# Solutions to Homework 1

MATH 316

1. Describe geometrically the sets of points $z$ in the complex plane defined by the following relations

$$\begin{aligned} 1/z &= \bar{z} & (1)\\ \operatorname{Re}(az+b) &> 0, \text{ where } a,b\in\mathbb{C} & (2)\\ \operatorname{Im}(z) &= c, \text{ with } c\in\mathbb{R} & (3)\end{aligned}$$

*Solution*:
(1) $\implies 1 = z\bar{z} = |z|^2$. This is the equation for the unit circle centered at the origin.
For (2), writing $z = x + iy$ we have

$$0 < \operatorname{Re}(az+b) = \operatorname{Re}(az) + \operatorname{Re}(b) = \operatorname{Re}(a)x - \operatorname{Im}(a)y + \operatorname{Re}(b)$$

which is the set of points below the line

$$y = \frac{\operatorname{Re}(a)}{\operatorname{Im}(a)}x - \frac{\operatorname{Re}(b)}{\operatorname{Im}(a)}$$

when a is not real or purely imaginary. If a is real, then this is just the set of points on the plane to the right of the vertical line

$$x = -\frac{\operatorname{Re}(b)}{\operatorname{Re}(a)}$$

On the other hand if $a$ is purely imaginary, then this is just the set of points below the horizontal line

$$y = \frac{\operatorname{Re}(b)}{\operatorname{Im}(a)}$$

For (3), we need the imaginary part of $z$ to equal $c$. Writing $z = x + iy$, we see that this is clearly the horizontal line passing through $c$, that is, $y = c$.

♣

With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where $n \in \mathbb{N}$. How many solutions are there?
*Solution*:
Writing $z$ in polar form $z = |z|e^{i\theta}$, we have

$$|z|^n e^{in\theta} = se^{i\varphi}$$

we obtain two equations, one for moduli and the other for arguments.

$$\begin{aligned} |z|^n &= s \quad (4) \\ n\theta &= \varphi + 2k\pi, \quad k \in \mathbb{Z} \quad (5) \end{aligned}$$

Whence, the set of solutions is given by

$$z = s^{1/n} e^{i\left(\varphi/n + 2k\pi/n\right)}, \quad k \in \mathbb{Z}$$

There are only n distinct solutions since $e^{i\theta}$ has a period of $2\pi$. Indeed,

$$e^{i\varphi + 2m\pi} = e^{i\varphi}, \forall m \in \mathbb{Z}$$

When $z$ is unit modulus ($|z| = 1$), these are called the $n^{th}$ *roots of unity*.

♣

Let $z, w \in \mathbb{C} : \bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| \leq 1 \quad (6)$$

for $|z| \leq 1$ and $|w| \leq 1$, with equality if $|z| = 1$ or $|w| = 1$.
*Solution*: We want to show that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| \leq 1 \iff |w - z|^2 < |1 - \bar{w}z|^2$$

$$\iff |w|^2 - z\bar{w} - w\bar{z} + |z|^2 \leq (1 - \bar{w}z)(1 - w\bar{z})$$

$$\iff 0 \leq (1 - |z|^2)(1 - |w|^2)$$

which is true and holds with equality if either $|z| = 1$ or $|w| = 1$.

♣

Prove that for a fixed $w$ in the unit disc $\mathbb{D}$, the mapping

$$F : z \longmapsto \frac{w - z}{1 - \bar{w}z} \quad (7)$$

satisfies

1. F maps the unit disc to itself and is holomorphic
2. F interchanges 0 and $w$
3. $|F(z)| = 1$ if $|z| = 1$
4. $F : \mathbb{D} \longrightarrow \mathbb{D}$ is bijective.

