MATH 381
HOMEWORK 2 SOLUTIONS

Question 1 (p.86 #8). If \(g(x)[e^{2y} - e^{-2y}]\) is harmonic, \(g(0) = 0, g'(0) = 1\), find \(g(x)\).

Solution. Let \(f(x, y) = g(x)[e^{2y} - e^{-2y}]\). Then

\[
\frac{\partial^2 f}{\partial x^2} = g''(x)[e^{2y} - e^{-2y}]
\]
\[
\frac{\partial^2 f}{\partial y^2} = 4g(x)[e^{2y} - e^{-2y}].
\]

Since \(f(x, y)\) is harmonic, \(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\) and we require

\[
g''(x) + 4g(x) = 0.
\]

Thus \(g(x)\) has the form \(A\sin(2x) + B\cos(2x)\) and by the initial conditions, \(A = 1/2\) and \(B = 0\). Therefore,

\[
g(x) = \frac{1}{2}\sin(2x).
\]

\(\square\)

Question 2 (p.86 #12). Find the harmonic conjugate of \(\tan^{-1}\left(\frac{z}{\pi}\right)\) where \(-\pi < \tan^{-1}\left(\frac{z}{\pi}\right) \leq \pi\).

Solution. Write \(u(x, y) = \tan^{-1}\left(\frac{x}{y}\right)\). Then by the Cauchy-Riemann equations,

(1)
\[
\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{x^2 + y^2} = \frac{1}{x^2 + y^2} \frac{\partial v}{\partial y}
\]
\[
-\frac{\partial u}{\partial y} = -\frac{y^2 - x^2}{x^2 + y^2} = \frac{x}{x^2 + y^2} \frac{\partial v}{\partial x}.
\]

By (1),

\[
v = \frac{1}{2}\log(x^2 + y^2) + C(x),
\]

and by (2)

\[
\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + C'(x) = \frac{x}{x^2 + y^2}
\]

so \(C'(x) = 0\) and \(C(x)\) is a constant, call it \(D\). Therefore,

\[
v(x, y) = \frac{1}{2}\log(x^2 + y^2) + D.
\]

\(\square\)

Question 3. (p.86 #13) Show, if \(u(x, y)\) and \(v(x, y)\) are harmonic functions, that \(u + v\) must be a harmonic function but that \(uv\) need not be a harmonic function. Is \(e^u e^v\) a harmonic function?

Solution. If \(u\) and \(v\) are harmonic, then \(u + v\) is harmonic since

\[
\frac{\partial^2 (u + v)}{\partial x^2} + \frac{\partial^2 (u + v)}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right)
\]
\[
= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0.
\]

To show that \(uv\) is not necessarily harmonic, it suffices to show that there exists \(u, v\) harmonic such that

\[
\frac{1}{2} \left(\frac{\partial^2 (uv)}{\partial x^2} + \frac{\partial^2 (uv)}{\partial y^2}\right) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0.
\]

Any \(u = v\) harmonic where \(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \neq 0\) will suffice. For instance, taking \(u = v = x\) will work, since it’s harmonic (both of its second-order partials vanish) but

\[
\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 1^2 \neq 0.
\]

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Now, in order for $e^u e^v$ to be harmonic, we need
\[
\frac{\partial^2 (e^u e^v)}{\partial x^2} + \frac{\partial^2 (e^u e^v)}{\partial y^2} = e^{u+v} \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] = 0.
\]
Thus, the existence of any $u, v$ harmonic such that \( \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \neq 0 \) will show that $e^u e^v$ is not harmonic. Again, taking $u = v = x$ gives us what we want as $e^{2x}$ is easily seen to be non-harmonic.

\[ \square \]

**Question 4** (p.106 #14). State the domain of analyticity of $f(z) = e^{iz}$. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy-Riemann equations, and find $f'(z)$ in terms of $z$.

