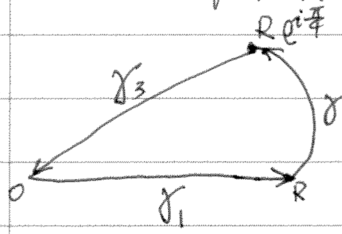
Math 366 HW 2 Solution

P64. #1.



By Cauchy's Thm,

$$\int_{\gamma_1} e^{-z^2}dz + \int_{\gamma_2} e^{-z^2}dz + \int_{\gamma_3} e^{-z^2}dz =$$

We have $\displaystyle\int_{\gamma_1} e^{-z^2}dz = \int_0^R e^{-x^2}dx \to \frac{1}{2}\sqrt{\pi}$ as $R \to \infty$

$$\left|\int_{\gamma_2} e^{-z^2}dz\right| \overset{z=Re^{it}}{=} \left|\int_0^{\frac{\pi}{4}} e^{-R^2 e^{2it}}\, iRe^{it}dt\right|$$

$$\leq \int_0^{\frac{\pi}{4}} e^{-R^2\cos 2t}\, R\,dt$$

$$\xrightarrow{R\to\infty} 0 \quad \text{(use D.C.T. when exchanging "lim" \& "}\textstyle\int\text{")}$$

$$\int_{\gamma_3} e^{-z^2}dz \overset{z=xe^{i\frac{\pi}{4}}}{=} -\int_0^R e^{-x^2 e^{i\frac{\pi}{2}}}e^{i\frac{\pi}{4}}dx$$

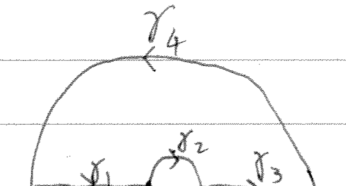
$$= -\frac{1+i}{\sqrt{2}}\int_0^R (\cos^2 x - i\sin^2 x)dx.$$

Thus, as $R\to\infty$,

$$\sqrt{\frac{\pi}{2}} = \int_0^\infty [\cos^2 x + \sin^2 x + i(\cos^2 x - \sin^2 x)]dx$$

Hence, $\displaystyle\int_0^\infty \sin^2 x\,dx = \int_0^\infty \cos^2 x\,dx = \frac{\sqrt{2\pi}}{4}$

P64. #2



By Cauchy's Thm,

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz =$$

where $f(z) = \dfrac{e^{iz}-1}{z}$. Moreover,

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = \int_{\varepsilon\le|x|\le R} \frac{e^{ix}-1}{x}dx$$

$$\int_{\gamma_2} f(z)dz + \int_{\gamma_4} f(z)dz = \int_0^\pi \left(\frac{e^{i\varepsilon e^{it}}-1}{\varepsilon e^{it}}\varepsilon + \frac{e^{iRe^{it}}-1}{Re^{it}}R\right) ie^{it}$$

$$= -i\int_0^\pi (e^{i\varepsilon e^{it}}-1)dt + i\int_0^\pi e^{iRe^{it}}dt - i\pi$$

Now, $\displaystyle\left|\int_0^\pi (e^{i\varepsilon e^{it}}-1)dt\right| \le \int_0^\pi |e^{i\varepsilon e^{it}}-1|\,dt \xrightarrow{\varepsilon\downarrow 0} 0$, and $\left| \int_0^{\pi} e^{iRe^{it}} dt \right| \le \int_0^{\pi} \left| e^{iR(\cos t + i\sin t)} \right| dt = \int_0^{\pi} e^{-}$

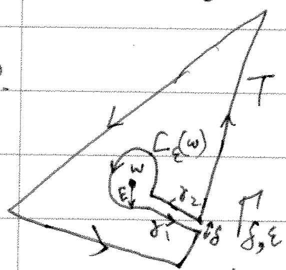
$\to 0$ as $R \to \infty$

Hence, as $R \to \infty$ and $\varepsilon \to 0^+$,

$$i\pi = \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x} dx = \int_{-\infty}^{\infty} \frac{\cos x + i\sin x - 1}{x} dx$$

Taking imaginary part gives $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

P65. #6.



Since f is holomorphic in Ω except at w

so it is holomorphic in an open set containing $\Gamma_{\delta,\varepsilon}$

By Cauchy's Thm,

$$\int_{\gamma_1 \cup \gamma_2} f(z) dz + \int_T f(z) dz + \int_{C_\varepsilon(w)} f(z) dz = 0$$

We have $\left| \int_{C_\varepsilon(w)} f(z) dz \right| \le 2\pi\varepsilon \cdot \sup_{z \in C_\varepsilon(w)} |f(z)| \le 2\pi\varepsilon M$

for some $M > 0$ since f is bounded near w.

