambda^{\frac{m-1}{2}}, \quad (m = \dim M).$$

The upper bound is achieved by zonal spherical harmonics. In [Sog] (see also [Sogb, Sogb2]) C.D. Sogge proved general L^p bounds:

THEOREM 12.1. (*Sogge, 1985*)

$$(137) \quad \sup_{\varphi \in V_\lambda} \frac{\|\varphi\|_p}{\|\varphi\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty$$

where

$$(138) \quad \delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

The upper bounds are sharp in the class of all (M, g) and are saturated on the round sphere:

- For $p > \frac{2(n+1)}{n-1}$, zonal (rotationally invariant) spherical harmonics saturate the L^p bounds. Such eigenfunctions also occur on surfaces of revolution.
- For L^p for $2 \leq p \leq \frac{2(n+1)}{n-1}$ the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic. Such eigenfunctions also occur on surfaces of revolution.

The zonal has high L^p norm due to its high peaks on balls of radius $\frac{1}{N}$. The balls are so small that they do not have high L^p norms for small p . The Gaussian beams are not as high but they are relatively high over an entire geodesic.

12.2. Lower bounds on L^1 norms. The L^p upper bounds are the only known tool for obtaining lower bounds on L^1 norms. We now prove Proposition 10:

Proof. Fix a function $\rho \in \mathcal{S}(\mathbb{R})$ having the properties that $\rho(0) = 1$ and $\hat{\rho}(t) = 0$ if $t \notin [\delta/2, \delta]$, where $\delta > 0$ is smaller than the injectivity radius of (M, g) . If we then set

$$T_\lambda f = \rho(\sqrt{-\Delta} - \lambda)f,$$

we have that $T_\lambda \varphi_\lambda = \varphi_\lambda$. Also, by Lemma 5.1.3 in [Sogb], T_λ is an oscillatory integral operator of the form

$$T_\lambda f(x) = \lambda^{\frac{n-1}{2}} \int_M e^{i\lambda r(x,y)} a_\lambda(x, y) f(y) dy,$$

with $|\partial_{x,y}^\alpha a_\lambda(x,y)| \leq C_\alpha$. Consequently, $\|T_\lambda \varphi_\lambda\|_{L^\infty} \leq C \lambda^{\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1}$, with C independent of λ , and so

$$1 = \|\varphi_\lambda\|_{L^2}^2 = \langle T_\lambda \varphi_\lambda, \varphi_\lambda \rangle \leq \|T_\lambda \varphi_\lambda\|_{L^\infty} \|\varphi_\lambda\|_{L^1} \leq C \lambda^{\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1}^2.$$

We can give another proof based on the eigenfunction estimates (Theorem 12.1), which say that

$$\|\varphi_\lambda\|_{L^p} \leq C \lambda^{\frac{(n-1)(p-2)}{4p}}, \quad 2 < p \leq \frac{2(n+1)}{n-1}.$$

If we pick such a $2 < p < \frac{2(n+1)}{n-1}$, then by Hölder's inequality, we have

$$1 = \|\varphi_\lambda\|_{L^2}^{1/\theta} \leq \|\varphi_\lambda\|_{L^1} \|\varphi_\lambda\|_{L^p}^{\frac{1}{\theta}-1} \leq \|\varphi_\lambda\|_{L^1} \left(C \lambda^{\frac{(n-1)(p-2)}{4p}} \right)^{\frac{1}{\theta}-1}, \quad \theta = \frac{p}{p-1} \left(\frac{1}{2} - \frac{1}{p} \right) = \frac{(p-2)}{2(p-1)},$$

which implies $\|\varphi_\lambda\|_{L^1} \geq c \lambda^{-\frac{n-1}{4}}$, since $(1 - \frac{1}{\theta}) \frac{(n-1)(p-2)}{4p} = \frac{n-1}{4}$. \square

We remark that this lowerbound for $\|\varphi_\lambda\|_{L^1}$ is sharp on the standard sphere, since L^2 -normalized highest weight spherical harmonics of degree k with eigenvalue $\lambda^2 = k(k+n-1)$ have L^1 -norms which are bounded above and below by $k^{(n-1)/4}$ as $k \rightarrow \infty$. Similarly, the L^p -upperbounds that we used in the second proof of this L^1 -lowerbound is also sharp because of these functions.

12.3. Riemannian manifolds with maximal eigenfunction growth. Although the general sup norm bound (136) is achieved by some sequences of eigenfunctions on some Riemannian manifolds (the standard sphere or a surface of revolution), it is very rare that (M, g) has such sequences of eigenfunctions. We say that such (M, g) have maximal eigenfunction growth. In a series of articles [SoZ, STZ, SoZ2], ever more stringent conditions are given on such (M, g) . We now go over the results.

Denote the eigenspaces by

$$V_\lambda = \{\varphi : \Delta \varphi = -\lambda^2 \varphi\}.$$

We measure the growth rate of L^p norms by

$$(139) \quad L^p(\lambda, g) = \sup_{\varphi \in V_\lambda : \|\varphi\|_{L^2} = 1} \|\varphi\|_{L^p}.$$

Definition: Say that (M, g) has maximal L^p eigenfunction growth if it possesses a sequence of eigenfunctions $\varphi_{\lambda_{j_k}}$ which saturates the L^p bounds. When $p = \infty$ we say that it has maximal sup norm growth.

PROBLEM 12.2. • Characterize (M, g) with maximal L^∞ eigenfunction growth. The same sequence of eigenfunctions should saturate all L^p norms with $p \geq p_n := \frac{2(n+1)}{n-1}$.

- Characterize (M, g) with maximal L^p eigenfunction growth for $2 \leq p \leq \frac{2(n+1)}{n-1}$.
- Characterize (M, g) for which $\|\varphi_\lambda\|_{L^1} \geq C > 0$.

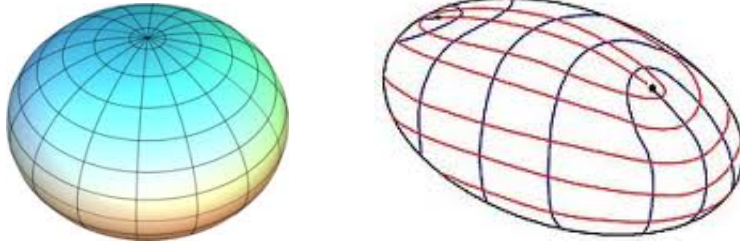
In [SoZ], it was shown that (M, g) of maximal L^p eigenfunction growth for $p \geq p_n$ have self-focal points. The terminology is non-standard and several different terms are used.

Definition:

We call a point p a *self-focal point* or *blow-down point* if all geodesics leaving p loop back to p at a common time T . That is, $\exp_p T\xi = p$ (They do not have to be closed geodesics.)

We call a point p a *partial self-focal point* if there exists a positive measure in S_x^*M of directions ξ which loop back to p .

The poles of a surface of revolution are self-focal and all geodesics close up smoothly (i.e. are closed geodesics). The umbilic points of an ellipsoid are self-focal but only two directions give smoothly closed geodesics (one up to time reversal).



In [SoZ] is proved:

THEOREM 12.3. *Suppose (M, g) is a C^∞ Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $\|\varphi_{\lambda_{j_k}}\|_{L^\infty} \geq C_0 \lambda_{j_k}^{(n-1)/2}$ for some $C_0 > 0$ depending only on (M, g) .*

Then there must exist a point $x \in M$ for which the set

$$(140) \quad \mathcal{L}_x = \{\xi \in S_x^*M : \exists T : \exp_x T\xi = x\}$$

*of directions of geodesic loops at x has positive measure in S_x^*M . Here, \exp is the exponential map, and the measure $|\Omega|$ of a set Ω is the one induced by the metric g_x on T_x^*M . For instance, the poles x_N, x_S of a surface of revolution (S^2, g) satisfy $|\mathcal{L}_x| = 2\pi$.*

Theorem 12.3, Theorem 9, as well as the results of [SoZ, STZ], are proved by studying the remainder term $R(\lambda, x)$ in the pointwise Weyl law,

$$(141) \quad N(\lambda, x) = \sum_{j:\lambda_j \leq \lambda} |\varphi_j(x)|^2 = C_m \lambda^m + R(\lambda, x).$$

The first term $N_W(\lambda) = C_m \lambda^m$ is called the Weyl term. It is classical that the remainder is of one lower order, $R(\lambda, x) = O(\lambda^{m-1})$. The relevance of the remainder to maximal eigenfunction growth is through the following well-known Lemma (see e.g. [SoZ]):

LEMMA 12.4. *Fix $x \in M$. Then if $\lambda \in \text{spec} \sqrt{-\Delta}$*

$$(142) \quad \sup_{\varphi \in V_\lambda} \frac{|\varphi(x)|}{\|\varphi\|_2} = \sqrt{R(\lambda, x) - R(\lambda - 0, x)}.$$

Here, for a right continuous function $f(x)$ we denote by $f(x+0) - f(x-0)$ the jump of f at x . Thus, Theorem 12.3 follows from

THEOREM 12.5. *Let $R(\lambda, x)$ denote the remainder for the local Weyl law at x . Then*

$$(143) \quad R(\lambda, x) = o(\lambda^{n-1}) \text{ if } |\mathcal{L}_x| = 0.$$

Additionally, if $|\mathcal{L}_x| = 0$ then, given $\varepsilon > 0$, there is a neighborhood \mathcal{N} of x and a $\Lambda = < \infty$, both depending on ε so that

$$(144) \quad |R(\lambda, y)| \leq \varepsilon \lambda^{n-1}, \quad y \in \mathcal{N}, \quad \lambda \geq \Lambda.$$

12.4. Theorem 9. However, Theorem 12.3 is not sharp: on a tri-axial ellipsoid (three distinct axes), the umbilic points are self-focal points. But the eigenfunctions which maximize the sup-norm only have L^∞ norms of order $\lambda^{\frac{n-1}{2}} / \log \lambda$. An improvement is given in [STZ].

