Analysis IV : Assignment 1 Solutions
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EXERCISE 1 The Cantor set $\mathcal{C}$ is totally disconnected and perfect.

PROOF By construction, $\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} \mathcal{C}_{ni}$, where $\mathcal{C}_{11} = [0, \frac{1}{3}]$, $\mathcal{C}_{12} = [\frac{2}{3}, 1]$, $\mathcal{C}_{21} = [0, \frac{1}{9}]$, $\mathcal{C}_{22} = [\frac{2}{9}, \frac{3}{9}]$, $\mathcal{C}_{23} = [\frac{6}{9}, \frac{7}{9}]$, $\mathcal{C}_{24} = [\frac{8}{9}, 1]$, etc. We have $|\mathcal{C}_{ni}| = |\mathcal{C}_{nj}| = \frac{1}{3^n}$ for every $1 \leq 1, j \leq 2^n$ and $\lim_{n\to\infty} |\mathcal{C}_{n1}| = 0$. Notice that the endpoints of the components $\mathcal{C}_{ni}$ belong to the Cantor set.
To show that the Cantor set is totally disconnected, we must show that the only connected subsets are the empty set or the singletons. It is enough to show that for any $x, y \in \mathcal{C}$, $x < y$, there is $z \in \mathcal{C}^{\mathsf{c}}$ such that $x < z < y$, because then if $\mathcal{O}$ is any subset of $\mathcal{C}$ containing x and y, we have $\mathcal{O} = \mathcal{O} \cap [0, y) \bigcup \mathcal{O} \cap (y, 1]$, a disjoint union of two nontrivial open subsets of the topology on $\mathcal{O}$. Choose N such that $|\mathcal{C}_{Ni}| < \frac{|x-y|}{2}$. Then $x \in \mathcal{C}_{N\alpha}$ and $y \in \mathcal{C}_{N\beta}$ for some $\alpha \neq \beta$. From the construction $d(\mathcal{C}_{N\alpha}, \mathcal{C}_{N\beta}) \geq \frac{1}{3^N}$, so it is obvious that there exists $z \in \mathcal{C}^{\mathsf{c}}$ with $x < z < y$. To show that the Cantor set has no isolated points, let $x \in \mathcal{C}$ and $\epsilon > 0$ be given. Choose N such that $|\mathcal{C}_{Ni}| < \frac{\epsilon}{2}$. Then $x \in \mathcal{C}_{N\alpha}$ for some α and $\mathcal{C}_{N\alpha} \bigcap B_\epsilon(x) = \mathcal{C}_{N\alpha}$. Now let y be an endpoint of $\mathcal{C}_{N\alpha}$ (if x is itself an endpoint of $\mathcal{C}_{N\alpha}$, take y to be the other endpoint) : $y \in \mathcal{C}$ and $|x - y| < \epsilon$. $\square$

EXERCISE 2 Let $E \subset \mathbb{R}^d$ be compact and $\mathcal{O}_n$ be the open sets : $\mathcal{O}_n = \{x : d(x, E) < 1/n\}$. Then

1. If E is compact then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$
2. However the conclusion is false if E is assumed to be closed and unbounded or open and bounded

PROOF

1. $\mathcal{O}_n$ is a decreasing sequence of sets and $\bigcap_{n=1}^{\infty} \mathcal{O}_n = \{x : d(x, E) = 0\} = E$. Moreover E bounded implies $m(\mathcal{O}_1) < \infty$ so by continuity from above $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.
2. A counterexample for the case when E is closed and unbounded would be $E = \{(n, 0, 0, ..., 0) \in \mathbb{R}^d : n \in \mathbb{N}\}$. Then $m(E) = 0$ but $m(\mathcal{O}_n) = \sum_{k=1}^{\infty} m(B_{\frac{1}{n}}(k)) = \sum_{k=1}^{\infty} |B_{\frac{1}{n}}(0)| = \infty$.
   A counterexample for the case when E is open and bounded would be the complement of a Cantor-like set $\hat{\mathcal{C}}$ having nonzero Lebesgue measure, constructed as follows: Start with the interval $[0, 1]$. At the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length l_k, chosen small enough so that $\sum_{k=1}^{\infty} 2^{k-1} l_k < 1$. (If $l_k = 1/3^k$ this construction corresponds to the original Cantor set and the sum equals 1.) Hence $\hat{\mathcal{C}} = \left(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^{k-1}} \hat{\mathcal{O}}_{ki} \right)^{\mathsf{c}} = \left(\bigcup_{k=1}^{\infty} \hat{\mathcal{O}}_k \right)^{\mathsf{c}}$. Let $E = \bigcup_{k=1}^{\infty} \hat{\mathcal{O}}_k$. Then E is open and bounded. The closure of E is $[0, 1]$ and so $m(E) < \lim_{n\to\infty} m(\mathcal{O}_n) = 1$.

