MATH 381
HOMEWORK 2 SOLUTIONS

Question 1 (p.86 #8). If $g(x)[e^{2y} - e^{2y}]$ is harmonic, $g(0) = 0, g'(0) = 1$, find $g(x)$.

Solution. Let $f(x,y) = g(x)[e^{2y} - e^{2y}]$. Then

$$\frac{\partial^2 f}{\partial x^2} = g''(x)[e^{2y} - e^{2y}]$$
$$\frac{\partial^2 f}{\partial y^2} = 4g(x)[e^{2y} - e^{2y}].$$

Since $f(x,y)$ is harmonic, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ and we require

$$g''(x) + 4g(x) = 0.$$

Thus $g(x)$ has the form $A\sin(2x) + B\cos(2x)$ and by the initial conditions, $A = 1/2$ and $B = 0$. Therefore,

$$g(x) = \frac{1}{2}\sin(2x).$$

□

Question 2 (p.86 #12). Find the harmonic conjugate of $\tan^{-1}\left(\frac{x}{y}\right)$ where $-\pi < \tan^{-1}\left(\frac{x}{y}\right) \leq \pi$.

Solution. Write $u(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$. Then by the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{y^2}{x^2+y^2}\frac{1}{y} = \frac{y}{x^2+y^2} = \frac{\partial v}{\partial y} \tag{1}$$

$$-\frac{\partial u}{\partial y} = -\frac{y^2}{x^2+y^2}\frac{-x}{y^2} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial x}. \tag{2}$$

By (1),

$$v = \frac{1}{2}\log(x^2+y^2) + C(x),$$

and by (2)

$$\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + C'(x) = \frac{x}{x^2+y^2}$$

so $C'(x) = 0$ and $C(x)$ is a constant, call it D. Therefore,

$$v(x,y) = \frac{1}{2}\log(x^2+y^2) + D.$$

□

Question 3. (p.86 #13) Show, if $u(x,y)$ and $v(x,y)$ are harmonic functions, that $u+v$ must be a harmonic function but that uv need not be a harmonic function. Is $e^u e^v$ a harmonic function?

Solution. If u and v are harmonic, then $u+v$ is harmonic since

$$\begin{aligned}\frac{\partial^2(u+v)}{\partial x^2} + \frac{\partial^2(u+v)}{\partial y^2} &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\right)\\ &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0.\end{aligned}$$

To show that uv is not necessarily harmonic, it suffices to show that there exists u, v harmonic such that

$$\frac{1}{2}\left(\frac{\partial^2(uv)}{\partial x^2} + \frac{\partial^2(uv)}{\partial y^2}\right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \neq 0.$$

Any $u = v$ harmonic where $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \neq 0$ will suffice. For instance, taking $u = v = x$ will work, since it's harmonic (both of its second-order partials vanish) but

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 1^2 \neq 0.$$

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Now, in order for $e^u e^v$ to be harmonic, we need

$$\frac{\partial^2(e^u e^v)}{\partial x^2} + \frac{\partial^2(e^u e^v)}{\partial y^2} = e^{u+v}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)^2\right] = 0.$$

Thus, the existence of any u, v harmonic such that $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)^2 \neq 0$ will show that $e^u e^v$ is not harmonic. Again, taking $u = v = x$ gives us what we want as e^{2x} is easily seen to be non-harmonic. □

Question 4 (p.106 #14). State the domain of analyticity of $f(z) = e^{iz}$. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy-Riemann equations, and find $f'(z)$ in terms of z.

Solution. By definition,

$$f(z) = e^{iz} = e^{ix}e^{-y} = e^{-y}[\cos x + i \sin x].$$

Therefore,

$$u(x, y) = e^{-y}\cos x$$
$$v(x, y) = e^{-y}\sin x.$$

These are continuous functions at all $(x, y) \in \mathbb{R}^2$. Now,

$$\frac{\partial u}{\partial x} = -e^{-y}\sin x = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^{-y}\cos x = -\frac{\partial v}{\partial x}$$

so u, v satisfy the C-R equations, and these derivatives are continuous for allx, y. Therefore, $f(z)$ is entire. Furthermore,

$$f'(z) = -e^{-y}\sin x + i(e^{-y}\cos x) = i(e^{-y}[\cos x + i \sin x]) = ie^{iz}.$$

□

Question 5 (p.106 #16). State the domain of analyticity of $f(z) = e^{e^z}$. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy-Riemann equations, and find $f'(z)$ in terms of z.