Recently, Sogge and the author have further improved the result in the case of real analytic (M, g) . In this case $|\mathcal{L}_x| > 0$ implies that $\mathcal{L}_x = S_x^* M$ and the geometry simplifies. In [SoZ2], we prove Theorem 9, which we restate in terms of the jumps of the remainder:

THEOREM 12.6. *Assume that U_x has no invariant L^2 function for any x . Then*

$$N(\lambda + o(1), x) - N(\lambda, x) = o(\lambda^{n-1}), \quad \text{uniformly in } x.$$

Equivalently,

$$(145) \quad R(\lambda + o(1), x) - R(\lambda, x) = o(\lambda^{n-1})$$

uniformly in x .

Before discussing the proof we note that the conclusion gives very stringent conditions on (M, g) . First, there are topological restrictions on manifolds possessing a self-focal point. If (M, g) has a focal point x_0 then the rational cohomology $H^*(M, \mathbb{Q})$ has a single generator (Berard-Bergery). But even in this case there are many open problems:

PROBLEM 12.7. *All known examples of (M, g) with maximal eigenfunction growth have completely integrable geodesic flow, and indeed quantum integrable Laplacians. Can one prove that maximum eigenfunction growth only occurs in the integrable case? Does it only hold if there exists a point p for which $\Phi_p = Id$?*

A related purely geometric problem: Do there exist (M, g) with $\dim M \geq 3$ possessing self-focal points with $\Phi_x \neq Id$. I.e. do there exist generalizations of umbilic points of ellipsoids in dimension two. There do not seem to exist any known examples; higher dimensional ellipsoids do not seem to have such points.

Despite these open questions, Theorem 12.6 is in a sense sharp. If there exists a self-focal point p with a smooth invariant function, then one can construct a *quasi-mode* of order zero which lives on the flow-out Lagrangian

$$\Lambda_p := \bigcup_{t \in [0, T]} G^t S_p^* M$$

where G^t is the geodesic flow and T is the minimal common return time. The ‘symbol’ is the flowout of the smooth invariant density.

PROPOSITION 12.8. *Suppose that (M, g) has a point p which is a self-focal point whose first return map Φ_x at the return time T is the identity map of S_p^*M . Then there exists a quasi-model of order zero associated to the sequence $\{\frac{2}{\pi}Tk + \frac{\beta}{2} : k = 1, 2, 3, \dots\}$ which concentrates microlocally on the flow-out of S_p^*M . (See §12.8 for background and more precise information).*

12.5. Sketch of proof of Theorem 9. We first outline the proof. A key issue is the uniformity of remainder estimates of $R(\lambda, x)$ as x varies. Intuitively it is obvious that the main points of interest are the self-focal points. But at this time of writing, we cannot exclude the possibility, even in the real analytic setting, that there are an infinite number of such points with twisted return maps. Points which isolated from the set of self-focal points are easy to deal with, but there may be non-self-focal points which lie in the closure of the self-focal points. We introduce some notation.

Definition: We say that $x \in M$

- is an \mathcal{L} point ($x \in \mathcal{L}$) if $\mathcal{L}_x = \pi^{-1}(x) \simeq S_x^*M$. Thus, x is a self-focal point.
- is a \mathcal{CL} point ($x \in \mathcal{CL}$) if $x \in \mathcal{L}$ and $\Phi_x = Id$. Thus, all of the loops at x are smoothly closed.
- is a \mathcal{TL} point ($x \in \mathcal{TL}$) if $x \in \mathcal{L}$ but $\Phi_x \neq Id$, i.e. x is a twisted self-focal point. Equivalently, $\mu_x\{\xi \in \mathcal{L}_x : \Phi_x(\xi) = \xi\} = 0$; All directions are loop directions, but almost none are directions of smoothly closed loops.

To prove Theorem 9 we may (and henceforth will) assume that $\mathcal{CL} = \emptyset$. Thus, $\mathcal{L} = \mathcal{TL}$. We also let $\overline{\mathcal{L}}$ denote the closure of the set of self-focal points. At this time of writing, we do not know how to exclude that $\overline{\mathcal{L}} = M$, i.e. that the set of self-focal points is dense.

PROBLEM 12.9. *Prove (or disprove) that if $\mathcal{CL} = \emptyset$ and if (M, g) is real analytic, then \mathcal{L} is a finite set.*

We also need to further specify times of returns. It is well-known and easy to prove that if all $\xi \in \pi^{-1}(x)$ are loop directions, then the time $T(x, \xi)$ of first return is constant on $\pi^{-1}(x)$. This is because an analytic function is constant on its critical point set.

Definition: We say that $x \in M$

- is a \mathcal{TL}_T point ($x \in \mathcal{TL}_T$) if $x \in \mathcal{TL}$ and if $T(x, \xi) \leq T$ for all $\xi \in \pi^{-1}(x)$. We denote the set of such points by \mathcal{TL}_T .

LEMMA 12.10. *If (M, g) is real analytic, then \mathcal{TL}_T is a finite set.*

There are several ways to prove this. One is to consider the set of all loop points,

$$\mathcal{E} = \{(x, \xi) \in TM : \exp_x \xi = x\},$$

where as usual we identify vectors and co-vectors with the metric. Then at a self-focal point p , $\mathcal{E} \cap T_pM$ contains a union of spheres of radii $kT(p)$, $k = 1, 2, 3, \dots$. The condition that $\Phi_p \neq I$ can be used to show that each sphere is a component of \mathcal{E} , i.e. is isolated from the rest of \mathcal{E} . Hence in the compact set $B_T^*M = \{(x, \xi) : |\xi| \leq T\}$, there can only exist a finite number of such components. Another way to prove it is to show that any limit point x , with $p_j \rightarrow x$ and $p_j \in \mathcal{TL}_T$ must be a \mathcal{TL}_T point whose first return map is the identity. Both proofs involve the study of Jacobi fields along the looping geodesics.

To outline the proof, let $\hat{\rho} \in C_0^\infty$ be an even function with $\hat{\rho}(0) = 1$, $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, and $\hat{\rho}_T(t) = \hat{\rho}(\frac{t}{T})$. The classical cosine Tauberian method to determine Weyl asymptotics is to study

One starts from the smoothed spectral expansion [DG, SV]

$$(146) \quad \begin{aligned} \rho_T * dN(\lambda, x) &= \int_{\mathbb{R}} \hat{\rho}(\frac{t}{T}) e^{i\lambda t} U(t, x, x) dt \\ &= a_0 \lambda^{n-1} + a_1 \lambda^{n-2} + \lambda^{n-1} \sum_{j=1}^{\infty} R_j(\lambda, x, T) + o_T(\lambda^{n-1}), \end{aligned}$$

with uniform remainder in x . The sum over j is a sum over charts needed to parametrize the canonical relation of $U(t, x, y)$, i.e. the graph of the geodesic flow. By the usual parametrix construction for $U(t) = e^{it\sqrt{\Delta}}$, one proves that there exist phases \tilde{t}_j and amplitudes a_{j0} such that

$$(147) \quad R_j(\lambda, x, T) \simeq \lambda^{n-1} \int_{S_x^* M} e^{i\lambda \tilde{t}_j(x, \xi)} ((\hat{\rho}_T a_{j0})) |d\xi| + O(\lambda^{n-2}).$$

As in [DG, Saf, SV] we use polar coordinates in T^*M , and stationary phase in $dt dr$ to reduce to integrals over $S_x^* M$. The phase \tilde{t}_j is the value of the phase $\varphi_j(t, x, x, \xi)$ of $U(t, x, x)$ at the critical point. The loop directions are those ξ such that $\nabla_\xi \tilde{t}_j(x, \xi) = 0$.

EXERCISE 6. Show that $\rho_T * dN(\lambda, x)$ is a semi-classical Lagrangian distribution in the sense of §15. What is its principal symbol?

To illustrate the notation, we consider a flat torus \mathbb{R}^n/Γ with $\Gamma \subset \mathbb{R}^n$ a full rank lattice. As is well-known, the wave kernel then has the form

$$U(t, x, y) = \sum_{\gamma \in \Gamma} \int_{\mathbb{R}^n} e^{i\langle x-y-\gamma, \xi \rangle} e^{it|\xi|} d\xi.$$

Thus, the indices j may be taken to be the lattice points $\gamma \in \Gamma$, and

$$\rho_T * dN(\lambda, x) = \sum_{\gamma \in \Gamma} \int_{\mathbb{R}} \int_0^\infty \int_{S^{n-1}} \hat{\rho}(\frac{t}{T}) e^{ir\langle \gamma, \omega \rangle} e^{itr} e^{-it\lambda r^{n-1}} dr dt d\omega$$

We change variables $r \rightarrow \lambda r$ to get a full phase $\lambda(r\langle \gamma, \omega \rangle + tr - t)$. The stationary phase points in (r, t) are $\langle \gamma, \omega \rangle = t$ and $r = 1$. Thus,

$$\tilde{t}_\gamma(x, \omega) = \langle \gamma, \omega \rangle.$$

The geometric interpretation of $t_\gamma^*(x, \omega)$ is that it is the value of t for which the geodesic $\exp_x t\omega = x + t\omega$ comes closest to the representative $x + \gamma$ of x in the γ th chart. Indeed, the line $x + t\omega$ is ‘closest’ to $x + \gamma$ when $t\omega$ is closest to γ , since $|\gamma - t\omega|^2 = |\gamma|^2 - 2t\langle \gamma, \omega \rangle + t^2$. On a general (M, g) without conjugate points,

$$\tilde{t}_\gamma(x, \omega) = \langle \exp_x^{-1} \gamma, \omega \rangle.$$

12.6. Size of the remainder at a self-focal point. The first key observation is that (147) takes a special form at a self-focal point. At a self-focal point x define U_x as in (39). Also define

$$(148) \quad U_x^\pm(\lambda) = e^{i\lambda T_x^\pm} U_x^\pm.$$

The following observation is due to Safarov [Saf] (see also [SV]).

