SOLUTIONS TO ASSIGNMENT 2 - MATH 355

Problem 1.
Recall that,

$$B_n = \{\omega \in [0,1] : |S_n(\omega)| > n\epsilon_n\},$$

$$N = \{\omega \in [0,1] : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0\},$$

and

$$m(B_n) \leq \frac{3}{n^2\epsilon^4}.$$

We want to show that $m(N^c) = 0$.

Let $\delta > 0$. We can pick $\epsilon_n^4 = \frac{c}{\sqrt{n}}$ with $c > \frac{3}{\delta}\sum_{n=1}^{\infty} n^{-3/2}$. Then,

$$\sum_{n=1}^{\infty} m(B_n) \leq \frac{3}{c}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} < \delta.$$

Now, if $\omega \in N^c$ then there exists $\mu > 0$ such that for all $n \in \mathbb{N}$, $|S_n(\omega)| \geq n\mu$. There exists $N \in \mathbb{N}$ such that $\epsilon_N < \mu$; hence $\omega \in B_N$. Thus, $N^c \subset \cup_{n=1}^{\infty} B_n$ and

$$m(N^c) < \delta.$$

Since $\delta > 0$ is arbitrary, $m(N^c) = 0$..

Problems from the textbook. Pages 90-93.

Problem 4. Suppose f is integrable on $[0,b]$ and

$$g(x) = \int_x^b \frac{f(t)}{t}dt,$$

$0 < x \leq b$. Prove that g is integrable on $[0,b]$ and

$$\int_0^b g(x)dx = \int_0^b f(t)dt.$$

Solution. We can assume that $f(x) \geq 0$ (and hence $g(x) \geq 0$), since $f = f^+ - f^-$ and the Lebesgue integral is linear.

$$\begin{aligned}\int_0^b g(x)dx &= \int_0^b \int_x^b \frac{f(t)}{t} dt dx = \int_0^b \int_0^b \frac{f(t)}{t} \chi_{\{t \geq x\}}(t) dx dt \\ &= \int_0^b \frac{f(t)}{t} \int_0^b \chi_{\{x \leq t\}}(x) dx\, dt = \int_0^b \frac{f(t)}{t} \int_0^t dx\, dt \\ &= \int_0^b \frac{f(t)}{t} t\, dt,\end{aligned}$$

where we used Tonelli's theorem to switch the order of integration.

Problem 5. Suppose F is a closed set in $\mathbb{R}$, whose complement has finite measure, and let $\delta(x) = \inf\{|x-y| : y \in F\}$. Let

$$I(x) := \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy.$$

(a) Show that, for all x, $x' \in \mathbb{R}$,

$$|\delta(x) - \delta(x')| \leq |x - x'|.$$

Solution. Let $\epsilon > 0$. There exists $y \in F$ such that $|x-y| \leq \delta(x) + \epsilon$. Then,

$$\delta(x') \leq |x' - y| \leq |x - x'| + |x - y| = |x - x'| + \delta(x) + \epsilon$$

so $\delta(x) - \delta(x') \leq |x - x'| + \epsilon$. Switching the roles of x and x' we get $\delta(x') - \delta(x) \leq |x - x'| + \epsilon$, so $|\delta(x) - \delta(x')| \leq |x - x'| + \epsilon$, and the result follows.

(b) Show that $I(x) = +\infty$ for all $x \notin F$.

Solution. Let $x \notin F$. Since F is closed, we have $\delta(x) = a > 0$. From the Lipschitz condition we have that $|\delta(y) - \delta(x)| \leq \frac{a}{2}$ whenever $|x - y| < \frac{a}{2}$. Then,

$$\begin{aligned}\int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy &\geq \int_{x-a/2}^{x+a/2} \frac{\delta(y)}{|x-y|^2} dy \geq \int_{x-a/2}^{x+a/2} \frac{a/2}{|x-y|^2} dy \\ &= \frac{a}{2} \int_{-a/2}^{a/2} \frac{1}{|y|^2} dy = \infty,\end{aligned}$$

where we used that $|\delta(y) - \delta(x)| \leq \frac{a}{2}$ iff $\delta(x) - \frac{a}{2} \leq \delta(y) - \leq \frac{a}{2} + \delta(x)$, so $\delta(y) \geq \frac{a}{2}$.

(c) Show that $I(x) < \infty$ for a.e. $x \in F$.