Solution. First, observe that f is an entire function of an entire function, so it is analytic everywhere. Now,

$$e^{e^z} = e^{e^x(\cos y + i \sin y)} = e^{e^x \cos y}\big(\cos(e^x \sin y) + i \sin(e^x \sin y)\big),$$

so

$$\begin{aligned}
u(x, y) &= e^{e^x \cos y}\cos(e^x \sin y)\\
v(x, y) &= e^{e^x \cos y}\sin(e^x \sin y)\\
\frac{\partial u}{\partial x} &= e^{e^x \cos y}(e^x \cos y)\cos(e^x \sin y) - e^{e^x \cos y}(e^x \sin y)\sin(e^x \sin y)\\
&= e^{e^x \cos y + x}(\cos y \cos(e^x \sin y) - \sin y \sin(e^x \sin y))\\
\frac{\partial v}{\partial y} &= e^{e^x \cos y}(-e^x \sin y)\sin(e^x \sin y) + e^{e^x \cos y}\cos(e^x \sin y)(e^x \cos y)\\
&= e^{e^x \cos y + x}(\cos y \cos(e^x \sin y) - \sin y \sin(e^x \sin y))\\
\frac{\partial u}{\partial y} &= e^{e^x \cos y}(-e^x \sin y)\cos(e^x \sin y) + e^{e^x \cos y}(e^x \cos y)(-\sin(e^x \sin y))\\
&= -e^{e^x \cos y + x}(\cos y \sin(e^x \sin y) + \sin y \cos(e^x \sin y))\\
\frac{\partial v}{\partial x} &= e^{e^x \cos y}(e^x \cos y)\sin(e^x \sin y) + e^{e^x \cos y}(e^x \sin y)\cos(e^x \sin y)\\
&= e^{e^x \cos y + x}(\cos y \sin(e^x \sin y) + \sin y \cos(e^x \sin y))
\end{aligned}$$

and f satisfies the C-R equations. Furthermore,

$$\begin{aligned}
f'(z) &= e^{e^x \cos y}e^x\big((\cos y \cos(e^x \sin y) - \sin y \sin(e^x \sin y)) + i(\cos y \sin(e^x \sin y) + \sin y \cos(e^x \sin y)\big)\\
&= e^{e^x \cos y}\Big(e^x \cos y\big(\cos(e^x \sin y) + i \sin(e^x \sin y)\big) + e^x \sin y\big(i \cos(e^x \sin y) - \sin(e^x \sin y)\big)\Big)\\
&= e^{e^x \cos y}\big(\cos(e^x \sin y) + i \sin(e^x \sin y)\big)e^x(\cos y + i \sin y)\\
&= e^{e^z}e^z.
\end{aligned}$$

□

Question 6 (p.106 #23).

(a) Prove the expression given in the text for the n^{th} derivative of $f(t) = \frac{t}{t^2+1} = \mathrm{Re}\left(\frac{1}{t-i}\right)$. (Note: $t \in \mathbb{R}$).
(b) Find similar expressions for the n^{th} derivative of $f(t) = \frac{1}{t^2+1} = \mathrm{Im}\left(\frac{1}{t-i}\right)$.(Note: $t \in \mathbb{R}$).

Solution.

(a) By the Lemma, for $n \geq 1$,

$$f^{(n)}(t) = \mathrm{Re}\left(\frac{\mathbf{d}^n}{\mathbf{d}t^n}\frac{1}{t-i}\right) = \mathrm{Re}\left(\frac{(-1)^n n!}{(t-i)^{n+1}}\right)$$

Now, observe that $\frac{1}{t-i} = \frac{t+i}{t^2+1}$, so by the binomial theorem

$$\frac{(-1)^n n!}{(t-i)^{n+1}} = \frac{(-1)^n n!(t+i)^{n+1}}{(t^2+1)^{n+1}} = \frac{(-1)^n n!(n+1)!}{(t^2+1)^{n+1}}\sum_{k=0}^{n+1}\frac{i^k t^{n+1-k}}{(n+1-k)!k!}.$$