LEMMA 12.11. *Suppose that x is a self-focal point. If $\hat{\rho} = 0$ in a neighborhood of $t = 0$ then*

$$(149) \quad \rho'_T * N(\lambda, x) = \lambda^{n-1} \sum_{k \in \mathbb{Z} \setminus 0} \int_{S_x^* M} \hat{\rho}\left(\frac{kT(\xi)}{T}\right) \overline{U_x(\lambda)^k} \cdot 1 d\xi + O(\lambda^{n-2}).$$

Here is the main result showing that $R(\lambda, x)$ is small at the self-focal points if there do not exist invariant L^2 functions. $T_x^{(k)}(\xi)$ is the k th return time of ξ for Φ_x .

PROPOSITION 12.12. *Assume that x is a self-focal point and that U_x has no invariant L^2 function. Then, for all $\eta > 0$, there exists T so that*

$$(150) \quad \frac{1}{T} \left| \int_{S_x^* M} \sum_{k=0}^{\infty} \hat{\rho}\left(\frac{T_x^{(k)}(\xi)}{T}\right) U_x^k \cdot 1 |d\xi| \right| \leq \eta.$$

This is a simple application of the von Neumann mean ergodic theorem to the unitary operator U_x . Indeed, $\frac{1}{N} \sum_{k=0}^N U_x^k \rightarrow P_x$, where $P_x : L^2(S_x^* M) \rightarrow L^2(S_x^* M)$ is the orthogonal projections onto the invariant L^2 functions for U_x . By our assumption, $P_x = 0$.

Proposition 12.12 is not apriori uniform as x varies over self-focal points, since there is no obvious relation between Φ_x at one self-focal point and another. It would of course be uniform if we knew that there only exist a finite number of self-focal points. As mentioned above, this is currently unknown. However, there is a second mechanism behind Proposition 12.12. Namely, if the first common return time $T_x^{(1)}(\xi)$ is larger than T , then there is only one term $k = 0$ in the sum with the cutoff $\hat{\rho}_T$ and the sum is $O(\frac{1}{T})$.

12.7. Decomposition of the remainder into almost loop directions and far from loop directions. We now consider non-self-focal points. Then the function $\tilde{t}_j(x, \xi)$ has almost no critical points in $S_x^* M$.

Pick $f \geq 0 \in C_0^\infty(\mathbb{R})$ which equals 1 on $|s| \leq 1$ and zero for $|s| \geq 2$ and split up the j th term into two terms using $f(\epsilon^{-2} |\nabla_\xi \tilde{t}_j|^2)$ and $1 - f(\epsilon^{-2} |\nabla_\xi \tilde{t}_j|^2)$:

$$(151) \quad R_j(\lambda, x, T) = R_{j1}(\lambda, x, T, \epsilon) + R_{j2}(\lambda, x, T, \epsilon),$$

where

$$(152) \quad R_{j1}(\lambda, x, T, \epsilon) := \int_{S_x^* M} e^{i\lambda \tilde{t}_j} f(\epsilon^{-2} |\nabla_\xi \tilde{t}_j(x, \xi)|^2) (\hat{\rho}(T_x(\xi))) a_0(T_x(\xi), x, \xi) d\xi$$

The second term R_{j2} comes from the $1 - f(\epsilon^{-2} |\nabla_\xi T_x(\xi)|^2)$ term. By one integration by parts, one easily has

LEMMA 12.13. *For all $T > 0$ and $\epsilon \geq \lambda^{-\frac{1}{2}} \log \lambda$ we have*

$$\sup_{x \in M} |R_2(\lambda, x, T, \epsilon)| \leq C(\epsilon^2 \lambda)^{-1}.$$

The f term involves the contribution of the almost-critical points of \tilde{t}_j . They are estimated by the measure of the almost-critical set.

LEMMA 12.14. *There exists a uniform positive constant C so that for all (x, ϵ) ,*

$$(153) \quad |R_{j1}(x; \epsilon)| \leq C \mu_x(\{\xi : 0 < |\nabla_\xi \tilde{t}_j(x, \xi)|^2 < \epsilon^2\}),$$

12.8. Points in $M \setminus \overline{\mathcal{TL}}$. If x is isolated from $\overline{\mathcal{TL}}$ then there is a uniform bound on the size of the remainder near x .

LEMMA 12.15. *Suppose that $x \notin \overline{\mathcal{TL}}$. Then given $\eta > 0$ there exists a ball $B(x, r(x, \eta))$ with radius $r(x, \eta) > 0$ and $\epsilon > 0$ so that*

$$\sup_{y \in B(x, r(x, \eta))} |R(\lambda, y, \epsilon)| \leq \eta.$$

Indeed, we pick $r(x, \eta)$ so that the closure of $B(x, r(x, \eta))$ is disjoint from $\overline{\mathcal{TL}}$. Then the one-parameter family of functions $F_\epsilon(y) \rightarrow \mu_y(\{\xi \in S_y^*M : 0 < |\nabla_\xi \tilde{t}_j(y, \xi)|^2 < \epsilon^2\})$ is decreasing to zero as $\epsilon \rightarrow 0$ for each y . By Dini's theorem, the family tends to zero uniformly on $\overline{B(x, r(x, \eta))}$.

12.9. Perturbation theory of the remainder. So far, we have good remainder estimates at each self-focal point and in balls around points isolated from the self-focal points. We still need to deal with the uniformity issues as p varies among self-focal points and points in $\overline{\mathcal{TL}}$.

We now compare remainders at nearby points. Although $R_j(\lambda, x, T)$ is oscillatory, the estimates on R_{j_1} and the ergodic estimates do not use the oscillatory factor $e^{i\lambda t}$, which in fact is only used in Lemma 20. Hence we compare absolute remainders $|R|(x, T)$, i.e. where we take the absolute under the integral sign. They are independent of λ . The integrands of the remainders vary smoothly with the base point and only involve integrations over different fibers S_x^*M of $S^*M \rightarrow M$.

LEMMA 12.16. *We have,*

$$||R|(x, T) - |R|(y, T)| \leq Ce^{aT} \text{dist}(x, y).$$

Indeed, we write the difference as the integral of its derivative. The derivative involves the change in Φ_x^n as x varies over iterates up to time T and therefore is estimated by the sup norm e^{aT} of the first derivative of the geodesic flow up to time T . If we choose a ball of radius δe^{-aT} around a focal point, we obtain'

COROLLARY 12.17. *For any $\eta > 0$, $T > 0$ and any focal point $p \in \mathcal{TL}$ there exists $r(p, \eta)$ so that*

$$\sup_{y \in \overline{B(p, r(p, \eta))}} |R(\lambda, y, T)| \leq \eta.$$

To complete the proof of Theorem 9 we prove

LEMMA 12.18. *Let $x \in \overline{\mathcal{TL}} \setminus \mathcal{TL}$. Then for any $\eta > 0$ there exists $r(x, \eta) > 0$ so that*

$$\sup_{y \in \overline{B(x, r(x, \eta))}} |R(\lambda, y, T)| \leq \eta.$$

Indeed, let $p_j \rightarrow x$ with $T(p_j) \rightarrow \infty$. Then the remainder is given at each p_j by the left side of (149). But for any fixed T , the first term of (149) has at most one term for j sufficiently large. Since the remainder is continuous, the remainder at x is the limit of the remainders at p_j and is therefore $O(T^{-1}) + O(\lambda^{-1})$.

By the perturbation estimate, one has the same remainder estimate in a sufficiently small ball around x .

12.10. **Conclusions.**

- No eigenfunction $\varphi_j(x)$ can be maximally large at a point x which is $\geq \lambda_j^{-\frac{1}{2}} \log \lambda_j$ away from the self-focal points.
- When there are no invariant measures, φ_j also cannot be large at a self-focal point.
- If φ_j is not large at any self-focal point, it is also not large near a self-focal point.

13. APPENDIX ON THE PHASE SPACE AND THE GEODESIC FLOW

Classical mechanics takes place in phase space T^*M (the cotangent bundle). Let (x, ξ) be Darboux coordinates on T^*M , i.e. x_j are local coordinates on M and ξ_j are the functions on T^*M which pick out components with respect to dx_j . Hamilton's equations for the Hamiltonian $H(x, \xi) = \frac{1}{2}|\xi|^2 + V(x) : T^*M \rightarrow \mathbb{R}$ are

$$\frac{dx}{dt} = \frac{\partial H}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial H}{\partial x}.$$

The Hamilton flow is $\Phi^t(x, \xi) = (x_t, \xi_t)$ with $(x_0, \xi_0) = (x, \xi)$. Let us recall the symplectic interpretation of Hamilton's equations.