Solution. If $y \in F$ we have $\delta(y) = 0$. Let $y \notin F$. For all $x \in F$, we have $|x - y| \geq \delta(y)$. Thus,

$$\int_F \frac{1}{|x-y|^2}dx \le \int_{-\infty}^{y-\delta(y)} \frac{1}{|x-y|^2}dx + \int_{y+\delta(y)}^{+\infty} \frac{1}{|x-y|^2}dx$$
$$= 2\int_{\delta(y)}^{+\infty} \frac{1}{|x|^2} = \frac{2}{\delta(y)}.$$

Then,

$$\int_{\mathbb{R}} I(x)dx = \int_{\mathbb{R}\times\mathbb{R}} \frac{\delta(y)}{|x-y|^2}\chi_F(x)dy\,dx = \int_{F^c} \delta(y) \int_F \frac{1}{|x-y|^2}dx\,dy$$
$$\le \int_{F^c} \delta(y)\frac{2}{\delta(y)}dy = 2m(F^c) < \infty.$$

Again, we could switch the order of integration thanks to Tonelli's theorem ($I(x) \ge 0$). Since $\int_{\mathbb{R}} I(x)dx < \infty$, $I(x) < \infty$ for almost all $x \in \mathbb{R}$.

Problem 6. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ such that f is integrable on $\mathbb{R}$, yet $\limsup_{x\to\infty} f(x) = \infty$.

Solution: One example is

$$f(x) = \begin{cases} 2n^4x - 2n(n^4-1) & x \in [n - \frac{1}{n^3}, n],\ n \ge 1 \\ -2n^4x + 2n(n^4+1) & x \in [n, n+\frac{1}{n^3}],\ n \ge 1; \\ 0 & \text{otherwise.} \end{cases}$$

Here, the graph of f consists on "triangles" of area $\frac{1}{2n^2}$ centred at every positive integer n. Then $\int_{\mathbb{R}} f dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Solution. Let $\epsilon > 0$. Since f uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x-y| < \delta$. Take $\delta < \frac{1}{2}$ Suppose $f(x) \not\to 0$ as $|x| \to \infty$. Then, there exists $x_1 \in \mathbb{R}$, s.t. $|f(x_1)| \ge \epsilon$. thus, $|f(y)| \ge \frac{\epsilon}{2}$ for all $y \in (x_1 - \delta, x_1 + \delta)$. Now, there exists $x_2 > x_1 + 1$ s.t. $|f(x_2)| \ge \epsilon$ and thus $|f(y)| \ge \frac{\epsilon}{2}$ for all $y \in (x_2 - \delta, x_2 + \delta)$ Note that $(x_1 - \delta, x_1 + \delta) \cap (x_2 - \delta, x_2 + \delta) = \emptyset$. Continue this way so as to find a sequence $x_n \to \infty$, $x_{n+1} > x_n + 1$, such that $|f(y)| \ge \frac{\epsilon}{2}$ whenever $y \in (x_n - \delta, x_n + \delta)$. Therefore,

$$\int_{\mathbb{R}} |f(x)|dx \ge \sum_{n=1}^{\infty} \int_{(x_n-\delta,x_n+\delta)} |f(x)|dx \ge \sum_{n=1}^{\infty} \delta\epsilon = \infty.$$

Problem 8. If f is integrable on $\mathbb{R}$, show that

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

is uniformly continuous.

Solution. Let $\epsilon > 0$ and $\delta := \epsilon \cdot (\int_{\mathbb{R}} |f|dt)^{-1} < \infty$, since f integrable. Let $x, y \in \mathbb{R}$ such that $x < y$, $|x-y| < \delta$. Then,

$$|F(x) - F(y)| \leq \int_{x}^{y} |f(t)|dt \leq |x-y| \int_{\mathbb{R}} |f(t)|dt < \epsilon.$$

Problem 15.
Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2}, & \text{if } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_n\}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with F a.e. is unbounded in any interval.

Solution. We first compute $\int_{\mathbb{R}} f dx$. The improper Riemann integral of f is given by

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_0^1 = 2.$$

We know that the proper Riemann and Lebesgue integrals coincide. Then,

$$\begin{aligned} \int_{\mathbb{R}} f(x)\,dx &= \sum_{n=1}^{\infty} \int_{(\frac{1}{n+1}, \frac{1}{n}]} f(x)dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} f(x)dx \\ &= 2\sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = 2. \end{aligned}$$

Now, by Monotone convergence theorem and translation invariance of the Lebesgue measure,

$$\begin{aligned}\int_{\mathbb{R}} F(x)\,dx &= \lim_{N\to\infty}\int_{\mathbb{R}}\sum_{n=1}^{N} 2^{-n} f(x-r_n)\,dx = \sum_{n=1}^{\infty}\int_{\mathbb{R}} 2^{-n} f(x-r_n)\,dx \\ &= \sum_{n=1}^{\infty} 2^{-n}\int_{\mathbb{R}} f(x)\,dx = \sum_{n=1}^{\infty} 2^{-n+1} = 2.\end{aligned}$$

Thus F is integrable and therefore finite a.e..