But notice that we only get contributions to the real part of this expression when k is even; *i.e.* when $i^k \in \mathbb{R}$. Summing over the even integers, $k = 2m$, we get for n odd that

$$\begin{aligned} f^{(n)}(t) &= \frac{(-1)^n n!(n+1)!}{(t^2+1)^{n+1}}\sum_{m=0}^{\frac{n+1}{2}}\frac{i^{2m}t^{n+1-2m}}{(n+1-2m)!(2m)!} \\ &= \frac{(-1)n!(n+1)!}{(t^2+1)^{n+1}}\sum_{m=0}^{\frac{n+1}{2}}\frac{(-1)^m t^{n+1-2m}}{(n+1-2m)!(2m)!} \end{aligned}$$

and for n even that

$$f^{(n)}(t) = \frac{n!(n+1)!}{(t^2+1)^{n+1}}\sum_{m=0}^{\frac{n}{2}}\frac{i^{2m}t^{n+1-2m}}{(n+1-2m)!(2m)!}$$

(b) In this case we want

$$f^{(n)}(t) = \mathrm{Im}\left(\frac{\mathbf{d}^n}{\mathbf{d}t^n}\frac{1}{t-i}\right) = \mathrm{Im}\left(\frac{(-1)^n n!}{(t-i)^{n+1}}\right).$$

By the work above, we want the imaginary part of

$$\frac{(-1)^n n!(n+1)!}{(t^2+1)^{n+1}}\sum_{k=0}^{n+1}\frac{i^k t^{n+1-k}}{(n+1-k)!k!}.$$

In this case we get contributions when k is odd, so we take the the sum over $k = 2m+1$ for $m \geq 0$. Note that $i^{2m+1} = (-1)^m i$. It follows that when n is odd,

$$f^{(n)}(t) = \frac{(-1)n!(n+1)!}{(t^2+1)^{n+1}}\sum_{m=0}^{\frac{n-1}{2}}\frac{(-1)^m t^{n-2m}}{(n-2m)!(2m+1)!}$$

and when n is even,

$$f^{(n)}(t) = \frac{n!(n+1)!}{(t^2+1)^{n+1}}\sum_{m=0}^{n/2}\frac{(-1)^m t^{n-2m}}{(n-2m)!(2m+1)!}.$$

□

Question 7 (p.106 #25). Let $P(\psi) = \sum_{n=0}^{N-1} e^{in\psi}$.

(a) Show that

$$|P(\psi)| = \left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right|.$$

(b) Find $\lim_{\psi\to 0}|P(\psi)|$.
(c) Plot $|P(\psi)|$ for $0 \leq \psi \leq 2$ and $N = 3$.

Solution.

(a) Note that

$$P(\psi) = \frac{1-e^{iN\psi}}{1-e^{i\psi}}.$$

Thus,

$$\begin{aligned}
P(\psi) &= \frac{e^{iN\psi}-1}{e^{i\psi}-1} \\
&= \frac{e^{iN\psi/2}}{e^{i\psi/2}}\left(\frac{e^{iN\psi/2}-e^{-iN\psi/2}}{e^{i\psi/2}-e^{-i\psi/2}}\right) \\
&= \frac{e^{iN\psi/2}}{e^{i\psi/2}}\left(\frac{\cos(N\psi/2)+i\sin(N\psi/2)-\cos(-N\psi/2)-i\sin(-N\psi/2)}{\cos(\psi/2)+i\sin(\psi/2)-\cos(-\psi/2)-i\sin(-\psi/2)}\right) \\
&= \frac{e^{iN\psi/2}}{e^{i\psi/2}}\frac{2i\sin(N\psi/2)}{2i\sin\psi/2}.
\end{aligned}$$

Thus,

$$|P(\psi)| = \left|\frac{e^{iN\psi/2}}{e^{i\psi/2}}\right|\left|\frac{\sin(N\psi/2)}{\sin\psi/2}\right| = \left|\frac{\sin(N\psi/2)}{\sin\psi/2}\right|.$$

(b) By l'Hopital's rule we get

$$\lim_{\psi\to 0}\frac{\sin(N\psi/2)}{\sin\psi/2} = \lim_{\psi\to 0}\frac{N/2\sin(N\psi/2)}{1/2\sin\psi/2} = N.$$

(c) If you have nothing else, just plug it in Wolfram Alpha.

□

Question 8 (p.112 #17). Show that $\sin z - \cos z = 0$ has solutions only for real values of z. What are the solutions?

Solution. In other words, for $z = x + iy$ we want

$$\sin x\cosh y + i\cos x\sinh y = \cos x\cosh y - i\sin x\sinh y.$$

Equating the real parts and imaginary parts we require

$$\sin x\cosh y = \cos x\cosh y \tag{3}$$

$$\cos x\sinh y = -\sin x\sinh y. \tag{4}$$

Suppose $y \neq 0$ and hence $\sinh y \neq 0$ and $\cosh y \neq 0$. Then in order to have solutions, by (3), we need $\cos x = \sin x$ and by (4) we need $\cos x = -\sin x$. These equations are only satisfied for $\sin x = \cos x = 0$, but no solutions for x exists. Therefore, if there are solutions to the original equation, we must have that $y = 0$.