The cotangent bundle T^*M of any manifold M carries a canonical action form α and symplectic form $\omega = d\alpha$. Given any local coordinates x_j and associated frame dx_j for T^*M , we put: $\alpha = \sum_j \xi_j dx_j$. The form is independent of the choice of coordinates and is called the action form.

It is invariantly defined as follows: At a covector $\xi \in T_x^*M$ define $\alpha_{x,\xi}(X) = \xi(\pi_*X)$ where X is a tangent vector to T^*M at (x, ξ) and $\pi : T^*M \rightarrow M$ is the projection $(x, \xi) \rightarrow x$. Another way to define α is that for any 1-form η , viewed as a section of T^*M , $\eta^*\alpha = \eta$.

By definition $\omega = d\alpha = \sum_{j=1}^n dx_j \wedge d\xi_j$.

A symplectic form is a non-degenerate closed 2-form. Thus it is an anti-symmetric bilinear form on tangent vectors at each point $(x, \xi) \in T^*M$. We note that

- $\omega(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) = 0$; one says that the vector space $\text{Span}\{\frac{\partial}{\partial x_j}\}$ at each $(x, \xi) \in T^*M$ is Lagrangian;
- $\omega(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial \xi_k}) = \delta_{jk}$; one says that $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial \xi_j}$ are symplectically paired;
- $\omega(\frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k}) = 0$. Thus, the vector space $\text{Span}\{\frac{\partial}{\partial \xi_j}\}$ is Lagrangian.

A Hamiltonian is a smooth function $H(x, \xi)$ on T^*M . We say it is homogeneous of degree p if $H(x, r\xi) = r^p H(x, \xi)$ for $r > 0$.

The Hamiltonian vector field Ξ_H of H is the symplectic gradient of H . That is, one takes dH , a 1-form on T^*M and uses the symplectic form ω to convert it to a vector field. That is,

$$\omega(\Xi_H, \cdot) = dH.$$

We claim:

$$\Xi_H = \sum_{j=1}^n \left(\frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

We note that $dH = \frac{\partial H}{\partial \xi_j} d\xi_j + \frac{\partial H}{\partial x_j} dx_j$, so the equation $\omega(\Xi_H, \cdot) = dH$ is equivalent to:

- $\omega\left(\frac{\partial}{\partial \xi_j}, \cdot\right) = -dx_j$;

The flow of the Hamiltonian vector field is the one-parameter group

$$\Phi^t : T^*M \rightarrow T^*M$$

defined by

$$\Phi^t(x_0, \xi_0) = (x_t, \xi_t)$$

where (x_t, ξ_t) solve the ordinary differential equation:

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j}, \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}. \end{cases},$$

with initial conditions $x(0) = x_0, \xi(0) = \xi_0$.

The symplectic form induces the following Lie bracket on functions:

$$\{f, g\}(x) = \Xi_f(g) = \sum_{j=1}^n \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

We see that $\{f, g\} = -\{g, f\}$.

Consider the coordinate functions x_j, ξ_k on T^*M . Exercise: Show:

- $\{x_j, x_k\} = 0$;
- $\{x_j, \xi_k\} = \delta_{jk}$;
- $\{\xi_j, \xi_k\} = 0$.

These are the ‘‘canonical commutation relations’’.

Classical phase space = cotangent bundle T^*M of M , equipped with its canonical symplectic form $\sum_i dx_i \wedge d\xi_i$. The metric defines the Hamiltonian $H(x, \xi) = |\xi|_g = \sqrt{\sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j}$ on T^*M , where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$. Hamilton’s equations:

$$\begin{cases} \frac{dx_j}{dt} = \frac{\partial H}{\partial \xi_j} \\ \frac{d\xi_j}{dt} = -\frac{\partial H}{\partial x_j}. \end{cases}$$

Its flow is the ‘geodesic flow’

$$G^t : S_g^*M \rightarrow S_g^*M$$

restricted to the energy surface $\{H = 1\} := S_g^*M$.

14. APPENDIX: WAVE EQUATION AND HADAMARD PARAMETRIX

The Cauchy problem for the wave equation on $\mathbb{R} \times M$ ($\dim M = n$) is the initial value problem (with Cauchy data f, g)

$$\begin{cases} \square u(t, x) = 0, \\ u(0, x) = f, \quad \frac{\partial}{\partial t} u(0, x) = g(x), \end{cases}.$$

The solution operator of the Cauchy problem (the “propagator”) is the wave group,

$$\mathcal{U}(t) = \begin{pmatrix} \cos t\sqrt{\Delta} & \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} \\ \sqrt{\Delta} \sin t\sqrt{\Delta} & \cos t\sqrt{\Delta} \end{pmatrix}.$$

The solution of the Cauchy problem with data (f, g) is $\mathcal{U}(t) \begin{pmatrix} f \\ g \end{pmatrix}$.

- Even part $\cos t\sqrt{\Delta}$ which solves the initial value problem

$$(154) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)u = 0 \\ u|_{t=0} = f \quad \frac{\partial}{\partial t} u|_{t=0} = 0 \end{cases}$$

- Odd part $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$ is the operator solving

$$(155) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)u = 0 \\ u|_{t=0} = 0 \quad \frac{\partial}{\partial t} u|_{t=0} = g \end{cases}$$

The forward half-wave group is the solution operator of the Cauchy problem

$$\left(\frac{1}{i} \frac{\partial}{\partial t} - \sqrt{-\Delta}\right)u = 0, \quad u(0, x) = u_0.$$

The solution is given by

$$u(t, x) = U(t)u_0(x),$$

with

$$U(t) = e^{it\sqrt{-\Delta}}$$

the unitary group on $L^2(M)$ generated by the self-adjoint elliptic operator $\sqrt{-\Delta}$.

A fundamental solution of the wave equation is a solution of

$$\square E(t, x, y) = \delta_0(t)\delta_x(y).$$

The right side is the Schwartz kernel of the identity operator on $\mathbb{R} \times M$.

There exists a unique fundamental solution with support in the forward light cone, called the advanced (or forward) propagator. It is given by

$$E_+(t) = H(t) \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}},$$

where $H(t) = \mathbf{1}_{t \geq 0}$ is the Heaviside step function.

14.1. **Hörmander parametrix.** We would like to construct a parametrix of the form

$$\int_{T_x^*M} e^{i\langle \exp_y^{-1} x, \eta \rangle} e^{it|\eta|_y} A(t, x, y, \eta) d\eta.$$

This is a homogeneous Fourier integral operator kernel (see §15).

Hörmander actually constructs one of the form

$$\int_{T_x^*M} e^{i\psi(x, y, \eta)} e^{it|\eta|} A(t, x, y, \eta) d\eta,$$

where ψ solves the Hamilton Jacobi Cauchy problem,

$$\begin{cases} q(x, d_x\psi(x, y, \eta)) = q(y, \eta), \\ \psi(x, y, \eta) = 0 \iff \langle x - y, \eta \rangle = 0, \\ d_x\psi(x, y, \eta) = \eta, \quad (\text{for } x = y) \end{cases}$$

The question is whether $\langle \exp_y^{-1} x, \eta \rangle$ solves the equations for ψ . Only the first one is unclear. We need to understand $\nabla_x \langle \exp_y^{-1} x, \eta \rangle$. We are only interested in the norm of the gradient at x but it is useful to consider the entire expression. If we write $\eta = \rho\omega$ with $|\omega|_y = 1$, then ρ can be eliminated from the equation by homogeneity. We fix $(y, \eta) \in S_y^*M$ and consider $\exp_y : T_yM \rightarrow M$. We wish to vary $\exp_y^{-1} x(t)$ along a curve. Now the level sets of $\langle \exp_y^{-1} x, \eta \rangle$ define a notion of local ‘plane waves’ of (M, g) near y . They are actual hyperplanes normal to ω in flat \mathbb{R}^n and in any case are far different from distance spheres. Having fixed (y, η) , $\nabla_x \langle \exp_y^{-1} x, \omega \rangle$ are normal to the plane waves defined by (y, η) . To determine the length we need to see how $\nabla_x \langle \exp_y^{-1} x, \omega \rangle$ changes in directions normal to plane waves.

The level sets of $\langle \exp_y^{-1} x, \eta \rangle$ are images under \exp_y of level sets of $\langle \xi, \eta \rangle = C$ in T_yM . These are parallel hyperplanes normal to η . The radial geodesic in the direction η is of course normal to the exponential image of the hyperplanes. Hence, this radial geodesic is parallel to $\langle \exp_y^{-1} x, \eta \rangle$ when $\exp_y t\eta = x$. It follows that $|\nabla_x \langle \exp_y^{-1} x, \eta \rangle$ at this point equals $\frac{\partial}{\partial t} \langle \exp_y^{-1} \exp_y t\frac{\eta}{|\eta|}, \eta \rangle = t|\eta|_y$. Hence $|\nabla_x \langle \exp_y^{-1} x, \eta \rangle|_x = 1$ at such points.