$\square$

Exercise 3 Let $\delta = (\delta_1, ..., \delta_d) \in \mathbb{R}^d$ where each $\delta_i > 0$. Define the operator (also denoted by δ) $\delta : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $\delta : (x_1, ..., x_d) \longrightarrow (\delta_1 x_1, ..., \delta_d x_d)$. Then $E \subset \mathbb{R}^d$ is Lebesgue measurable iff δE is Lebesgue measurable and $m(\delta E) = \delta_1 ... \delta_d m(E)$.

Remark We follow the definitions $m_*(E) = \inf\{\sum_{i=1}^{\infty} |Q_i| : Q_i$ are cubes and $\bigcup_{i=1}^{\infty} Q_i \supset E\}$ and E is Lebesgue measurable iff $\forall \epsilon > 0\ \exists \mathcal{O}$ open, with $\mathcal{O} \supset E$ and $m_*(\mathcal{O} \setminus E) < \epsilon$.

Proof As a matrix, $\delta = \text{diag}(\delta_1, ..., \delta_d)$ and $\det\delta = \prod_{i=1}^{d} \delta_i > 0$. So δ is an invertible bounded linear operator. In particular δ is a homeomorphism. Notice that if $Q \subset \mathbb{R}^d$ is a cube, $Q = [a, b]^d$, then δQ is the rectangle $[\delta_1 a, \delta_1 b] \times ... \times [\delta_d a, \delta_d b]$ and $|\delta Q| = \det\delta\ |Q|$.

If $E \subset \mathbb{R}^d$ is measurable then $\exists\ \mathcal{O}$ open, with $\mathcal{O} \supset E$ and $m(\mathcal{O} \setminus E) < \frac{\epsilon}{\det\delta}$. $\mathcal{O} \setminus E$ is measurable so there are are cubes $\{Q_i\}$ such that $\bigcup_{i=1}^{\infty} Q_i \supset \mathcal{O} \setminus E$ and $\sum_{i=1}^{\infty} |Q_i| < \frac{\epsilon}{\det\delta}$. Then $\delta\mathcal{O}$ is open and $\delta\mathcal{O} \supset \delta E$. Further $\bigcup_{i=1}^{\infty} \delta Q_i = \delta(\bigcup_{i=1}^{\infty} Q_i) \supset \delta(\mathcal{O} \setminus E) \supset \delta\mathcal{O} \setminus \delta E$. Thus $\epsilon > \det\delta \sum_{i=1}^{\infty} |Q_i| = \sum_{i=1}^{\infty} m(\delta Q_i) = \sum_{i=1}^{\infty} m_*(\delta Q_i) \geq m_*(\bigcup_{i=1}^{\infty} \delta Q_i) \geq m_*(\delta\mathcal{O} \setminus \delta E)$. The same argument repeated with δ^{-1} gives the converse.

Finally, given E measurable, let Q_i be cubes such that $\bigcup_{i=1}^{\infty} Q_i \supset E$. Then $m(\delta E) \leq m(\delta \bigcup_{i=1}^{\infty} Q_i) \leq \sum_{i=1}^{\infty} m(\delta Q_i) = \det\delta \sum_{i=1}^{\infty} |Q_i|$. Taking the infimum over all collections $\{Q_i\}$ yields $m(\delta E) \leq (\det\delta) m(E)$. Repeating the same argument with δ^{-1} gives: $m(E) = m(\delta^{-1}(\delta E)) \leq (\det\delta^{-1}) m(\delta E) = (\det\delta)^{-1} m(\delta E)$. $\square$

Exercise 4 Let A be the subset of $[0, 1]$ which consists of all numbers which don't have the digit 4 appearing in their decimal expansion. Then $m(A) = 0$.