~~So $\to 0$ as $\varepsilon \to 0$,~~

Also, $\int_{\gamma_1} f(z) dz$ cancels out with $\int_{\gamma_2} f(z) dz$ as $\delta \to 0$

So, $\int_T f(z) dz = 0$ if we let $\delta \to 0$ and $\varepsilon \to 0$.

P65. #8. For $x \in \mathbb{R}$, consider $C = C(x, \frac{1}{2}) \subset$ the strip $-1 < y < 1$.

By Cauchy inequality,

$$\left| f^{(n)}(x) \right| \le 2^n n! \sup_{z \in C} |f(z)| \le 2^n n! A \sup_{z \in C} (1+|z|)^{\eta}$$

If $\eta \ge 0$, $\forall z \in C$, $(1+|z|)^{\eta} \le (\frac{3}{2})^{\eta} (1+|x|)^{\eta}$

So, $\left| f^{(n)}(x) \right| \le 2^n n! A (\frac{3}{2})^{\eta} (1+|x|)^{\eta}$

Take $A_n = 2^n n! A (\frac{3}{2})^{\eta} > 0$.

If $\eta < 0$, $\forall z \in C$, $(1+|z|)^{\eta} \le \frac{1}{2^{\eta}} (1+|x|)^{\eta}$

So, $\left| f^{(n)}(x) \right| \le \frac{1}{2^{\eta}} (1+|x|)^{\eta} \cdot 2^n n! A$

Take $A_n = \frac{2^n}{2^{\eta}} n! A > 0$

Let $R < R_0$ and $|z| < R$.

P66 #11 (a) $\left|\frac{R^2}{\bar{z}}\right| > R$ for $|z| < R$, so the disc D_R does not conta[in]
singularities of $\frac{f(\zeta)}{\zeta - R^2/\bar{z}}$.
By Cauchy's Thm, $\int_{C_R} \frac{f(\zeta)}{\zeta - R^2/\bar{z}} d\zeta = 0$,
where C_R is the circle centered at $\vec{O}$ with radius R,
oriented counter-clockwise. Using Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - R^2/\bar{z}} d\zeta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{it}) \frac{R^2 - z\bar{z}}{R^2 - \bar{z}Re^{it} - zRe^{-it} + z\bar{z}} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{it}) \frac{R^2 - z\bar{z}}{|Re^{it} - z|^2} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{it}) \operatorname{Re}\left(\frac{Re^{it} + z}{Re^{it} - z}\right) dt$$

(b) $$\operatorname{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r}\right) = \frac{1}{2}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r} + \frac{Re^{-i\gamma} + r}{Re^{-i\gamma} - r}\right)$$

$$= \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}$$

P66. #12. (a) Let $g(z) = \frac{2\partial u}{\partial z}$, and let $z = x + iy$.
Then $g(z) = 2 \cdot \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y) u$

~~Since g is holom~~

Furthermore, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u\right) = u_{xx}$, $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -u_{yy}$

Since $\Delta u = 0$ so $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$.

Also, $\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$.

Thus, the Cauchy-Riemann equations hold for g on $\mathbb{D}$
and so g is holomorphic on $\mathbb{D}$.

By Thm 2.1, g has a primitive F on $\mathbb{D}$, i.e. $F'(z) = g(z)$.
Write $F'(z) = \tilde{u}(x,y) + i\tilde{v}(x,y)$.
Then $F'(z) = g(z) \Rightarrow \frac{2\partial\tilde{u}}{\partial z} = 2\frac{\partial u}{\partial z} \Rightarrow \frac{\partial\tilde{u}}{\partial x} - i\frac{\partial\tilde{u}}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$
Taking real and imaginary parts respectively gives $\tilde{u} = u + C$
for some constant $C \in \mathbb{R}$. Take $f(z) = F(z) - C$, then $\operatorname{Re}(f) = u$

(b) Let $R=1$ and $z=re^{i\theta}$ in P66. #11. Then,

$$\begin{aligned} u(z) &= \mathrm{Re}(f(z)) \\ &= \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)dt\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re} f(e^{it})\,\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\cancel{\theta}-t)\,u(e^{it})\,dt \end{aligned}$$

P67. #13 Write $\overline{\mathbb{D}}$ for the closed unit disc in $\mathbb{C}$. By assumption, for each $z_0 \in \overline{\mathbb{D}}$, $\exists N_{z_0} \in \mathbb{N}$ st. $C_{N_{z_0}} = 0$ in the expansion

$$f(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n$$

Note that $C_n = \frac{1}{n!} f^{(n)}(z_0)$. Then there ~~are~~ is at least an $n_0 \in \mathbb{N}$ st. $f^{(n_0)}(z)=0$ for infinitely many $z \in \overline{\mathbb{D}}$, since $\overline{\mathbb{D}}$ is uncountable while $\mathbb{N}$ is countable.

