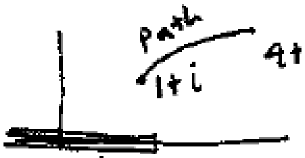
MATH - 381

Homework#3 Solutions

7] $\dfrac{z}{z^2-1} = \dfrac{1}{2}\left[\dfrac{1}{z-1} + \dfrac{1}{z+1}\right]$.

Now $\dfrac{d}{dz}\left[\dfrac{1}{2}\,\mathrm{Log}(z-1) + \dfrac{1}{2}\,\mathrm{Log}(z+1)\right] = \dfrac{z}{z^2-1}$,

The Function $\frac{1}{2}\left[\mathrm{Log}(z-1) + \mathrm{Log}(z+1)\right]$ is analytic in a simply connected domain containing the path



branch cut for $\mathrm{Log}(z-1) + \mathrm{Log}(z+1)$

answer = $\dfrac{1}{2}\left[\mathrm{Log}(z-1) + \mathrm{Log}(z+1)\right]_{1+i}^{4+2i}$ =

$\frac{1}{2}\left[\mathrm{Log}(3+2i) + \mathrm{Log}[5+2i] - \mathrm{Log}(i) - \mathrm{Log}(2+i)\right]$

(continued next page)

7 <u>continued</u>

$\frac{1}{2}\left[\mathrm{Log}(3+2i) - \mathrm{Log}\,i + \mathrm{Log}[5+2i) - \mathrm{Log}(2+i)\right]$

$= \dfrac{1}{2}\left[\mathrm{Log}\left(\dfrac{3+2i}{i}\right) + \mathrm{Log}\left(\dfrac{5+2i}{2+i}\right)\right] =$

$\dfrac{1}{2}\left[\mathrm{Log}\quad \dfrac{11+16i}{-1+2i}\right]$

10] Note: $\frac{d}{dz}\frac{(\mathrm{Log}\, z)^2}{2} = \frac{\mathrm{Log}\, z}{z}$ analytic in any domain not containing $z=0$ or points on the negative real axis. Our contour can lie in such a domain. Thus $\int_{1+i}^{-1-i} \frac{\mathrm{Log}\, z}{z}\, dz = \frac{1}{2}(\mathrm{Log}\, z)^2\Big|_{1+i}^{-1-i}$

$= \frac{1}{2}\left\{\left[\mathrm{Log}\sqrt{2} - i3\frac{\pi}{4}\right]^2 - \left(\mathrm{Log}\sqrt{2} + i\frac{\pi}{4}\right)^2\right\} = -\,i\pi\,\mathrm{Log}\,2 - \pi^2/2$

12] As above $\int_1^i z^{1/2}\, dz = \frac{2}{3} z * z^{1/2}\Big|_1^i$

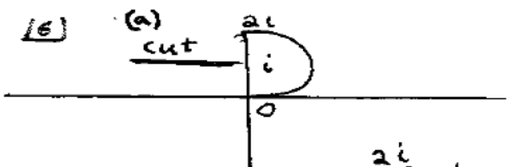
$z^{1/2} = \sqrt{|z|}\ \angle\frac{\theta}{2} + k\pi$, take $k=1$, $-\pi < \theta < \pi$

$i^{1/2} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $1^{1/2} = (-1)$

$\int_1^i z^{1/2}\, dz = \left(\frac{2}{3}\right) i \left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] - \frac{2}{3}(1) * (-1)$

$= \boxed{\frac{2}{3} + \frac{2}{3}\frac{1}{\sqrt{2}} - \frac{2}{3}\frac{i}{\sqrt{2}}}$

16] (a)



$\frac{d}{dz}\mathrm{Log}(z-i) = \frac{1}{z-i}$

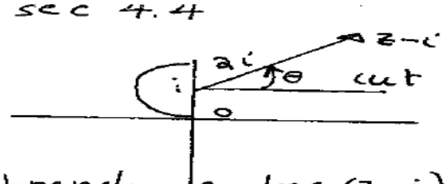
use princ branch cut as shown

$\int_0^{2i} \left(\frac{1}{z-i}\right) dz = \mathrm{Log}(z-i)\Big|_0^{2i} = \mathrm{Log}\, i - \mathrm{Log}\, {-i}$

$= \boxed{i\pi}$

16] sec 4.4

(b) continued



$\theta = \arg(z-i)$

than in part a.