Proof This is similar to the calculation of the measure of the Cantor set. We construct a Cantor-like set as follows: Start with the interval $[0, 1]$. At the k^{th} stage of the construction one removes 9^{k-1} open intervals each of length $1/10^k$ and situated at four tenths of their respective components. These open intervals are disjoint, hence $m(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{9^{k-1}} \mathcal{O}_{ki}) = \sum_{k=1}^{\infty} \sum_{i=1}^{9^{k-1}} m(\mathcal{O}_{ki}) = \sum_{k=1}^{\infty} \frac{9^{k-1}}{10^k} = 1$. $A = \left(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{9^{k-1}} \mathcal{O}_{ki} \right)^{\mathsf{c}}$ and so $m(A) = 0$.

$\square$

Exercise 5 *(Borel-Cantelli Lemma #1)* Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets in $\mathbb{R}^d$ satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then $m(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = 0$.

Remark $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \{x \in \bigcup_{k=1}^{\infty} E_k : x$ belongs to infinitely many $E_k\}$.

Proof Countable unions and countable intersections of measurable sets are measurable, hence E is measurable.

Let $\epsilon > 0$ be given an choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\sum\limits_{k=n}^{\infty} m(E_k) < \epsilon$. Then $m(\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} E_k) \leq m(\bigcup\limits_{k=N}^{\infty} E_k) \leq \sum\limits_{k=N}^{\infty} m(E_k) < \epsilon$. Now let $\epsilon \to 0$. $\square$

EXERCISE 6 There exists a continuous function that maps a Lebesgue measurable set to a non-measurable set.
PROOF We take for granted the existence of a non-measurable set $A \subset [0,1]$ (Theorem 3.6 in Stein & Shakarchi). Recall that the Cantor set $\mathcal{C} = \{\sum\limits_{k=1}^{\infty} \frac{a_k}{3^k} : a_k = 0 \text{ or } 2\}$. Consider the Cantor-Lebesgue function defined by $F : \mathcal{C} \longrightarrow [0,1]$, $F : \sum\limits_{k=1}^{\infty} \frac{a_k}{3^k} \longrightarrow \sum\limits_{k=1}^{\infty} \frac{a_k/2}{2^k}$. It is readily seen that F is surjective, based on the observation that every number in $[0,1]$ admits a binary expansion $\sum\limits_{k=1}^{\infty} \frac{a_k}{2^k}$, where $a_k = 0$ or 1. Now we show that F is continuous. Let $x = \sum\limits_{k=1}^{\infty} \frac{s_k}{3^k} \in \mathcal{C}$ and $\epsilon > 0$ be given. Choose N so that $1/2^N < \epsilon$ and then let $\delta = 1/3^N$. If $y = \sum\limits_{k=1}^{\infty} \frac{t_k}{3^k} \in B_{1/3^N}(x) \cap \mathcal{C}$, then referring to the notation in Exercise 1, there exists α such that x and y belong to the same component $\mathcal{C}_{N\alpha}$. Then it must be that $s_k = t_k$ for $1 \leq k \leq N$. Hence $|F(x) - F(y)| = |\sum\limits_{k=N+1}^{\infty} \frac{(s_k - t_k)/2}{2^k}| \leq \sum\limits_{k=N+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^N} < \epsilon$ and F is continous at x. Finally the claim is that the function $F' = F \restriction_{F^{-1}(A)}$ is a continuous map from a Lebesgue measurable set to a non-measurable set. The only thing left to prove is that $F^{-1}(A)$ is Lebesgue measurable. But $m(F^{-1}(A)) \leq m(\mathcal{C}) = 0$. $\square$