**Solution.** By definition,
\[
f(z) = e^{iz} = e^{ix} - e^{-iy} = e^{-y}[\cos x + i \sin x].
\]

Therefore,
\[
u(x, y) = e^{-y} \cos x \\
v(x, y) = e^{-y} \sin x.
\]

These are continuous functions at all $(x, y) \in \mathbb{R}^2$. Now,
\[
\begin{align*}
\frac{\partial u}{\partial x} &= -e^{-y} \sin x = \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -e^{-y} \cos x = -\frac{\partial v}{\partial x}
\end{align*}
\]

so $u, v$ satisfy the C-R equations, and these derivatives are continuous for all $x, y$. Therefore, $f(z)$ is entire. Furthermore,
\[
f'(z) = -e^{-y} \sin x + i(e^{-y} \cos x) = i(e^{-y}[\cos x + i \sin x]) = ie^{iz}.
\]

\[ \square \]

**Question 5** (p.106 #16). State the domain of analyticity of $f(z) = e^{e^z}$. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy-Riemann equations, and find $f'(z)$ in terms of $z$.

**Solution.** First, observe that $f$ is an entire function of an entire function, so it is analytic everywhere. Now,
\[
e^{e^z} = e^{e^{x+y} \sin y} = e^{e^x \cos y} \left( \cos(e^x \sin y) + i \sin(e^x \sin y) \right),
\]
so
\[
u(x, y) = e^{e^x \cos y} \cos(e^x \sin y) \\
v(x, y) = e^{e^x \cos y} \sin(e^x \sin y)
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= e^{e^x \cos y} (e^x \cos y) \cos(e^x \sin y) - e^{e^x \cos y} (e^x \sin y) \sin(e^x \sin y) \\
&= e^{e^x \cos y} (e^y \cos(e^x \sin y) - e^y \sin(e^x \sin y)) \\
\frac{\partial v}{\partial y} &= e^{e^x \cos y} (-e^x \sin y) \sin(e^x \sin y) + e^{e^x \cos y} \cos(e^x \sin y) (e^x \cos y) \\
&= e^{e^x \cos y} (e^y \cos(e^x \sin y) - e^y \sin(e^x \sin y)) \\
\frac{\partial u}{\partial y} &= e^{e^x \cos y} (-e^x \sin y) \cos(e^x \sin y) + e^{e^x \cos y} (e^x \cos y) (-e^x \sin y) \\
&= -e^{e^x \cos y} (e^y \sin(e^x \sin y) + e^y \cos(e^x \sin y)) \\
\frac{\partial v}{\partial x} &= e^{e^x \cos y} (e^x \cos y) \sin(e^x \sin y) + e^{e^x \cos y} (e^x \sin y) \cos(e^x \sin y) \\
&= e^{e^x \cos y} (e^y \sin(e^x \sin y) + e^y \cos(e^x \sin y))
\end{align*}
\]

and $f$ satisfies the C-R equations. Furthermore,
\[
f'(z) = e^{e^x \cos y} e^z \left( (e^y \cos(e^x \sin y) - e^y \sin(e^x \sin y)) + i(e^y \sin(e^x \sin y) + e^y \cos(e^x \sin y)) \right) \\
= e^{e^x \cos y} \left( e^x \cos y (e^x \sin y + i \sin(e^x \sin y)) + e^x \sin y (i \cos(e^x \sin y) - e^x \sin y) \right) \\
= e^{e^x \cos y} (\cos(e^x \sin y) + i \sin(e^x \sin y)) e^{e^x} (e^y + i e^y)
\]

\[ \square \]
Question 6 (p.106 #23).

(a) Prove the expression given in the text for the $n^{th}$ derivative of $f(t) = \frac{t}{e^{t^2/4}} = \Re\left(\frac{1}{t^{i+1}}\right)$. (Note: $t \in \mathbb{R}$).

(b) Find similar expressions for the $n^{th}$ derivative of $f(t) = \frac{1}{e^{t^2/4}} = \Im\left(\frac{1}{t^{i+1}}\right)$. (Note: $t \in \mathbb{R}$).

Solution.

(a) By the Lemma, for $n \geq 1$,

$$f^{(n)}(t) = \Re\left(\frac{d^n}{dt^n} \frac{1}{t - i}\right) = \Re\left(\frac{(-1)^n n!}{(t - i)^{n+1}}\right).$$

Now, observe that $\frac{1}{t - i} = \frac{i}{t^2 + 1}$, so by the binomial theorem

$$\frac{(-1)^n n!}{(t - i)^{n+1}} = \frac{(-1)^n n! (t + i)^{n+1}}{(t^2 + 1)^{n+1}} = \frac{(-1)^n n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{k=0}^{n+1} \frac{i^k t^{n+1-k}}{(n + 1 - k)!}.$$  