14.2. **Wave group:** $r^2 - t^2$. We now review the construction of a Hadamard parametrix,

$$(156) \quad U(t)(x, y) = \int_0^\infty e^{i\theta(r^2(x, y) - t^2)} \sum_{k=0}^\infty W_k(x, y) \theta^{\frac{n-3}{2} - k} d\theta \quad (t < \text{inj}(M, g))$$

where $U_o(x, y) = \Theta^{-\frac{1}{2}}(x, y)$ is the volume 1/2-density, where the higher coefficients are determined by transport equations, and where θ^r is regularized at 0 (see below). This formula is only valid for times $t < \text{inj}(M, g)$ but using the group property of $U(t)$ it determines the wave kernel for all times. It shows that for fixed (x, t) the kernel $U(t)(x, y)$ is singular along the distance sphere $S_t(x)$ of radius t centered at x , with singularities propagating along geodesics. It only represents the singularity and in the analytic case only converges in a neighborhood of the characteristic conoid.

Closely related but somewhat simpler is the even part of the wave kernel, $\cos t\sqrt{\Delta}$ which solves the initial value problem

$$(157) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)u = 0 \\ u|_{t=0} = f \quad \frac{\partial}{\partial t}u|_{t=0} = 0 \end{cases}$$

Similar, the odd part of the wave kernel, $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}$ is the operator solving

$$(158) \quad \begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta)u = 0 \\ u|_{t=0} = 0 \quad \frac{\partial}{\partial t}u|_{t=0} = g \end{cases}$$

These kernels only really involve Δ and may be constructed by the Hadamard-Riesz parametrix method. As above they have the form

$$(159) \quad \int_0^\infty e^{i\theta(r^2-t^2)} \sum_{j=0}^\infty W_j(x, y) \theta_{reg}^{\frac{n-1}{2}-j} d\theta \quad \text{mod } C^\infty$$

where W_j are the Hadamard-Riesz coefficients determined inductively by the transport equations

$$(160) \quad \begin{aligned} \frac{\Theta'}{2\Theta}W_0 + \frac{\partial W_0}{\partial r} &= 0 \\ 4ir(x, y) \left\{ \left(\frac{k+1}{r(x, y)} + \frac{\Theta'}{2\Theta} \right) W_{k+1} + \frac{\partial W_{k+1}}{\partial r} \right\} &= \Delta_y W_k. \end{aligned}$$

The solutions are given by:

$$(161) \quad \begin{aligned} W_0(x, y) &= \Theta^{-\frac{1}{2}}(x, y) \\ W_{j+1}(x, y) &= \Theta^{-\frac{1}{2}}(x, y) \int_0^1 s^k \Theta(x, x_s)^{\frac{1}{2}} \Delta_2 W_j(x, x_s) ds \end{aligned}$$

where x_s is the geodesic from x to y parametrized proportionately to arc-length and where Δ_2 operates in the second variable.

According to [GS], page 171,

$$\int_0^\infty e^{i\theta\sigma} \theta_+^\lambda d\lambda = ie^{i\lambda\pi/2} \Gamma(\lambda + 1) (\sigma + i0)^{-\lambda-1}.$$

One has,

$$(162) \quad \int_0^\infty e^{i\theta(r^2-t^2)} \theta_+^{\frac{d-3}{2}-j} d\theta = ie^{i(\frac{n-1}{2}-j)\pi/2} \Gamma\left(\frac{n-3}{2} - j + 1\right) (r^2 - t^2 + i0)^{j - \frac{n-3}{2} - 2}$$

Here there is a problem when n is odd since $\Gamma(\frac{n-3}{2} - j + 1)$ has poles at the negative integers.

One then uses

$$\Gamma(\alpha + 1 - k) = (-1)^{k+1} (-1)^{[\alpha]} \frac{\Gamma(\alpha + 1 - [\alpha]) \Gamma([\alpha] + 1 - \alpha)}{\alpha + 1} \frac{1}{\alpha - [\alpha]} \frac{1}{\Gamma(k - \alpha)}.$$

We note that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Here and above t^{-n} is the distribution defined by $t^{-n} = Re(t + i0)^{-n}$ (see [Be], [G.Sh., p.52,60].) We recall that $(t + i0)^{-n} = e^{-i\pi\frac{n}{2}} \frac{1}{\Gamma(n)} \int_0^\infty e^{itx} x^{n-1} dx$.

We also need that $(x + i0)^\lambda$ is entire and

$$(x + i0)^\lambda = \begin{cases} e^{i\pi\lambda}|x|^\lambda, & x < 0 \\ x_+^\lambda, & x > 0. \end{cases}$$

The imaginary part cancels the singularity of $\frac{1}{\alpha - [\alpha]}$ as $\alpha \rightarrow \frac{d-3}{2}$ when $n = 2m + 1$. There is no singularity in even dimensions. In odd dimensions the real part is $\cos \pi\lambda x_-^\lambda + x_+^\lambda$ and we always seem to have a pole in each term!

But in any dimension, the imaginary part is well-defined and we have

$$(163) \quad \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}(x, y) = C_o \operatorname{sgn}(t) \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \frac{(r^2 - t^2)_-^{j - \frac{n-3}{2} - 1}}{4^j \Gamma(j - \frac{n-3}{2})} \mod C^\infty$$

By taking the time derivative we also have,

$$(164) \quad \cos t\sqrt{\Delta}(x, y) = C_o |t| \sum_{j=0}^{\infty} (-1)^j w_j(x, y) \frac{(r^2 - t^2)_-^{j - \frac{n-3}{2} - 2}}{4^j \Gamma(j - \frac{n-3}{2} - 1)} \mod C^\infty$$

where C_o is a universal constant and where $W_j = \tilde{C}_o e^{-ij\frac{\pi}{2}} 4^{-j} w_j(x, y)$.

14.3. Exact formula in spaces of constant curvature. The Poisson kernel of \mathbb{R}^{n+1} is the kernel of $e^{-t\sqrt{\Delta}}$, given by

$$\begin{aligned} K(t, x, y) &= t^{-n} (1 + |\frac{x-y}{t}|^2)^{-\frac{n+1}{2}} \\ &= t (t^2 + |x - y|^2)^{-\frac{n+1}{2}}. \end{aligned}$$

It is defined only for $t > 0$, although formally it appears to be odd.

Thus, the kernel of $e^{it\sqrt{\Delta}}$ is

$$U(t, x, y) = (it) (|x - y|^2 - t^2)^{-\frac{n+1}{2}}.$$

One would conjecture that the Poisson kernel of any Riemannian manifold would have the form

$$(165) \quad K(t, x, y) = t \sum_{j=0}^{\infty} (t^2 + r(x, y)^2)^{-\frac{n+1}{2} + j} U_j(x, y)$$

for suitable U_j .

14.4. \mathbb{S}^n . One can determine the kernel of $e^{it\sqrt{\Delta}}$ on \mathbb{S}^n from the Poisson kernel of the unit ball $B \subset \mathbb{R}^{n+1}$. We recall that the Poisson integral formula for the unit ball is:

$$u(x) = C_n \int_{\mathbb{S}^n} \frac{1 - |x|^2}{|x - \omega'|^2} f(\omega') dA(\omega').$$

Write $x = r\omega$ with $|\omega| = 1$ to get:

$$P(r, \omega, \omega') = \frac{1 - r^2}{(1 - 2r\langle \omega, \omega' \rangle + r^2)^{\frac{n+1}{2}}}.$$

A second formula for $u(r\omega)$ is

$$u(r, \omega) = r^{A - \frac{n-1}{2}} f(\omega),$$

where $A = \sqrt{\Delta + (\frac{n-1}{4})^2}$. This follows from by writing the equation $\Delta_{\mathbb{R}^{n+1}} u = 0$ as an Euler equation:

$$\left\{ r^2 \frac{\partial^2}{\partial r^2} + nr \frac{\partial}{\partial r} - \Delta_{\mathbb{S}^n} \right\} u = 0.$$

Therefore, the Poisson operator e^{-tA} with $r = e^{-t}$ is given by

$$\begin{aligned} P(t, \omega, \omega') &= C_n \frac{\sinh t}{(\cosh t - \cos r(\omega, \omega'))^{\frac{n+1}{2}}} \\ &= C_n \frac{\partial}{\partial t} \frac{1}{(\cosh t - \cos r(\omega, \omega'))^{\frac{n-1}{2}}}. \end{aligned}$$

Here, $r(\omega, \omega')$ is the distance between points of \mathbb{S}^n .

We analytically continue the expressions to $t > 0$ and obtain the wave kernel as a boundary value:

$$\begin{aligned} e^{itA} &= \lim_{\epsilon \rightarrow 0^+} C_n i \sin t (\cosh \epsilon \cos t - i \sinh \epsilon \sin t - \cos r(\omega, \omega'))^{-\frac{n+1}{2}} \\ &= \lim_{\epsilon \rightarrow 0^+} C_n i \sinh(it - \epsilon) (\cosh(it - \epsilon) - \cos r(\omega, \omega'))^{-\frac{n+1}{2}}. \end{aligned}$$

If we formally put $\epsilon = 0$ we obtain:

$$e^{itA} = C_n i \sin t (\cos t - \cos r(\omega, \omega'))^{-\frac{n+1}{2}}.$$

This expression is singular when $\cos t = \cos r$. We note that $r \in [0, \pi]$ and that it is singular on the cut locus $r = \pi$. Also, $\cos : [0, \pi] \rightarrow [-1, 1]$ is decreasing, so the wave kernel is singular when $t = \pm r$ if $t \in [-\pi, \pi]$.

When n is even, the expression appears to be pure imaginary but that is because we need to regularize it on the set $t = \pm r$. When n is odd, the square root is real if $\cos t \geq \cos r$ and pure imaginary if $\cos t < \cos r$.