Use a different branch of $\log(z-i)$. Take

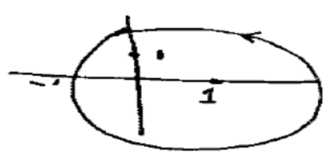
$\frac{d}{dz}\log(z-i) = \frac{1}{(z-i)}$ Now $\log(z-i) = \mathrm{Log}|z-i| + i\arg(z-i)$

Take $0 < \arg(z-i) < 2\pi$

$\int_0^{2i} \frac{1}{(z-i)}\, dz = \log(z-i)\Big|_0^{2i} = \log((z-i)-i)$ $(z = 2i)$

$- \log[0 - i] = i\frac{\pi}{2} - i\frac{3\pi}{2} = \boxed{-i\pi}$

8]



$\frac{1}{2\pi i}\oint_{|z-1|=2} \frac{e^{iz}}{(z-i)^2}\, dz = \frac{d}{dz} e^{iz}\Big|_{z=i}$

$= i\, e^{iz}\Big|_i = \boxed{i e^{-1}}$

12) $z_0 = 0$, $\quad f^{n}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$f(z) = \sin 2z$, $n = 14$

$$\left.\frac{\frac{d^{14}}{dz^{14}}\sin 2z}{14!}\right|_{z=0} = \frac{1}{2\pi i}\oint \frac{\sin(2z)}{z^{15}}\,dz$$

Note, any even derivative of $\sin(2z)$ has the form (constant) $*\ \sin(2z)$ [whose value is zero at $z=0$] $\therefore$ answer is zero.

16) a) If $|a| > 1$, $\frac{1}{z-a}$ is analytic on and inside $|z|=1$. Thus $\oint \frac{dz}{(z-a)} = \boxed{0}$ [Cauchy-Goursat]

Now if $|a| < 1$, $\oint \frac{dz}{z-a} = 2\pi i \Big|_{z=a} = \boxed{2\pi i,}$ Cauchy Inter Formula

b) $\bar{z}$ is nowhere analytic, $\frac{1}{\bar{z}-a}$ is nowhere analytic

Thus neither the Cauchy-Goursat Theorem nor the Cauchy Integral Formula apply.

Suppose $|a| > 1$. $\quad \oint \frac{dz}{\bar{z}-a} = \oint \frac{dz}{\frac{1}{z}-a} = \oint \frac{z\,dz}{1-az}$

$$= \frac{1}{-a}\oint \frac{z}{z-\frac{1}{a}}\,dz = \frac{2\pi i}{-a}\, z\Big/_{1/a} = \boxed{-\frac{2\pi i}{a^2}} \quad |a| > 1$$

Suppose $|a| < 1$

$\oint \frac{dz}{\bar{z}-a} = \frac{-1}{a}\oint \frac{z\,dz}{z-\frac{1}{a}}$ as above. The integrand is analytic in and on C. $\therefore$ answer $= \boxed{0}$ for $|a| < 1$

The sets of answers to (a) and (b) are completely different

sec 4.6

1) $\frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}}\,d\theta$ use (4.6-1), $z_0 = 0$, $f(z) = e^z$, $r = 1$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}}\,d\theta = e^{z}\Big|_0 = 1$$

2) $\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = \int_0^{2\pi}\cdots\,d\theta$

use Eq. (4.6-1). Consider previous problem

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta + i\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}\left[\cos[\sin\theta] + i\sin(\sin\theta)\right]d\theta$$

$= 1$, use real part only

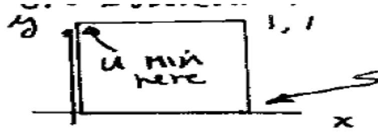
$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}\cos[\sin\theta]\,d\theta = 1. \quad \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi$$

11) $|f(z)| = e^x$

$\boxed{\text{max value} = e^2 \quad \text{at } x=2, y=1}$

$\boxed{\text{min value} = e^0 = 1 \quad \text{at } x=0, y=1}$

15)



Note u is harmonic

u max here

$\boxed{\text{max at } 1, 0}$ $\boxed{\text{min at } 0, 1}$

3) $|u_n| = |2iz|^n = 2^n |z|^n$

suppose $|z| = \frac{1}{2}$, then $|u_n| = 1$ all n, and $|u_n|$ does not go to zero as $n \to \infty$. ∴ series div

suppose $|z| > \frac{1}{2}$, then $2|z| > 1$. $|u_n| = (2|z|)^n$ goes to ∞ as $n \to \infty$. Since this is not zero seq. diverges

5) $|u_n| = \dfrac{n(\sqrt{2})^n}{|z-2i|^n}$. If $|z-2i| = \sqrt{2}$, $|u_n| = n$

$|u_n| \to \infty$ as $n \to \infty$. Series diverges.

If $|z-2i| < \sqrt{2}$, $\dfrac{\sqrt{2}}{|z-2i|} > 1$ and $n\left|\dfrac{\sqrt{2}}{z-2i}\right|^n \to \infty$ as $n \to \infty$

Series diverges.

7) $\left|\dfrac{u_{n+1}}{u_n}\right| = \dfrac{(n+1)^2}{n^2} \dfrac{|z+\frac{1}{2}|^{n+1}}{|z+1/2|^n} = \left(1+\frac{1}{n}\right)^2 \left|z+\frac{1}{2}\right| = |z+1/2|$ as $n \to \infty$

∴ series is abs. conv. $|z+1/2| < 1$ and diverges for $|z+1/2| > 1$.