Suppose $y = 0$. Then since $\cosh 0 = 1$ and $\sinh 0 = 0$ we simply need solutions to $\sin x = \cos x$. Thus we have solutions if and only if

$$z = \frac{\pi}{4} + k\pi, \qquad k \in \mathbb{Z}.$$

□

Question 9 (p.112 #21). Where does the function $f(z) = \frac{1}{\sqrt{3}\sin z - \cos z}$ fail to be analytic?

Solution. Since $\sin z$ and $\cos z$ are both analytic, $f(z)$ will fail to be analytic when $\sqrt{3}\sin z - \cos z = 0$. In other words, when we have solutions to

$$\sqrt{(}\sin x\cosh y + i\cos x\sinh y) = \cos x\cosh y - i\sin x\sinh y.$$

Equating the real parts and imaginary parts we require

$$\sqrt{3}\sin x\cosh y = \cos x\cosh y \tag{5}$$

$$\sqrt{3}\cos x\sinh y = -\sin x\sinh y. \tag{6}$$

By the same argument as the previous question, there are no solutions when $y \neq 0$. Suppose $y = 0$. Then since $\cosh 0 = 1$ and $\sinh 0 = 0$ we simply need solutions to $\sqrt{3}\sin x = \cos x$, that is to $\tan x = \frac{1}{\sqrt{3}}$. So $f(z)$ is not analytic when

$$z = \frac{\pi}{6} + k\pi, \qquad k \in \mathbb{Z}.$$

□

Question 10 (p.112 #22). Let $f(z) = \sin\left(\frac{1}{z}\right)$.

(a) Express this function in the form $u(x, y) + iv(x, y)$. Where in the complex plane is this function analytic?

(b) What is the derivative of $f(z)$? Where in the complex plane is $f'(z)$ analytic?

Solution.

(a) Since $\sin z$ is entire, and $\frac{1}{z}$ is analytic for $z \neq 0$, it follows that $f(z)$ is analytic for $z \neq 0$.

$$\begin{aligned}\sin\left(\frac{1}{z}\right) &= \sin\left(\frac{x-iy}{x^2+y^2}\right)\\ &= \sin\left(\frac{x}{x^2+y^2}\right)\cosh\left(\frac{-y}{x^2+y^2}\right) + i\cos\left(\frac{x}{x^2+y^2}\right)\sinh\left(\frac{-y}{x^2+y^2}\right)\\ &= \sin\left(\frac{x}{x^2+y^2}\right)\cosh\left(\frac{y}{x^2+y^2}\right) - i\cos\left(\frac{x}{x^2+y^2}\right)\sinh\left(\frac{y}{x^2+y^2}\right).\end{aligned}$$

(b) For $z \neq 0$,

$$\frac{\mathbf{d}}{\mathbf{d}z}\sin\left(\frac{1}{z}\right) = \left(\cos\frac{1}{z}\right)\left(\frac{-1}{z^2}\right)$$

which is analytic for all $z \neq 0$.

□

Question 11 (p.112 #25). Show that $|\cos z| = \sqrt{\sinh^2 y + \cos^2 x}$.

Solution.

$$\begin{aligned}|\cos z| &= |\cos x\cosh y - i\sin x\sinh y|\\ &= \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y}\\ &= \sqrt{\cos^2 x(1+\sinh^2 y) + \sin^2 x\sinh^2 y}\\ &= \sqrt{\cos^2 x + \sinh^2 y(\cos x^2 + \sin^2 x)}\\ &= \sqrt{\cos^2 x + \sinh^2 y}\end{aligned}$$

□

Question 12 (p.119 #16). Use logarithms to find solutions to $e^z = e^{iz}$.

Solution. We want solutions to $e^{z(1-i)} = 1$, so taking logs on both sides we get for any $k \in \mathbb{Z}$, $z(1-i) = 2\pi ik$, so

$$z = \frac{2\pi ik}{1-i} = \frac{(i+1)2\pi ik}{2} = (i-1)k\pi.$$

□

Question 13 (p.119 #18). Use logarithms to find solutions to $e^z = (e^z - 1)^2$.