We see that the kernels of $\cos tA$, $\frac{\sin tA}{A}$ are supported inside the light cone $|r| \leq |t|$. On the other hand, e^{itA} has no such support property (it has infinite propagation speed). On odd dimensional spheres, the kernels are supported on the distance sphere (sharp Huyghens phenomenon).

The Poisson kernel of the unit sphere is then

$$e^{-tA} = C_n \sinh t (\cosh t - \cos r(\omega, \omega'))^{-\frac{n+1}{2}}.$$

It is singular on the complex characteristic conoid when $\cosh t - \cos r(\zeta, \bar{\zeta}') = 0$.

14.5. Analytic continuation into the complex. If we write out the eigenfunction expansions of $\cos t\sqrt{\Delta}(x, y)$ and $\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}(x, y)$ for $t = i\tau$, we would not expect convergence since the eigenvalues are now exponentially growing. Yet the majorants argument seems to indicate that these wave kernels admit an analytic continuation into a complex neighborhood of the complex characteristic conoid. Define the characteristic conoid in $\mathbb{R} \times M \times M$ by $r(x, y)^2 - t^2 = 0$. For simplicity of visualization, assume x is fixed. Then analytically the

conoid to $\mathbb{C} \times M_{\mathbb{C}} \times M_{\mathbb{C}}$. By definition $(\zeta, \bar{\zeta}, 2\tau)$ lies on the complexified conoid. That is, the series also converge after analytic continuation, again if $r^2 - t^2$ is small. If $t = i\tau$ then we need $r(\zeta, y)^2 + \tau^2$ to be small, which either forces $r(\zeta, y)^2$ to be negative and close to τ or else forces both τ and $r(\zeta, y)$ to be small.

If we wish to use orthogonality relations on M to sift out complexifications of eigenfunctions, then we need $U(i\tau, \zeta, y)$ to be holomorphic in ζ no matter how far it is from y !. So far, we do not have a proof that $U(i\tau, \zeta, y)$ is globally holomorphic for $\zeta \in M_\tau$ for every y .

Regimes of analytic continuation. Let $E(t, x, y)$ be any of the above kernels. Then analytically continue to $E(i\tau, \zeta, \bar{\zeta}')$ where $r(\zeta, \zeta')^2 + \tau^2$ is small. For instance if $\zeta' = \zeta$ and $\sqrt{\rho}(\zeta) = \frac{\tau}{2}$, then $r(\zeta, \zeta')^2 + \tau^2 = 0$.

Is there a neighborhood of the characteristic conoid into which the analytic continuation is possible? We need to have

$$r^2(\zeta, \zeta') \pm \tau^2 \ll \epsilon.$$

If we analytically continue in ζ and anti-analytically continue in ζ' , we seem to get a neighborhood of the conoid.

We would like to analytically continue the Hadamard parametrix to a small neighborhood of the characteristic conoid. It is singular on the conoid.

15. APPENDIX: LAGRANGIAN DISTRIBUTIONS, QUASI-MODES AND FOURIER INTEGRAL OPERATORS

In this section, we go over the definitions of Lagrangian distributions, both semi-classical and homogeneous. A very detailed treatment of the homogeneous Lagrangian distributions (and Fourier integral operators) can be found in [HoIV]. The semi-classical case is almost the same and the detailed treatments can be found in [D, DSj, GSj, CV2, Zw]. We also continue the discussion in §2.10 of quasi-modes.

15.1. Semi-classical Lagrangian distributions and Fourier integral operators. Semi-classical Fourier integral operators with large parameter $\lambda = \frac{1}{\hbar}$ are operators whose Schwartz kernels are defined by semi-classical Lagrangian distributions,

$$I_\lambda(x, y) = \int_{\mathbb{R}^N} e^{i\lambda\varphi(x, y, \theta)} a(\lambda, x, y, \theta) d\theta.$$

More generally, semi-classical Lagrangian distributions are defined by oscillatory integrals (see [D]),

$$(166) \quad u(x, \hbar) = \hbar^{-N/2} \int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\varphi(x, \theta)} a(x, \theta, \hbar) d\theta.$$

We assume that $a(x, \theta, \hbar)$ is a semi-classical symbol,

$$a(x, \theta, \hbar) \sim \sum_{k=0}^{\infty} \hbar^{\mu+k} a_k(x, \theta).$$

The critical set of the phase is given by

$$C_\varphi = \{(x, \theta) : d_\theta\varphi = 0\}.$$

The phase is called non-degenerate if

$$d\left(\frac{\partial\varphi}{\partial\theta_1}\right), \dots, d\left(\frac{\partial\varphi}{\partial\theta_N}\right)$$

are independent on C_φ . Thus, the map

$$\varphi'_\theta := \left(\left(\frac{\partial\varphi}{\partial\theta_1} \right), \dots, \left(\frac{\partial\varphi}{\partial\theta_N} \right) \right) : X \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

is locally a submersion near 0 and $(\varphi'_\theta)^{-1}(0)$ is a manifold of codimension N whose tangent space is $\ker D\varphi'_\theta$. Then

$$T_{(x_0, \theta_0)}C_\varphi = \ker d_{x, \theta}d\varphi.$$

We write a tangent vector to $M \times \mathbb{R}^N$ as $(\delta_x, \delta_\theta)$. The kernel of

$$D\varphi'_\theta = \begin{pmatrix} \varphi''_{\theta x} & \varphi''_{\theta\theta} \end{pmatrix}$$

is $T_{(x, \theta)}C_\varphi$. I.e. $(\delta_x, \delta_\theta) \in TC_\varphi$ if and only if $\varphi''_{\theta x}\delta_x + \varphi''_{\theta\theta}\delta_\theta = 0$. Indeed, φ'_θ is the defining function of C_φ and $d\varphi_\theta$ is the defining function of TC_φ . From [HoIV] Definition 21.2.5: The number of linearly independent differentials $d\frac{\partial\varphi}{\partial\theta}$ at a point of C_φ is $N - e$ where e is the excess. Then $C \rightarrow \Lambda$ is locally a fibration with fibers of dimension e . So to find the excess we need to compute the rank of $\begin{pmatrix} \varphi''_{x\theta} & \varphi''_{\theta\theta} \end{pmatrix}$ on $T_{x, \theta}(\mathbb{R}^N \times M)$.

Non-degeneracy is thus the condition that

$$\begin{pmatrix} \varphi''_{\theta x} & \varphi''_{\theta\theta} \end{pmatrix} \text{ is surjective on } C_\varphi \iff \begin{pmatrix} \varphi''_{\theta x} \\ \varphi''_{\theta\theta} \end{pmatrix} \text{ is injective on } C_\varphi.$$

If φ is non-degenerate, then $\iota_\varphi(x, \theta) = (x, \varphi'_x(x, \theta))$ is an immersion from $C_\varphi \rightarrow T^*X$. Note that

$$d\iota_\varphi(\delta_x, \delta_\theta) = (\delta_x, \varphi''_{xx}\delta_x + \varphi''_{x\theta}\delta_\theta).$$

So if $\begin{pmatrix} \varphi''_{\theta x} \\ \varphi''_{\theta\theta} \end{pmatrix}$ is injective, then $\delta_\theta = 0$.

If $(\lambda_1, \dots, \lambda_n)$ are any local coordinates on C_φ , extended as smooth functions in neighborhood, the delta-function on C_φ is defined by

$$d_{C_\varphi} := \frac{|d\lambda|}{|D(\lambda, \varphi'_\theta)/D(x, \theta)|} = \frac{d\text{vol}_{T_{x_0}M} \otimes d\text{vol}_{\mathbb{R}^N}}{d\frac{\partial\varphi}{\partial\theta_1} \wedge \dots \wedge d\frac{\partial\varphi}{\partial\theta_N}}$$

where the denominator can be regarded as the pullback of $d\text{Vol}_{\mathbb{R}^N}$ under the map

$$d_{\theta, x}d_\theta\varphi(x_0, \theta_0).$$

The symbol $\sigma(\nu)$ of a Lagrangian (Fourier integral) distributions is a section of the bundle $\Omega_{\frac{1}{2}} \otimes \mathcal{M}_{\frac{1}{2}}$ of the bundle of half-densities (tensor the Maslov line bundle). In terms of a Fourier integral representation it is the square root $\sqrt{d_{C_\varphi}}$ of the delta-function on C_φ defined by $\delta(d_\theta\varphi)$, transported to its image in T^*M under ι_φ

Definition: The principal symbol $\sigma_u(x_0, \xi_0)$ is

$$\sigma_u(x_0, \xi_0) = a_0(x_0, \xi_0)\sqrt{d_{C_\varphi}}.$$

It is a $\frac{1}{2}$ density on $T_{(x_0, \xi_0)}\Lambda_\varphi$ which depends on the choice of a density on $T_{x_0}M$.

15.2. Homogeneous Fourier integral operators. A homogeneous Fourier integral operator $A : C^\infty(X) \rightarrow C^\infty(Y)$ is an operator whose Schwartz kernel may be represented by an oscillatory integral

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\varphi(x, y, \theta)} a(x, y, \theta) d\theta$$

where the phase φ is homogeneous of degree one in θ . We assume $a(x, y, \theta)$ is a zeroth order classical polyhomogeneous symbol with $a \sim \sum_{j=0}^{\infty} a_j$, a_j homogeneous of degree $-j$. We refer to [DSj, GS2] and especially to [HoIV] for background on Fourier integral operators. We use the notation $I^m(X \times Y, C)$ for the class of Fourier integral operators of order m with wave front set along the canonical relation C , and $WF'(F)$ to denote the canonical relation of a Fourier integral operator F .