EXERCISE 7 There exists a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable.
PROOF Consider a Cantor-like set $\hat{\mathcal{C}}$ with $m(\hat{\mathcal{C}}) > 0$. Such a set is described in part 2 of Exercise 2. We can write $\hat{\mathcal{C}} = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=1}^{2^k} \hat{\mathcal{C}}_{kn}$ where $|\hat{\mathcal{C}}_{kn}| = |\hat{\mathcal{C}}_{km}|$ for all $1 \leq n, m \leq 2^k$. Consider also the standard Cantor set $\mathcal{C} = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{n=1}^{2^k} \mathcal{C}_{kn}$. At each step of the construction we have a bijection $\Phi_k : \bigcup\limits_{n=1}^{2^k} \hat{\mathcal{C}}_{kn} \longrightarrow \bigcup\limits_{n=1}^{2^k} \mathcal{C}_{kn}$ such that $\Phi_k(\hat{\mathcal{C}}_{kn}) = \mathcal{C}_{kn}$ is a linear function for all $1 \leq n \leq 2^k$, mapping the left (resp. right) endpoint of $\hat{\mathcal{C}}_{kn}$ to the left (resp. right) endpoint of $\mathcal{C}_{kn}$. For all $k \geq 1$ we have $\Phi_k \restriction_{\bigcup\limits_{n=1}^{2^{k+1}} \hat{\mathcal{C}}_{(k+1)n}} = \Phi_{k+1} \restriction_{\bigcup\limits_{n=1}^{2^{k+1}} \hat{\mathcal{C}}_{(k+1)n}}$. In this way the sequence $\{\Phi_k\}$ induces a bijection $\Phi : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$. Let $A \subset \hat{\mathcal{C}}$ be a non-measurable set (to prove the existence of such a set, mimic the proof of Theorem 3.6 in Stein & Shakarchi). Let $f = \chi_{\Phi(A)}$, the characteristic (or indicator) function supported on $\Phi(A)$. Then $f \circ \Phi : \hat{\mathcal{C}} \longrightarrow \{0,1\}$ is a non-measurable function, since $(f \circ \Phi)^{-1}(\{1\}) = A$. Note also that although A is not measurable, $\Phi(A)$ is measurable since $m_*(\Phi(A)) \leq m_*(\mathcal{C}) = 0$. $\square$

Exercise 8 Let $\Gamma \subset \mathbb{R}^2$ be a curve given by the continuous function $y = f(x)$. Then $m(\Gamma) = 0$.

Proof Decompose Γ into $\Gamma = \bigcup_{n\in\mathbb{Z}} \Gamma \cap [n, n+1] \times \mathbb{R} = \bigcup_{n\in\mathbb{Z}} \Gamma_n$. It is enough to show that $m(\Gamma_n) = 0$ for all n. Wlog we show $m(\Gamma_0) = 0$. Let $\epsilon > 0$ be given and choose $\delta > 0$ so that $|x-y| < 2\delta \Rightarrow |f(x) - f(y)| < \epsilon$. Let $N = \lfloor \frac{1}{\delta} \rfloor$. Then for $0 \leq i \leq N$, there exists a rectangle R_i of size $\delta \times \epsilon$ such that $\Gamma_0 \bigcap [i\delta, (i+1)\delta] \times \mathbb{R} \subset R_i$. Then $m(\Gamma_0) \leq \sum_{i=0}^{N} m(R_i) = (N+1)\delta\epsilon \leq (1+\delta)\epsilon$. Now let $\epsilon \to 0$. □

Exercise 9 Let $\omega \in [0,1]$. Then ω can be written in the form $\sum_{j=1}^{\infty} \frac{a_j}{2^j}$ where $a_j = 0, 1$. Moreover this expansion is unique when we restrict to nonterminating series.

Proof (taken from Rodrick Kuate Defo) Let $\omega \in [0,1)$. If $\omega \in [0, \frac{1}{2})$, let $a_1 = 0$ and if $\omega \in [\frac{1}{2}, 1)$. Suppose that a_j has been determined for $j = 1, 2, ...n-1$. Then $\omega \in [\sum_{j=1}^{n-1} \frac{a_j}{2^j}, \sum_{j=1}^{n-1} \frac{a_j}{2^j} + \frac{1}{2^{n-1}})$. If $\omega \in [\sum_{j=1}^{n-1} \frac{a_j}{2^j}, \sum_{j=1}^{n-1} \frac{a_j}{2^j} + \frac{1}{2^n})$, let $a_n = 0$, otherwise let $a_n = 1$. Let $\omega_n = \sum_{j=1}^{n} \frac{a_j}{2^j}$. The sequence $(\omega_n)_{n=1}^{\infty}$ is Cauchy in $(\mathbb{R}, |\cdot|)$, since for $m > n$, $|\omega_m - \omega_n| = |\sum_{j=1}^{m} \frac{a_j}{2^j} - \sum_{j=1}^{n} \frac{a_j}{2^j}| = \sum_{j=n+1}^{m} \frac{a_j}{2^j} \leq \sum_{j=n+1}^{\infty} \frac{a_j}{2^j} = \frac{1}{2^n}$. Hence the sequence converges to unique element in $\mathbb{R}$. This element is ω, since by construction, at the n^{th} step $|\omega - \omega_n| < \frac{1}{2^n}$. This proves $\omega = \lim_{n\to\infty} \omega_n = \sum_{j=1}^{\infty} \frac{a_j}{2^j}$.