Solution. In other words, we want solutions to $e^{2z} - 3e^z + 1 = 0$. By the quadratic formula, we get that

$$e^z = \frac{3 \pm \sqrt{4-9}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

Taking logs gives that

$$z = \log\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right) + 2\pi ik$$

for $k \in \mathbb{Z}$. □

Question 14 (p.119 #21). Use logarithms to find solutions to $e^{e^z} = 1$.

Solution. First, taking logs we get $e^z = 2\pi ik$ for $k \in \mathbb{Z}$. Now for $k > 0$, the argument of $2\pi ik$ is $\frac{\pi}{2} + 2\pi m$ where $m \in \mathbb{Z}$, and for $k < 0$, the argument of $2\pi ik$ is $\frac{3\pi}{2} + 2\pi m$ (again $m \in \mathbb{Z}$). Thus, for $k > 0$,

$$z = \log(2\pi k) + i\left(\frac{\pi}{2} + 2m\pi\right)$$

and for $k < 0$,

$$z = \log(2\pi k) + i\left(\frac{-\pi}{2} + 2m\pi\right).$$

□

Question 15 (p.119 #23). Show that

$$\operatorname{Re}\left(\log(1+e^{i\theta})\right) = \log\left|2\cos\left(\frac{\theta}{2}\right)\right|$$

where $\theta \in \mathbb{R}$ and $e^{i\theta} \neq -1$.

Solution.

$$\begin{aligned}
\mathrm{Re}\left(\log(1+e^{i\theta})\right) &= \log\left|1+e^{i\theta}\right| \\
&= \frac{1}{2}\log\left((1+\cos\theta)^2+\sin^2\theta\right) \\
&= \frac{1}{2}\log(2+2\cos\theta) \\
&= \frac{1}{2}\log\left(2\cos^2\left(\frac{\theta}{2}\right)+2\sin^2\left(\frac{\theta}{2}\right)+2\cos^2\left(\frac{\theta}{2}\right)-2\sin^2\left(\frac{\theta}{2}\right)\right) \\
&= \log\left|2\cos\left(\frac{\theta}{2}\right)\right|
\end{aligned}$$

□

Question 16 (p.170 #9). Intergrate

$$\int_1^{-1}\frac{1}{z}\mathbf{d}z$$

along $|z|=1$, in the lower half plane.

Solution. Let $z=e^{it}$, then we are integrating along the interval $t\in[0,-\pi]$. Now, $\mathbf{d}z=ie^{it}\mathbf{d}t$ so

$$\int_1^{-1}\frac{1}{z}\mathbf{d}z=\int_0^{-\pi}\frac{1}{e^it}ie^it\mathbf{d}t=-i\pi.$$

□

Question 17 (p.170 #11). Show that $x=2\cos t, y=\sin t$, where t ranges from 0 to 2π, yields a parametric representation of the ellipse $\frac{x^2}{4}+y^2=1$. Use this representation to evaluate $\int_2^i\bar{z}\mathbf{d}z$ along the portion of the ellipse in the first quadrant.

Solution. Note that

$$\frac{(2\cos t)^2}{4}+\sin^2 t=\cos^2 t+\sin^2 t=1$$

and furthermore $2\cos 0=2\cos 2\pi=2$ and $\sin 0=\sin 2\pi=0$. To see that we get all of the ellipse, note that $x=2\cos t$ has solutions $t\in[0,2\pi]$ for all $x\in[-2,2]$ and $y=\sin t$ has solutions $t\in[0,2\pi]$ for all $y\in[-1,1]$. Furthermore, the parametrization is 1:1 except for when $x=2, y=0$.

Setting $z=x+iy=2\cos t+i\sin t$, we get $\mathbf{d}z=(i\cos t-2\sin t)\mathbf{d}t$, and

$$\begin{aligned}
\int_2^i\bar{z}\mathbf{d}z &= \int_0^{\frac{\pi}{2}}(2\cos t-i\sin t)(i\cos t-2\sin t)\mathbf{d}t \\
&= \int_0^{\frac{\pi}{2}}(2i-3\sin t\cos t)\mathbf{d}t \\
&= -\frac{3}{2}+i\pi.
\end{aligned}$$

□

Question 18 (p.170 #14). Consider $I=\int_0^{2+i}e^{z^2}\mathbf{d}z$ taken along the line $x=2y$. Without actually doing the integration, show that $|I|\le\sqrt{5}e^3$.