Let such points be $(z_k)_{k\geq 1}$.

Since $\overline{\mathbb{D}}$ is compact, so $(z_k)_{k\geq1}$ has a limit point in $\overline{\mathbb{D}}$.

Then by Thm 4.8, $f^{(n_0)}$ is zero ~~or~~ everywhere in $\mathbb{C}$.

Hence, f is a polynomial of degree at most n_0.

P67. #14. Let D_{S+1} be an open disc with radius $1+S$ for $S>0$, centered at $\vec{0}$, and contained in f's holomorphic dom ~~By assumption, there exists a Let D_S be the open unit disc in~~

By assumption, f is holomorphic in D_{S+1} and has a pole at z_0.

So $f(z) = \frac{1}{z-z_0} g(z)$ where $g(z)$ is holomorphic on D_{S+1}

and hence it is analytic. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{z-z_0}\sum_{n=0}^{\infty} b_n z^n$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} a_n z^n\right)(z-z_0) = \sum_{n=0}^{\infty} b_n z^n$$

RHS converges as g is analytic, so LHS also converges.

Hence $\lim_{n\to\infty} |a_{n-1} - a_n z_0| = 0$, or $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$

P67. #15 First define an entire and bounded extension of $f(z)$. Let

$$F(z) = \begin{cases} f(z) & \text{for } |z| \leq 1 \\ \dfrac{1}{\overline{f(\frac{1}{\bar{z}})}} & \text{for } |z| > 1 \end{cases}$$

Note that $\lim_{|z| \downarrow 1} F(z) = \lim_{|z| \uparrow 1} \frac{1}{\overline{f(z)}} \stackrel{\leftarrow}{=} 1$ by assumption.

and that $\lim_{|z| \uparrow 1} F(z) = 1$.

So $F(z)$ extend continuously on ~~$|z| \leq 1$ to~~ $\partial \mathbb{D}$.

With a similar argument as in Schwarz reflection principle, we have $F(z)$ is entire.

Secondly, note that $F(z) \neq 0$ is continuous on $\overline{\mathbb{D}}$ so it is bounded from above and also away from zero. And for $|z| > 1$, $F(z)$ is clearly bounded from above and away from zero.

By Liouville Thm, $F(z)$ is a constant, so is $f(z)$.

P68. #2 Since $2 \leq d(n) \leq n$, so $\limsup_{n\to\infty} 2^{\frac{1}{n}} \leq \limsup_{n\to\infty} d(n)^{\frac{1}{n}} \leq \limsup_{n\to\infty} n^{\frac{1}{n}}$

Hence $\limsup_{n\to\infty} d(n)^{\frac{1}{n}} = 1$. Thus Radius of Convergence for $F(z)$ is also 1.

Note that $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z^{nm}$ for $|z| < 1$.

For a fixed $k \in \mathbb{N}$, the number of appearances of the term z^k in the above series is equal to the number of ways of writing $k = nm$ for $n, m \in \mathbb{N}$. This is equal to the number of divisors of k.

Hence the identity.

$$F(r) = \sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \geq \sum_{n=1}^{\infty} \frac{r^n}{n(1-r)} = \frac{1}{1-r} \sum_{n=1}^{\infty} \frac{r^n}{n} = \frac{1}{1-r} \log\left(\frac{1}{1-r}\right)$$

Let $\theta = \frac{2\pi p}{q}$ for $p, q \in \mathbb{Z}^+$, and let $z = re^{i\theta}$, the

$$F(re^{i\theta}) = \sum_{n=1}^{\infty} d(n) r^n e^{in\theta}$$

$$= \sum_{n=1}^{\infty} d(1+nq) r^{1+nq} e^{i\theta} + \sum_{n=0}^{\infty} d(2+nq) r^{2+nq} e^{i2\theta}$$

$$+ \cdots + \sum_{n=0}^{\infty} d(q+nq) r^{q+nq} e^{iq\theta}$$

Since $|z| \geq \mathrm{Re}(z)$ for $z \in \mathbb{C}$, so

$$|F(re^{i\theta})| \geq \sum_{n=0}^{\infty} d(1+nq) r^{1+nq} \cos\theta + \cdots + \sum_{n=0}^{\infty} d(q+nq) r^{q+nq} \cos q$$

$$\geq C_{p/q} \sum_{n=1}^{\infty} d(n) r^n = C_{p/q} F(r) \geq \frac{C_{p/q}}{1-r} \log\left(\frac{1}{1-r}\right)$$

~~The~~

Since $\frac{1}{1-r} \log\left(\frac{1}{1-r}\right) \to \infty$ as $r \to 1^-$, so F cannot be continu analytically past the unit disc.