Now suppose ω admits two distinct expansions $\sum_{j=1}^{\infty} \frac{a_j}{2^j} = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$. Suppose that $\sum_{j=1}^{\infty} \frac{a_j}{2^j}$ is a nonterminating expansion (by that we mean that there are infinitely many a_j's taking the values 0 and 1). Let k^* be the smallest integer such that $a_{k^*} \neq b_{k^*}$. Wlog $a_{k^*} = 1$ and $b_{k^*} = 0$. Then $\sum_{j=1}^{\infty} \frac{a_j}{2^j} - \sum_{j=1}^{\infty} \frac{b_j}{2^j} = 2^{-k^*} + \sum_{j=k^*+1}^{\infty} \frac{a_j - b_j}{2^j} > 2^{-k^*} - \sum_{j=k^*+1}^{\infty} \frac{1}{2^j} = 0$, where the strict inequality comes from the fact that there is $j \geq k^* + 1$ such that $a_j = 1$, making $a_j - b_j \geq 0$. This contradiction shows uniqueness for nonterminating series. On the other hand, if ω has a terminating series then it is the endpoint of one of the intervals in the construction. In this case, ω will admit two distinct representations, both being terminating series. For example, $\frac{1}{2} = 0.01..._2 = \sum_{j=2}^{\infty} \frac{1}{2^j} = 0.10..._2 = \sum_{j=1}^{1} \frac{1}{2^j}$.

□

Notation Given a function $f : A \longrightarrow \mathbb{R}$, by $\{f > \alpha\}$ we mean $\{x \in A : f(x) > \alpha\}$.

Exercise 10 *(Chebyshev's inequality)* Let $f : [0,1] \longrightarrow \mathbb{R}$ be a non-negative monotone function. Then $m(\{f > \alpha\}) \leq \frac{1}{\alpha} \int_0^1 f dx$ with the integral on the right being the Riemann integral.

Proof Let $[a, b]$ be a finite interval. If $f : [a,b] \longrightarrow \mathbb{R}$ is monotone then f is Riemann integrable, i.e. $\int_a^b f dx < \infty$ (for a proof of this, see for example theorem 7.2.7 in Bartle & Sherbert. In this case the Riemann and Lebesgue integrals coincide and $\int_a^b f dx = \int_{[a,b]} f dm$. Hence $\int_0^1 f dx = \int_{[0,1]} f dm \geq \int_{\{f>\alpha\}} f dm \geq \int_{\{f>\alpha\}} \alpha dm = \alpha m(\{f > \alpha\})$. □

EXERCISE 11 A gambler has an initial stake of one dollar. Calculate the probability of ruin at times 1, 3 and 5. Show that the chance of eventual ruin is at least 70%.

PROOF (taken from Rodrick Kuate Defo) The idea is the following: the gambler has to bet on the outcome of a game of which there are 2 possible outcomes (e.g. flipping a coin). If he is right, he wins (W) one dollar, if he is wrong he loses (L) a dollar. To play the game, he must have an initial stake and he is secretely forced to play the game until he is ruined. After the 1^{st} iteration, the sample space, or the collection of all possible events is $\{\{W\},\{L\}\}$. The probability of each event being the same, $\mathbb{P}(\{W\}) = \mathbb{P}(\{L\}) = 1/2$. After the 2^{nd} iteration, the sample space is $\{\{WW\},\{LL\},\{WL\},\{LW\}\}$ and each event has probability of $1/4$. In the k^{th} iteration there are 2^k possible events each with probability $1/2^k$. At the k^{th} iteration of the game, set $R_k = 1$ if the gamer wins and $R_k = -1$ if he loses. For example if ω is the event $\{WWL\}$, then $R_1(\omega) = 1, R_2(\omega) = 1, R_3(\omega) = -1$. The amount money made after N iterations is $S_N(\omega) = \sum_{k=1}^{N} R_k(\omega)$. Given an initial stake of x, the probability of ruin after N iterations equals

$$\mathbb{P}(\{\omega : S_k(\omega) > -x \text{ for all } 1 \le k < N,\ S_N(\omega) = -x\})$$