But notice that we only get contributions to the real part of this expression when $k$ is even, i.e. when $i^k \in \mathbb{R}$. Summing over the even integers, $k = 2m$, we get for $n$ odd that

$$f^{(n)}(t) = \frac{(-1)^n n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{m=0}^{\frac{n+1}{2}} \frac{i^{2m} t^{n+1-2m}}{(n + 1 - 2m)!}.$$  

and for $n$ even that

$$f^{(n)}(t) = \frac{n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{m=0}^{\frac{n}{2}} \frac{i^{2m} t^{n+1-2m}}{(n + 1 - 2m)!}.$$  

(b) In this case we want

$$f^{(n)}(t) = \Im\left(\frac{d^n}{dt^n} \frac{1}{t - i}\right) = \Im\left(\frac{(-1)^n n!}{(t - i)^{n+1}}\right).$$

By the work above, we want the imaginary part of

$$\frac{(-1)^n n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{k=0}^{n+1} \frac{i^k t^{n+1-k}}{(n + 1 - k)!}.$$  

In this case we get contributions when $k$ is odd, so we take the sum over $k = 2m + 1$ for $m \geq 0$. Note that $i^{2m+1} = (-1)^m i$. It follows that when $n$ is odd,

$$f^{(n)}(t) = \frac{(-1)^n n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{m=0}^{\frac{n+1}{2}} \frac{(-1)^m t^{n-2m}}{(n - 2m)!}.$$  

and when $n$ is even,

$$f^{(n)}(t) = \frac{n! (n + 1)!}{(t^2 + 1)^{n+1}} \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m t^{n-2m}}{(n - 2m)!}.$$  

\[\square\]

Question 7 (p.106 #25). Let $P(\psi) = \sum_{n=0}^{N-1} e^{in\psi}$.

(a) Show that

$$|P(\psi)| = \left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right|.$$  

(b) Find $\lim_{\psi \to 0} |P(\psi)|$.

(c) Plot $|P(\psi)|$ for $0 \leq \psi \leq 2$ and $N = 3$.

Solution.

(a) Note that

$$P(\psi) = \frac{1 - e^{iN\psi}}{1 - e^{i\psi}}.$$
Thus,
\[ P(\psi) = \frac{e^{iN\psi} - 1}{e^{i\psi} - 1} = \frac{e^{in\psi/2} - e^{-in\psi/2}}{e^{i\psi/2} - e^{-i\psi/2}} = \frac{\cos(n\psi/2) + isin(n\psi/2) - \cos(-n\psi/2) - isin(-n\psi/2)}{\cos(\psi/2) + isin(\psi/2) - \cos(-\psi/2) - isin(-\psi/2)} = \frac{2isin(N\psi/2)}{2isin(\psi/2)}. \]

Thus,
\[ |P(\psi)| = \left| \frac{e^{in\psi/2}}{e^{i\psi/2}} \right| = \left| \frac{\cos(n\psi/2)}{\sin(\psi/2)} \right| = \left| \frac{\sin(N\psi/2)}{\sin(\psi/2)} \right|. \]

(b) By l’Hopital’s rule we get
\[ \lim_{\psi \to 0} \frac{\sin(N\psi/2)}{\sin(\psi/2)} = \lim_{\psi \to 0} \frac{N/2 \sin(N\psi/2)}{1/2 \sin(\psi/2)} = N. \]

(c) If you have nothing else, just plug it in Wolfram Alpha.

\[ \square \]

Question 8 (p.112 #17). Show that \( \sin z - \cos z = 0 \) has solutions only for real values of \( z \). What are the solutions?

Solution. In other words, for \( z = x + iy \) we want
\[ \sin x \cosh y + i \cos x \sinh y = \cos x \cosh y - i \sin x \sinh y. \]

Equating the real parts and imaginary parts we require
\[ \begin{align*}
\sin x \cosh y &= \cos x \cosh y \\
\cos x \sinh y &= -\sin x \sinh y.
\end{align*} \]

Suppose \( y \neq 0 \) and hence \( \sinh y \neq 0 \) and \( \cosh y \neq 0 \). Then in order to have solutions, by (3), we need \( \cos x = \sin x \) and by (4) we need \( \cos x = -\sin x \). These equations are only satisfied for \( \sin x = \cos x = 0 \), but no solutions for \( x \) exists. Therefore, if there are solutions to the original equation, we must have that \( y = 0 \).