Solution. Let M be the maximal value attained by $\left|e^{z^2}\right|$ along the path of integration. Now, for $x=2y$,

$$\left|e^{z^2}\right|=\left|e^{x^2-y^2+2ixy}\right|=e^{3y^2}$$

which attains its maximum when y attains a maximum—that is, when $z=2+i$. Therefore $M=e^3$. By the pythagorean theorem, the length of the path is $\sqrt{2^2+1^2}=\sqrt{5}$, so by the ML inequality, $|I|\le ML=\sqrt{5}e^3$. □

Question 19 (p.170 #16). Consider $I=\int_i^1 e^{i\log\bar{z}}\mathbf{d}z$ taken along the parabola $y=1-x^2$. Without doing the integration, show that $|I|\le 1.479e^{\pi/2}$.

Solution. Letting $\theta=\arg z$

$$\begin{aligned}
\left|e^{i\log\bar{z}}\right| &= \left|e^{i(\log|z|-i\theta)}\right| \\
&= \left|e^{i\log|z|}e^{\theta}\right|=e^{\theta}.
\end{aligned}$$

Along the given path, this attains a maximum when $\theta=\pi/2$, so let $M=e^{\pi/2}$.

Now, we need to find the length of the path of integration. So since $\mathbf{d}y = -2x\mathbf{d}x$,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{\mathbf{d}y}{\mathbf{d}x}\right)^2}\mathbf{d}x \\ &= \int_0^1 \sqrt{1+4x^2}\mathbf{d}x \\ &< 1.479. \end{aligned}$$

The ML inequality then gives the desired result. □

Question 20 (p.180 #2). Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z|=1} \frac{\sin z}{z+2i}\mathbf{d}z$?

Solution. Since $\frac{\sin z}{z+2i}$ is analytic everywhere except for $z = -2i$ which is not in the unit circle, the C-G theorem is directly applicable. □

Question 21 (p.180 #6). Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z-i-1|=1} \log z\mathbf{d}z$?

Solution. Since 0 is not in the unit circle about $i+1$, $\log z$ is analytic in the desired region so the C-G theorem is directly applicable. □

Question 22 (p.180 #7). Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z|=1/2} \frac{1}{(z-1)^4+1}\mathbf{d}z$?

Solution. Observe that we have a singularity when $z-1$ is a primitive 8^{th} root of unity—that is, when $(z-1)^4 = -1$. These roots of unity lie on the unit circle, so shifting over by 1, we need to determine if the roots closest to the origin, at $z = e^{i\pi 3/4}+1$ and $z = e^{i\pi 5/4}+1$ have absolute value greater than 1/2. By geometry (right angle triangles), it can be seen that at these points,

$$|z| = \sqrt{\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} > 1/2$$

so the C-G theorem directly applies. □

Question 23 (p.180 #9). Is the Cauchy-Goursat theorem directly applicable to

$$\oint_{|z|=b} \frac{1}{z^2+bz+1}\mathbf{d}z$$

where $0 < b < 1$?

Solution. In this case the singularities are at the roots of the equation x^2+bx+1, that is, when

$$z = \frac{-b \pm \sqrt{b^2-4}}{2} = \frac{-b}{2} \pm i\frac{\sqrt{4-b^2}}{2}.$$

Here,

$$|z| = \sqrt{\frac{b^2}{4} + \frac{4-b^2}{4}} = 1 > b$$

therefore C-G applies directly. □

Question 24 (p.180 #13). Prove that

$$\int_0^{2\pi} e^{\cos\theta}\big(\sin(\sin\theta+\theta)\big)\mathbf{d}\theta = 0.$$

Begin with $\oint e^z\mathbf{d}z$ performed around $|z|=1$. Use the parametric representation $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Separate your equation into real and imaginary parts.

Solution. Let $z = e^{i\theta} = \cos\theta + i\sin\theta$, so $\mathbf{d}z = e^{i\theta}i\mathbf{d}\theta$. Since e^z is analytic,

$$\oint_{|z|=1} e^z\mathbf{d}z = \int_0^{2\pi} e^{\cos\theta + i\sin\theta}e^{i\theta}i\mathbf{d}\theta = 0.$$

But then

$$\int_0^{2\pi} e^{\cos\theta + i(\sin\theta+\theta)}\mathbf{d}\theta = \int_0^{2\pi} e^{\cos\theta}(\cos(\theta+\sin\theta) + i\sin(\theta+\sin\theta))\mathbf{d}\theta = 0$$

so by equating the imaginary part with zero we get the desired result. □