Let $I \subset \mathbb{R}$ be an interval. Since rationals are dense in $\mathbb{R}$ there exists $N \in \mathbb{N}$ such that $r_N \in I$. Since $F(x)$ is a sum of positive terms,

$$\lim_{x\to r_N} F(x) \geq \lim_{x\to r_N} 2^{-N} f(x-r_N) = +\infty.$$

Let $\bar{F}$ coincide with F almost everywhere. Let $M > 0$. Since $\lim_{x\to r_N} F(x) = +\infty$, There exists $\epsilon > 0$ such that $F(x) > M$ for all $x \in (r_N - \epsilon, r_N + \epsilon)$. Then $\bar{F}(x) > M$ almost everywhere on $(r_N - \epsilon, r_N + \epsilon)$, so $\bar{F}(x_0) > M$ for some $x_0 \in (r_N - \epsilon, r_N + \epsilon)$. Since M is arbitrarily large, $\bar{F}$ is unbounded on I.

Problem 17. Suppose f is defined on $\mathbb{R}^2$ as follows

$$f(x,y) = \begin{cases} a_n, & n \leq x < n+1 \text{ and } n \leq y < n+1,\ n \geq 0; \\ -a_n & n \leq x < n+1 \text{ and } n+1 \leq y < n+2,\ n \geq 0; \\ 0, & \text{otherwise.}\end{cases}$$

Here, $a_n = \sum_{k\leq n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s \leq \infty$.

(a) Verify that each slice $f^y(x)$ and $f_x(y)$ is integrable. Also, for all x, $\int f_x(y)dy = 0$, and hence $\int\int f(x,y)dy\,dx = 0$.

Solution. Let $y \in \mathbb{R}$. There exists a unique $n \in \mathbb{N}$ such that $n \leq y < n+1$. Then

$$\int_{\mathbb{R}} |f^y(x)|dx = \int_{[n-1,n)} |f^y(x)|dx + \int_{[n,n+1)} |f^y(x)|dx = a_{n-1} + a_n.$$

Let $x \in \mathbb{R}$. There exists a unique $n \in \mathbb{N}$ such that $n \leq x < n+1$. Then

$$\int_{\mathbb{R}} |f_x(y)|dy = \int_{[n,n+1)} |f_x(y)|dy + \int_{[n+1,n+2)} |f_x(y)|dy = 2a_n.$$

Moreover,

$$\int_{\mathbb{R}} f_x(y)dy = \int_{[n,n+1)} f_x(y)dy + \int_{[n+1,n+2)} f_x(y)dy = a_n - a_n = 0.$$

It follows that,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dy\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f_x(y)dy\,dx = 0$$

(b) For $0 \le y < 1$,

$$\int_{\mathbb{R}} f^y(x)dx = \int_{[0,1)} f^y(x)dx = a_0.$$

If $y \ge 1$,

$$\int_{\mathbb{R}} f^y(x)dx = \int_{[n-1,n)} f^y(x)dx + \int_{[n,n+1)} f^y(x)dx = a_n - a_{n-1}.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} f^y(x)dx\,dy &= \sum_{n=0}^{\infty}\int_{[n,n+1)}\left(\int_{[n-1,n)} f^y(x)dx + \int_{[n,n+1)} f^y(x)dx\right)dy\\ &= \sum_{n=0}^{\infty}[a_n - a_{n-1}] = \sum_{n=0}^{\infty} b_n = s.\end{aligned}$$

(c) By Tonelli,

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \sum_{n=0}^{\infty}\int_{[n,n+1)}\int_{\mathbb{R}} |f_x(y)|dy\,dx = \sum_{n=0}^{\infty} 2a_n = \infty,$$

since a_n is increasing.

Problem 19. Suppose f is integrable in $\mathbb{R}^d$. For each $\alpha > 0$ let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^{+\infty} m(E_\alpha)d\alpha.$$

Solution:
First note that

$$|f(x)| = \int_0^{|f(x)|} 1d\alpha.$$

Then, by Tonelli,

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)| dx &= \int_{\mathbb{R}^d} \int_0^{|f(x)|} d\alpha \, dx = \int_{\mathbb{R}^d} \int_0^{+\infty} \chi_{\{\alpha < |f(x)|\}}(\alpha) d\alpha \, dx \\ &= \int_0^{+\infty} \int_{\mathbb{R}^d} \chi_{\{\alpha < |f(x)|\}}(x) dx \, d\alpha \\ &= \int_0^{+\infty} m(E_\alpha) d\alpha.\end{aligned}$$