Suppose \( y = 0 \). Then since \( \cosh 0 = 1 \) and \( \sinh 0 = 0 \) we simply need solutions to \( \sin x = \cos x \). Thus we have solutions if and only if
\[ z = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}. \]

\[ \square \]

Question 9 (p.112 #21). Where does the function \( f(z) = \frac{1}{\sqrt{3} \sin z - \cos z} \) fail to be analytic?

Solution. Since \( \sin z \) and \( \cos z \) are both analytic, \( f(z) \) will fail to be analytic when \( \sqrt{3} \sin z - \cos z = 0 \). In other words, when we have solutions to
\[ \sqrt{3} \sin x \cosh y + i \cos x \sinh y = \cos x \cosh y - i \sin x \sinh y. \]

Equating the real parts and imaginary parts we require
\[ \begin{align*}
\sqrt{3} \sin x \cosh y &= \cos x \cosh y \\
\sqrt{3} \cos x \sinh y &= -\sin x \sinh y.
\end{align*} \]

By the same argument as the previous question, there are no solutions when \( y \neq 0 \). Suppose \( y = 0 \). Then since \( \cosh 0 = 1 \) and \( \sinh 0 = 0 \) we simply need solutions to \( \sqrt{3} \sin x = \cos x \), that is to \( \tan x = \frac{1}{\sqrt{3}} \). So \( f(z) \) is not analytic when
\[ z = \frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}. \]

\[ \square \]

Question 10 (p.112 #22). Let \( f(z) = \sin \left( \frac{z}{2} \right) \).

(a) Express this function in the form \( u(x,y) + iv(x,y) \). Where in the complex plane is this function analytic?

(b) What is the derivative of \( f(z) \)? Where in the complex plane is \( f'(z) \) analytic?

Solution.
(a) Since \( \sin z \) is entire, and \( \frac{1}{z} \) is analytic for \( z \neq 0 \), it follows that \( f(z) \) is analytic for \( z \neq 0 \).

\[
\sin \left( \frac{1}{z} \right) = \sin \left( \frac{x - iy}{x^2 + y^2} \right) \\
= \sin \left( \frac{x}{x^2 + y^2} \right) \cosh \left( \frac{-y}{x^2 + y^2} \right) + i \cos \left( \frac{x}{x^2 + y^2} \right) \sinh \left( \frac{-y}{x^2 + y^2} \right) \\
= \sin \left( \frac{x}{x^2 + y^2} \right) \cosh \left( \frac{y}{x^2 + y^2} \right) - i \cos \left( \frac{x}{x^2 + y^2} \right) \sinh \left( \frac{y}{x^2 + y^2} \right).
\]

(b) For \( z \neq 0 \),

\[
\frac{d}{dz} \sin \left( \frac{1}{z} \right) = (\cos \frac{1}{z}) \left( -\frac{1}{z^2} \right)
\]

which is analytic for all \( z \neq 0 \).

\[\square\]

**Question 11** (p.112 #25). Show that \( |\cos z| = \sqrt{\sinh^2 y + \cos^2 x} \).

**Solution.**

\[
|\cos z| = |\cos x \cosh y - i \sin x \sinh y| \\
= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\
= \sqrt{\cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y} \\
= \sqrt{\cos^2 x + \sinh^2 y (\cos^2 x + \sin^2 x)} \\
= \sqrt{\cos^2 x + \sinh^2 y}
\]

\[\square\]

**Question 12** (p.119 #16). Use logarithms to find solutions to \( e^z = e^{i\pi} \).

**Solution.** We want solutions to \( e^z (1 - i) = 1 \), so taking logs on both sides we get for any \( k \in \mathbb{Z} \), \( z(1 - i) = 2\pi i k \), so

\[
z = \frac{2\pi i k}{1 - i} = \left( \frac{1}{2} + i \frac{3\pi}{2} \right) = (i - 1)k\pi.
\]

\[\square\]

**Question 13** (p.119 #18). Use logarithms to find solutions to \( e^z = (e^z - 1)^2 \).

**Solution.** In other words, we want solutions to \( e^{2z} - 3e^z + 1 = 0 \). By the quadratic formula, we get that

\[
e^z = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm \sqrt{1}}{2}.
\]