When

$$\iota_\varphi : C_\varphi \rightarrow \Lambda_\varphi \subset T^*(X, Y), \quad \iota_\varphi(x, y, \theta) = (x, d_x\varphi, y, -d_y\varphi)$$

is an embedding, or at least an immersion, the phase is called non-degenerate. Less restrictive, although still an ideal situation, is where the phase is clean. This means that the map $\iota_\varphi : C_\varphi \rightarrow \Lambda_\varphi$, where Λ_φ is the image of ι_φ , is locally a fibration with fibers of dimension e . From [HoIV] Definition 21.2.5, the number of linearly independent differentials $d\frac{\partial\varphi}{\partial\theta}$ at a point of C_φ is $N - e$ where e is the excess.

We recall that the order of $F : L^2(X) \rightarrow L^2(Y)$ in the non-degenerate case is given in terms of a local oscillatory integral formula by $m + \frac{N}{2} - \frac{n}{4}$, where $n = \dim X + \dim Y$, where m is the order of the amplitude, and N is the number of phase variables in the local Fourier integral representation (see [HoIV], Proposition 25.1.5); in the general clean case with excess e , the order goes up by $\frac{e}{2}$ ([HoIV], Proposition 25.1.5'). Further, under clean composition of operators of orders m_1, m_2 , the order of the composition is $m_1 + m_2 - \frac{e}{2}$ where e is the so-called excess (the fiber dimension of the composition); see [HoIV], Theorem 25.2.2.

The definition of the principal symbol is essentially the same as in the semi-classical case. As discussed in [SV], (see (2.1.2) and ((2.2.5) and Definition 2.7.1)), if an oscillatory integral is represented as

$$\mathcal{I}_{\varphi, a}(t, x, y) = \int e^{i\varphi(t, x, y, \eta)} a(t, x, y, \eta) \zeta(t, x, y, \eta) d_\varphi(t, x, y, \eta) d\eta,$$

where

$$d_\varphi(t, x, y, \eta) = |\det \varphi_{x, \eta}|^{\frac{1}{2}}$$

and where the number of phase variables equals the number of x variables, then the phase is non-degenerate if and only if $(\varphi_{x\eta}(t, x, y, \eta))$ is non-singular. Then $a_0 |\det \varphi_{x, \eta}|^{-\frac{1}{2}}$ is the symbol.

The behavior of symbols under pushforwards and pullbacks of Lagrangian submanifolds are described in [GS2], Chapter IV. 5 (page 345). The main statement (Theorem 5.1, loc. cit.) states that the symbol map $\sigma : I^m(X, \Lambda) \rightarrow S^m(\Lambda)$ has the following pullback-pushforward properties under maps $f : X \rightarrow Y$ satisfying appropriate transversality conditions,

$$(167) \quad \begin{cases} \sigma(f^*\nu) = f^*\sigma(\nu), \\ \sigma(f_*\mu) = f_*\sigma(\mu), \end{cases}$$

Here, $f_*\sigma(\mu)$ is integration over the fibers of f when f is a submersion. In order to define a pushforward, f must be a “morphism” in the language of [GSt2], i.e. must be accompanied by a map $r(x) : |\wedge|^{\frac{1}{2}}TY_{f(x)} \rightarrow |\wedge|^{\frac{1}{2}}TX_x$, or equivalently a half-density on $N^*(\text{graph}(f))$, the co-normal bundle to the graph of f which is constant long the fibers of $N^*(\text{graph}(f)) \rightarrow \text{graph}(f)$.

15.3. Quasi-modes. We consider here Lagrangian quasi-modes of order zero, i.e. semi-classical oscillatory integrals (166) which solves $\Delta u_k = O(1)$. If we pass Δ under the integral sign we obtain a leading order term whose amplitude contains the factor $k^2|\nabla_x\varphi(x, \theta)|^2$. The integral of this term must vanish and hence must vanish on the stationary phase set $\nabla_\theta\varphi(x, \theta) = 0$. Hence φ must generate a Lagrangian submanifold $\Lambda_\varphi \subset S^*M$. Such a Lagrangian submanifold must be invariant under the geodesic flow $G^t : S^*M \rightarrow S^*M$ because the Hamilton vector field Ξ_H of a function H which is constant on a Lagrangian submanifold is tangent to Λ . The Lagrangian submanifold may only be locally defined and its global extension might be dense in S^*M . The amplitude a must then be assumed to be compactly supported and the distribution is $O(k^{-M})$ for all M outside of its support.

The $O(k)$ term has an invariant interpretation as $\mathcal{L}_{\Xi_H} a_0 \sqrt{d_{C_\varphi}}$, the Lie derivative of the principal symbol, which is a half-density on Λ_φ . To define a quasi-mode of order zero, the principal symbol must define a global half-density invariant under the geodesic flow. Here we suppress the role of the Maslov bundle and refer to [D, DSj, Zw] for its definition.

To obtain a quasi-mode of higher order, one must solve recursively a sequence of transport equations, which are homogeneous equations of the form $\mathcal{L}_{\Xi_H} a_{j+1} = \mathcal{D}_j a_j$ for various operators \mathcal{D}_k . In general there are obstructions to solving the inhomogeneous equations. The integral of the left side over Λ with respect to the invariant density equals zero, and therefore so must the right side. Here we express all half-densities in terms of the invariant one.

In sum, a zeroth order quasi-mode is an oscillatory integral (166) quantizing a pair (Λ, σ) where $\Lambda \subset S^*M$ is a closed invariant Lagrangian submanifold and σ is a G^t -invariant half-density along it.

There are additional quasi-modes associated to isotropic submanifolds of S^*M , i.e. manifolds on which the symplectic form vanishes but which have dimension $< \dim M$. Gaussian beams are of this kind. We refer to [BB, R1, R2] for their definition. They are constructed along stable elliptic closed geodesics, and may be regarded as Lagrangian distributions with complex phase.

In the case of toric completely integrable systems, the joint eigenfunctions are automatically semi-classical Lagrangian distributions. Indeed, they may be expressed as Fourier coefficients of the unitary Fourier integral operator quantizing the torus action. It is also true that joint eigenfunctions in the general case of quantum integrable systems are Lagrangian. We refer to [TZ3] for background and references.

We should emphasize that it is very rare that eigenfunctions of the Laplacian are quasi-modes or that quasi-modes are genuine eigenfunctions of the Laplacian. In §16.2 we will see that the standard eigenfunctions on S^2 are Lagrangian quasi-modes.

16. APPENDIX ON SPHERICAL HARMONICS

Spherical harmonics furnish the extremals for L^p norms of eigenfunctions φ_λ as (M, g) ranges over Riemannian manifolds and φ_λ ranges over its eigenfunctions. They are not

unique in this respect: surfaces of revolution and their higher dimensional analogues also give examples where extremal eigenfunction bounds are achieved. In this appendix we review the definition and properties of spherical harmonics.

Eigenfunctions of the Laplacian Δ_{S^n} on the standard sphere S^n are restrictions of harmonic homogeneous polynomials on \mathbb{R}^{n+1} .

Let $\Delta_{\mathbb{R}^{n+1}} = -(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2})$ denote the Euclidean Laplacian. In polar coordinates (r, ω) on \mathbb{R}^{n+1} , we have $\Delta_{\mathbb{R}^{n+1}} = -\left(\frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \Delta_{S^n}$. A polynomial $P(x) = P(x_1, \dots, x_{n+1})$ on \mathbb{R}^{n+1} is called:

- homogeneous of degree k if $P(rx) = r^k P(x)$. We denote the space of such polynomials by \mathcal{P}_k . A basis is given by the monomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_{n+1} = k.$$

- Harmonic if $\Delta_{\mathbb{R}^{n+1}} P(x) = 0$. We denote the space of harmonic homogeneous polynomials of degree k by \mathcal{H}_k .

