Mathematics 566 : Homework Problems (partial list)

1. Show that projective space $\mathbf{P}^1$ is diffeomorphic to the Riemann sphere $(\mathbf{S}^2; \{\phi_N, \phi_S\})$.

2. Let $\omega_1, \omega_2 \in \mathbf{C}$ be $\mathbf{R}$-independent vectors. Consider the lattice $L = \{n\omega_1 + m\omega_2; n, m \in \mathbf{Z}\}$. Show that the quotient space $X := \mathbf{C}/L$ can be the structure of a Riemann surface.

3. Show that the set $\{[x, y, z] \in \mathbf{P}^2; zx^2 = y(y-z)(y-2z)\}$ can be given a complex structure and is therefore a Riemann surface.

4. Show that the definition of $\int_\Gamma \frac{df}{f}$ given in class is independent of the choice of cutoff functions χ_j.

5. Let X, Y be compact, connected Riemann surfaces and $F : X \to Y$ a holomorphic map. Show that F has finitely-many ramification points.

6. Let $f \in \mathcal{M}(X)$ and $F = F_f : X \to \mathbf{P}^2$ the associated holomorphic map. Show that
(a) If $p \in X$ is not a pole of f then $Mult_p(F) = Ord_p(f - f(p))$.
(b) If $p \in X$ is a pole of f, then $Mult_p(F) = -Ord_p(f)$.

7. Let $R(F) = \sum_{x \in X}(Mult_x(F) - 1)$ be the ramification degree of F. Show that $R(F)$ is even.

8. Suppose $F \in O(D)$ where $D = \{z \in C; |z| < 1\}$ with $F(-z) = F(z)$. Show that there exists $h \in O(D)$ such that $F(z) = h(z^2)$ for all $z \in D$.

9. Suppose $u \in C^2(X)$ is globally harmonic on a compact, connected Riemann surface X. Show that $u = constant$.

10. Show that a complex line bundle $L \cong M \times \mathbf{C}$ if and only if it has a nowhere-vanishing section.

11. Show that if $\omega_1, \omega_2 \in \Gamma(X, T_X^* \otimes \mathbf{C})$ are global one-forms, $\omega_1 \wedge \omega_2$ is a globally-defined complex 2-form.

12. Show that if $\omega = f dx + g dy$ is a smooth 1-form on $\mathbf{R}^2$ and $\Omega \subset \mathbf{R}^2$ is a smooth, bounded domain, then

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

is nothing but Green's formula.

13. Compute $H_1(P^1), H^1_{dR}(P^1)$ and $H_1(C/\Lambda), H^1_{dR}(C/\Lambda)$.

14. Suppose $V \subset \mathbf{C}$ bounded, open with compact closure $\overline{V}$ and $\partial\overline{V}$ smooth. Then, for $h \in C^1(\overline{V})$,

$$h(z) = \frac{1}{2\pi i}\int_{\partial V} \frac{h(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi}\int\int_V \frac{\partial h}{\partial\bar{\zeta}} \frac{d\zeta d\bar{\zeta}}{\zeta - z}.$$