Taking logs gives that

\[
z = \log \left( \frac{3}{2} \pm \frac{1}{2} \right) + 2\pi i k
\]

for \( k \in \mathbb{Z} \).

\[\square\]

**Question 14** (p.119 #21). Use logarithms to find solutions to \( e^z = 1 \).

**Solution.** First, taking logs we get \( e^z = 2\pi i k \) for \( k \in \mathbb{Z} \). Now for \( k > 0 \), the argument of \( 2\pi i k \) is \( \frac{\pi}{2} + 2\pi m \) where \( m \in \mathbb{Z} \), and for \( k < 0 \), the argument of \( 2\pi i k \) is \( \frac{3\pi}{2} + 2\pi m \) (again \( m \in \mathbb{Z} \)). Thus, for \( k > 0 \),

\[
z = \log(2\pi k) + i \left( \frac{\pi}{2} + 2m\pi \right)
\]

and for \( k < 0 \),

\[
z = \log(2\pi k) + i \left( -\frac{\pi}{2} + 2m\pi \right).
\]

\[\square\]

**Question 15** (p.119 #23). Show that

\[
\text{Re} \left( \log(1 + e^{i\theta}) \right) = \log \left| 2 \cos \left( \frac{\theta}{2} \right) \right|
\]

where \( \theta \in \mathbb{R} \) and \( e^{i\theta} \neq -1 \).
Along the given path, this attains a maximum when

\[ \text{Solution.} \]

Integration, show that

\[ \text{Question} \]

Theorem, the length of the path is

\[ \text{Solution.} \]

Parametrization is 1:1 except for when

\[ \text{Has solutions} \]

And furthermore, \( 2 \cos \theta \)

\[ \text{Solution.} \]

Ellipse in the first quadrant.

\[ \text{Question} \]

\[ \text{Solution.} \]

Letting \( z = e^{it} \), then we are integrating along the interval \( t \in [0, \pi] \). Now, \( dz = ie^{it} dt \) so

\[ \int_{1}^{-1} \frac{1}{z} \, dz = \int_{0}^{-\pi} \frac{1}{e^{it}} ie^{it} dt = -i\pi. \]

\[ \text{Question 17 (p.170 #11).} \]

Show that \( x = 2 \cos t, y = \sin t \), where \( t \) ranges from 0 to \( 2\pi \), yields a parametric representation of the ellipse \( \frac{x^2}{4} + y^2 = 1 \). Use this representation to evaluate \( \int_{\frac{3}{2}}^{\frac{3}{2}} \bar{z} \, dz \) along the portion of the ellipse in the first quadrant.

\[ \text{Solution.} \]

Note that

\[ \frac{(2 \cos t)^2}{4} + \sin^2 t = \cos^2 t + \sin^2 t = 1 \]

and furthermore \( 2 \cos 0 = 2 \cos 2\pi = 2 \) and \( \sin 0 = \sin 2\pi = 0 \). To see that we get all of the ellipse, note that \( x = 2 \cos t \) has solutions \( t \in [0, 2\pi] \) for all \( x \in [-2, 2] \) and \( y = \sin t \) has solutions \( t \in [0, 2\pi] \) for all \( y \in [-1, 1] \). Furthermore, the parametrization is 1:1 except for when \( x = 2, y = 0 \).

Setting \( z = x + iy = 2 \cos t + i \sin t \), we get \( dz = (i \cos t - 2 \sin t) dt \) and

\[ \int_{\frac{3}{2}} \bar{z} \, dz = \int_{0}^{\frac{3}{2}} (2 \cos t - i \sin t) (i \cos t - 2 \sin t) \, dt \]

\[ = \int_{0}^{\frac{3}{2}} (2i - 3 \sin t \cos t) \, dt \]

\[ = \frac{3}{2} + i\pi. \]

\[ \text{Question 18 (p.170 #14).} \]

Consider \( I = \int_{0}^{2+1} e^{i \bar{z}} \, dz \) taken along the line \( x = 2y \). Without actually doing the integration, show that \( |I| \leq \sqrt{5}e^{3} \).

\[ \text{Solution.} \]