There is a connection between measure theory and probability. Consider the interval $[0,1]$ and write each $\omega \in [0,1]$ as $\omega = 0.\omega_1\omega_2\omega_3...$, with $\omega_i \in \{0,1\}$. We associate the event $\{W\}$ with the set $\{\omega \in [0,1] : \omega_1 = 1\} = [\frac{1}{2}, 1]$, $\{L\}$ with the set $\{\omega \in [0,1] : \omega_1 = 0\} = [0, \frac{1}{2}]$, $\{WL\}$ with the set $\{\omega \in [0,1] : \omega_1 = 1, \omega_2 = 0\} = [\frac{1}{2}, \frac{3}{4}]$, etc. Thus $\mathbb{P}(\{W\}) = m([\frac{1}{2}, 1]) = 1/2$, $\mathbb{P}(\{L\}) = m([0, \frac{1}{2}]) = 1/2$, $\mathbb{P}(\{WL\}) = m([\frac{1}{2}, \frac{3}{4}]) = 1/4$. Define $R_k(\omega) = 2\omega_k - 1$ and $S_N(\omega) = \sum_{k=1}^{N} R_k(\omega)$. Then the probability of ruin after N iterations equals

$$m(\{\omega \in [0,1] : S_k(\omega) > -x \text{ for all } 1 \le k < N,\ S_N(\omega) = -x\})$$

- The probability of ruin after time 1 is given by $\mathbb{P}(\{R_1 = -1\}) = 1/2$.
  Recall the notation: $\mathbb{P}(\{R_1 = -1\}) = \mathbb{P}(\{\omega : R_1(\omega) = -1\})$.

- The probability of ruin after time 3 is given by $\mathbb{P}(\{S_k > -1 \text{ for all } 1 \le k < 3,\ S_3 = -1\})$. Since it is game over at time 3 but not before, it is clear that in this event he must win on the first iteration and lose on the 3rd iteration. Thus $S_3(\omega) = 1 + R_2(\omega) - 1 = -1 \Rightarrow R_2(\omega) = -1$. This event is $\{WLL\}$ and has probability $1/8$.

- The probability of ruin after time 5 is given by $\mathbb{P}(\{S_k > -1 \text{ for all } 1 \le k < 5,\ S_5 = -1\})$. Again we know that $R_1(\omega) = 1$ and $R_5(\omega) = -1$, i.e. $\omega_1 = 1$ and $\omega_5 = 0$. Also notice that the overall number of times the gamer loses is once more than the number of times he wins. If $R_2(\omega) = -1$, this forces $R_3(\omega) = 1$ and $R_4(\omega) = -1$. If $R_2(\omega) = 1$, this forces $R_3(\omega) = -1$ and $R_4(\omega) = -1$. So the probability of ruin after time 5 $=$ $\mathbb{P}(\{WLWLL\}) + \mathbb{P}(\{WWLLL\}) = 2\frac{1}{32}$.

- The probability that he is eventually ruined is

$$\begin{aligned}
&m(\{\omega \in [0,1] : \exists N \text{ such that } S_k(\omega) > -1 \text{ for all } 1 \le k < N,\ S_N(\omega) = -x\}) \\
&= m(\bigcup_{N=1}^{\infty} \{\omega \in [0,1] : \text{ such that } S_k(\omega) > -1 \text{ for all } 1 \le k < N,\ S_N(\omega) = -x\}) \\
&= \sum_{N=1}^{\infty} m(\{\omega \in [0,1] : \text{ such that } S_k(\omega) > -1 \text{ for all } 1 \le k < 2N+1,\ S_{2N+1}(\omega) = -x\})
\end{aligned}$$

  because the sets are disjoint and because $S_N(\omega) = -1$ is not possible when N is even. The probability of ruin after time 7 is $\ge \mathbb{P}(\{WLWLWLL\}) + \mathbb{P}(\{WWWLLLL\}) = 2\frac{1}{128}$. Hence the probability that he is eventually ruined is $\ge \frac{1}{2} + \frac{1}{8} + 2\frac{1}{32} + 2\frac{1}{128} \ge 0.7$.

$\square$