Suppose that $P(x)$ is a homogeneous harmonic polynomial of degree k on \mathbb{R}^{n+1} . Then,

$$\begin{aligned} 0 &= \Delta_{\mathbb{R}^{n+1}} P = -\left\{\frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r}\right\} r^k P(\omega) + \frac{1}{r^2} \Delta_{S^n} P(\omega) \\ &\implies \Delta_{S^n} P(\omega) = (k(k-1) + nk) P(\omega). \end{aligned}$$

Thus, if we restrict $P(x)$ to the unit sphere S^n we obtain an eigenfunction of eigenvalue $k(n+k-1)$. Let $\mathcal{H}_k \subset L^2(S^n)$ denote the space of spherical harmonics of degree k . Then:

- $L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. The sum is orthogonal.
- $Sp(\Delta_{S^n}) = \{\lambda_k^2 = k(n+k-1)\}$.
- $\dim \mathcal{H}_k$ is given by

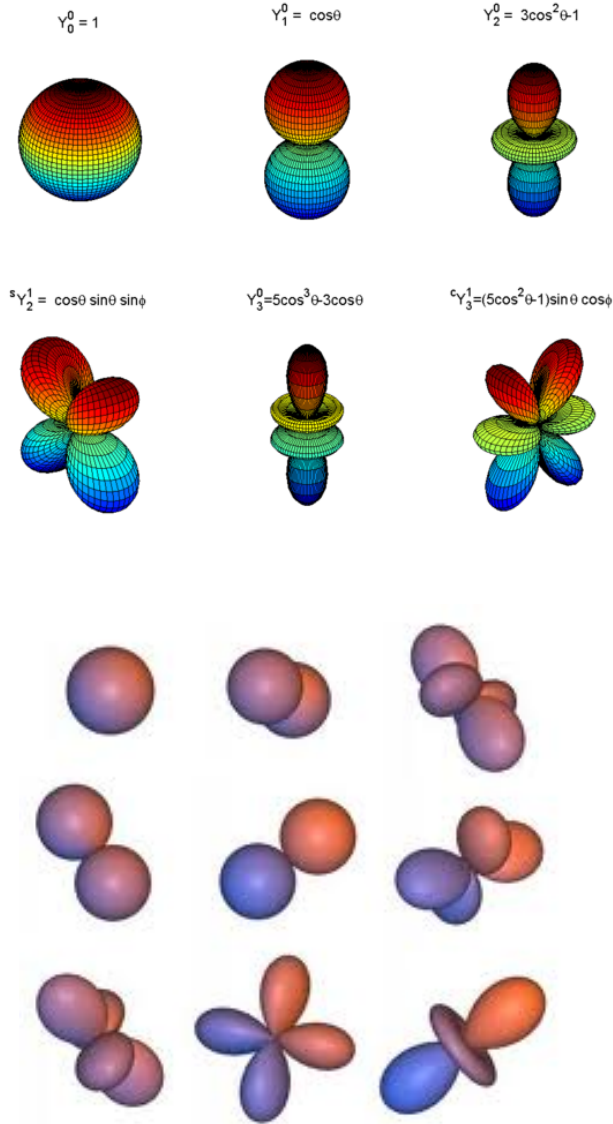
$$d_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$$

The Laplacian Δ_{S^n} is quantum integrable. For simplicity, we restrict to S^2 . Then the group $SO(2) \subset SO(3)$ of rotations around the x_3 -axis commutes with the Laplacian. We denote its infinitesimal generator by $L_3 = \frac{\partial}{i\partial\theta}$. The standard basis of spherical harmonics is given by the joint eigenfunctions ($|m| \leq k$)

$$\begin{cases} \Delta_{S^2} Y_m^k = k(k+1) Y_m^k, \\ \frac{\partial}{i\partial\theta} Y_m^k = m Y_m^k. \end{cases}$$

Two basic spherical harmonics are:

- The highest weight spherical harmonic Y_k^k . As a homogeneous polynomial it is given up to a normalizing constant by $(x_1 + ix_2)^k$ in \mathbb{R}^3 with coordinates (x_1, x_2, x_3) . It is a ‘Gaussian beam’ along the equator $\{x_3 = 0\}$, and is also a quasi-mode associated to this stable elliptic orbit.
- The zonal spherical harmonic Y_0^k . It may be expressed in terms of the orthogonal projection $\Pi_k : L^2(S^2) \rightarrow \mathcal{H}_k$.



We now explain the last statement: For any n , the kernel $\Pi_k(x, y)$ of Π_k is defined by

$$\Pi_k f(x) = \int_{S^n} \Pi_k(x, y) f(y) dS(y),$$

where dS is the standard surface measure. If $\{Y_m^k\}$ is an orthonormal basis of \mathcal{H}_k then

$$\Pi_k(x, y) = \sum_{m=1}^{d_k} Y_m^k(x) Y_m^k(y).$$

Thus for each y , $\Pi_k(x, y) \in \mathcal{H}_k$. We can L^2 normalize this function by dividing by the square root of

$$\|\Pi_k(\cdot, y)\|_{L^2}^2 = \int_{S^n} \Pi_k(x, y) \Pi_k(y, x) dS(x) = \Pi_k(y, y).$$

We note that $\Pi_k(y, y) = C_k$ since it is rotationally invariant and $O(n + 1)$ acts transitively on S^n . Its integral is $\dim \mathcal{H}_k$, hence, $\Pi_k(y, y) = \frac{1}{\text{Vol}(S^n)} \dim \mathcal{H}_k$. Hence the normalized

projection kernel with ‘peak’ at y_0 is

$$Y_0^k(x) = \frac{\Pi_k(x, y_0) \sqrt{\text{Vol}(S^n)}}{\sqrt{\dim \mathcal{H}_k}}.$$

Here, we put y_0 equal to the north pole $(0, 0, \dots, 1)$. The resulting function is called a zonal spherical harmonic since it is invariant under the group $O(n+1)$ of rotations fixing y_0 .

One can rotate $Y_0^k(x)$ to $Y_0^k(g \cdot x)$ with $g \in O(n+1)$ to place the ‘pole’ or ‘peak point’ at any point in S^2 .

16.1. Highest weight spherical harmonics. As mentioned above, the highest weight spherical harmonic is an extremal for the L^6 norm and for all L^p norms with $2 < p < 6$ it is the unique extremal. Let us verify that it achieves the maximum.

We claim that $\|(x + iy)^k\|_{L^2(S^2)} \sim k^{-1/4}$. Indeed we compute it using Gaussian integrals:

$$\begin{aligned} \int_{\mathbb{R}^3} (x^2 + y^2)^k e^{-(x^2+y^2+z^2)} &= \|(x + iy)^k\|_{L^2(S^2)}^2 \int_0^\infty r^{2k} e^{-r^2} r^2 dr, \\ \int_{\mathbb{R}^3} (x^2 + y^2)^k e^{-(x^2+y^2+z^2)} &= \int_{\mathbb{R}^2} (x^2 + y^2)^k e^{-(x^2+y^2)} \\ &= \int_0^\infty r^{2k} e^{-r^2} r dr, \implies \|(x + iy)^k\|_{L^2(S^2)}^2 = \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} \sim k^{-1/2}. \end{aligned}$$

Thus, $k^{1/4}(x + iy)^k$ is the L^2 -normalized highest weight spherical harmonic. It achieves its L^∞ norm at $(1, 0, 0)$ where it has size $k^{1/4}$.

To see that it is an extremal for L^p for $2 \leq p \leq 6$, we use Gaussian integrals:

$$\begin{aligned} \int_{\mathbb{R}^3} (x^2 + y^2)^{3k} e^{-(x^2+y^2+z^2)} &= \|(x + iy)^k\|_{L^6(S^2)}^6 \int_0^\infty r^{6k} e^{-r^2} r^2 dr, \\ \int_{\mathbb{R}^3} (x^2 + y^2)^{3k} e^{-(x^2+y^2+z^2)} &= \int_{\mathbb{R}^2} (x^2 + y^2)^{3k} e^{-(x^2+y^2)} = \int_0^\infty r^{6k} e^{-r^2} r dr, \\ \implies \|(x + iy)^k\|_{L^6(S^2)}^6 &= \frac{\Gamma(6k+1)}{\Gamma(6k+\frac{3}{2})} \sim k^{-1/2}. \end{aligned}$$

Hence, the L^6 norm of $k^{1/4}(x + iy)^k$ equals

$$k^{1/4} k^{-1/12} = k^{1/6}.$$

Since $\lambda_k \sim k$ and $\delta(6) = \frac{1}{6}$ in dimension 2, we see that it is an extremal.

Problem: If a surface (M^2, g) has maximal L^p growth for $2 < p < 6$, must it have a Gaussian beam? Here we may insist that the Gaussian beam be a sequence of eigenfunctions or we may relax the definition and allow it to be a quasi-mode. We might also ask, if there exists a quasi-mode Gaussian beam, does (M^2, g) has maximal L^p growth for $2 < p < 6$?

16.2. Spherical harmonics as quasi-modes. The normalized joint eigenfunctions on the standard sphere are given by

$$(168) \quad Y_m^N(\theta, \varphi) = \sqrt{(2N+1) \frac{(N-m)!}{(N+m)!}} P_m^N(\cos \varphi) e^{im\theta},$$

where

$$P_m^N(\cos \varphi) = \frac{1}{2\pi} \int_0^{2\pi} (i \sin \varphi \cos \theta + \cos \varphi)^N e^{-im\theta} d\theta$$

are the Legendre polynomials.

To obtain a Lagrangian distribution, we consider a sequence of Y_m^N with $\frac{m}{N} \rightarrow C$ for some C . I.e. we consider pairs $k(m_0, N_0)$ lying on a ray in the lattice in \mathbb{Z}^2 of (m, N) with $|m| \leq N$.

The Lagrangian submanifold of T^*S^2 associated to Y_m^N with $m/N \rightarrow c$ is the torus in S^*S^2 defined by

$$p_\theta(x, \xi) := \langle \xi, \frac{\partial}{\partial \theta} \rangle = c.$$

This is a level set of the Clairaut integral

$$p_\theta : S^*S^2 \rightarrow [-1, 1].$$

Examples:

- (i) The Lagrangian submanifold associated to the zonal spherical harmonic is the “meridian torus” consisting of geodesics from the north to south poles.
- (ii) The Gaussian beam is associated to the unit vectors along the equator– a degenerate Lagrangian torus of dimension 1.

Note that we also express the standard spherical harmonics as

$$Y_m^N(x) = \int_0^{2\pi} \Pi_N(x, r_\theta y) e^{-im\theta} d\theta,$$

where r_θ are rotations around the third axis. Since $\Pi_N(x, y)$ is a semi-classical Lagrangian quasi-mode, this also exhibits Y_m^N as one.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA
E-mail address: zelditch@math.northwestern.edu