Let \( M \) be the maximal value attained by \( |e^{i \bar{z}}| \) along the path of integration. Now, for \( x = 2y \),

\[ |e^{i \bar{z}}| = |e^{2y^2 - 2xy} - 2i\bar{y}y| = e^{3y^2} \]

which attains its maximum when \( y \) attains a maximum—that is, when \( z = 2 + i \). Therefore \( M = e^{3} \). By the pythagorean theorem, the length of the path is \( \sqrt{2^2 + 1^2} = \sqrt{5} \), so by the ML inequality, \( |I| \leq ML = \sqrt{5}e^{3} \).

\[ \text{Question 19 (p.170 #16).} \]

Consider \( I = \int_{1}^{1} e^{i \log \bar{z}} \, dz \) taken along the parabola \( y = 1 - x^2 \). Without doing the integration, show that \( |I| \leq 1.479e^{\pi/2} \).

\[ \text{Solution.} \]

Letting \( \theta = \arg z \)

\[ |e^{i \log \bar{z}}| = |e^{i \log |z| - i\theta}| = |e^{i \log |z|}e^{i \theta}| = e^{i \theta}. \]

Along the given path, this attains a maximum when \( \theta = \pi/2 \), so let \( M = e^{\pi/2} \).
Now, we need to find the length of the path of integration. So since $dy = -2x\,dx$,
\[
L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
= \int_0^1 \sqrt{1 + 4x^2} \, dx
< 1.479.
\]
The ML inequality then gives the desired result. \qed

\textbf{Question 20 (p.180 #2).} Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z|=1} \frac{\sin z}{z^2} \, dz$?

\textbf{Solution.} Since $\frac{\sin z}{z^2}$ is analytic everywhere except for $z = -2\pi$ which is not in the unit circle, the C-G theorem is directly applicable. \qed

\textbf{Question 21 (p.180 #6).} Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z-i|=1} \log z \, dz$?

\textbf{Solution.} Since $0$ is not in the unit circle about $i+1$, $\log z$ is analytic in the desired region so the C-G theorem is directly applicable. \qed

\textbf{Question 22 (p.180 #7).} Is the Cauchy-Goursat theorem directly applicable to $\oint_{|z|=1/2} \frac{1}{(z-1)^4+1} \, dz$?

\textbf{Solution.} Observe that we have a singularity when $z-1$ is a primitive 8th root of unity—that is, when $(z-1)^4 = -1$. These roots of unity lie on the unit circle, so shifting over by $1$, we need to determine if the roots closest to the origin, at $z = e^{\pi i/4} + 1$ and $z = e^{3\pi i/4} + 1$ have absolute value greater than $1/2$. By geometry (right angle triangles), it can be seen that at these points,
\[
|z| = \sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} > 1/2
\]
so the C-G theorem directly applies. \qed

\textbf{Question 23 (p.180 #9).} Is the Cauchy-Goursat theorem directly applicable to
\[
\oint_{|z|=b} \frac{1}{z^2 + bz + 1} \, dz
\]
where $0 < b < 1$?

\textbf{Solution.} In this case the singularities are at the roots of the equation $x^2 + bx + 1$, that is, when
\[
z = \frac{-b \pm \sqrt{b^2 - 4}}{2} = \frac{-b}{2} \pm i \frac{\sqrt{4 - b^2}}{2}.
\]
Here,
\[
|z| = \sqrt{\frac{b^2}{4} + \frac{4 - b^2}{4}} = 1 > b
\]
therefore C-G applies directly. \qed

\textbf{Question 24 (p.180 #13).} Prove that
\[
\int_0^{2\pi} e^{\cos \theta} (\sin(\sin \theta + \theta)) \, d\theta = 0.
\]
Begin with $e^z \, dz$ performed around $|z| = 1$. Use the parametric representation $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Separate your equation into real and imaginary parts.

\textbf{Solution.} Let $z = e^{i\theta} = \cos \theta + i\sin \theta$, so $dz = e^{i\theta} \, d\theta$. Since $e^z$ is analytic,
\[
\oint_{|z|=1} e^z \, dz = \int_0^{2\pi} e^{\cos \theta + i\sin \theta} \, e^{i\theta} \, d\theta = 0.
\]
But then
\[
\int_0^{2\pi} e^{\cos \theta + i(\sin \theta + \theta)} \, d\theta = \int_0^{2\pi} e^{e^{i\theta}} (\cos(\theta + \sin \theta) + i\sin(\theta + \sin \theta)) \, d\theta = 0
\]
so by equating the imaginary part with zero we get the desired result. \qed