*Solution*: We have already proven above that $F$ maps the unit disc to itself. Since it is a ratio of polynomials it is clear that it is holomorphic. That $F$ interchanges 0 and $w$ is clear by explicit computation. That $F$ maps the unit circle to itself is also clear from the previous proof, where we showed that $|z| = 1 \implies |F(z)| = 1$. We have already shown that $F$ is onto. To show that it is also injective we show that $F \circ F = \mathrm{id}_{\mathbb{D}}$

$$\frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} = \frac{w(\cancel{1} - \bar{w}z) - \cancel{w} + z}{1 - \cancel{\bar{w}z} - \bar{w}w + \cancel{\bar{w}z}} = z$$

♣

Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{8}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \tag{9}$$

Use these equations to show that the logarithm function defined by

$$\log z := \log r + i\theta, \quad \text{where } z = re^{i\theta}, -\pi < \theta < \pi \tag{10}$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.
*Solution*:
The Cauchy-Riemann equations for $u(x, y)$, $v(x, y)$ are

$$u_x = v_y, \quad u_y = -v_x$$

The transition matrix for polar coordinates can be obtained by applying chain rule to

$$x = r\cos\theta, \quad y = r\sin\theta$$

We obtain

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ \frac{1}{r}u_\theta \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Similarly,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_r \\ \frac{1}{r}v_\theta \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

So that the Cauchy-Riemann equations now read

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ \frac{1}{r}u_\theta \end{bmatrix} = \begin{bmatrix} v_y \\ -v_x \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} v_r \\ \frac{1}{r}v_\theta \end{bmatrix}$$

Whence,

$$\begin{bmatrix} u_r \\ \frac{1}{r}u_\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_r \\ \frac{1}{r}v_\theta \end{bmatrix}$$

That is,

$$u_r = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = -v_r$$

as desired. To check $\log z$ is holomorphic, we write it in polar coordinates

$$\log z = \log r + i\theta$$

so that $u = \text{Re}(z) = \log r$ and $v = \text{Im}(z) = \theta$. Then,

$$u_r = \frac{1}{r} = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = 0 = -v_r$$

♣

2. Suppose $f$ is a holomorphic function on an open set $\Omega$. Prove that if any of the following hold, $f$ is a constant.

1. $\text{Re}(f)$ is constant
2. $\text{Im}(f)$ is constant

*Solution*:
Either of the conditions along with the Cauchy-Riemann equations imply that

$$f'(z) = \left[(u_x + iv_x) + (v_y - iu_y)\right] = 0$$

♣

Determine the radius of convergence of the power series with coefficients:

1. $a_n := (\log n)^2$
2. $a_n := n!$
3. $a_n := \frac{n^2}{4^n + 3n}$

Using the ratio test, we have for (1)

$$R = \lim_n \frac{\log n}{\log(n+1)} = \lim_n \frac{n+1}{n} = 1$$

For (2),

$$R = \lim_n \frac{n!}{n+1!} = \lim_n \frac{1}{n+1} = 0$$

For (3),

$$R = \lim_n \frac{n^2/(4^n + 3n)}{(n+1)^2/(4^{n+1} + 3n + 3)} = 4$$

♣

3. Let $f$ be a power series centered at the origin. Prove that $f$ has a power series expansion around any point in its disc of convergence.

*Solution*:
Suppose

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

and let $|z_0| < R$. Set $z = z_0 + (z - z_0)$. The binomial expansion of $z^n$ gives

$$z^n = \sum_{n=0}^{\infty} \left( \begin{array}{c} n \\ k \end{array} \right) z_0^{n-k} (z - z_0)^k$$

whence, $f$ can be written as

$$f(z) = \sum_{k=0}^{\infty} \left[ \sum_{n>k} \left( \begin{array}{c} n \\ k \end{array} \right) z_0^{n-k} \right] (z - z_0)^k$$

♣

Prove that

1. $\sum nz^n$ does not converge on the unit circle
2. $\sum z^n/n^2$ converges on the unit circle

*Solution:*
Since $|z| = 1$, the first sequence $nz^n$ diverges, so the series cannot converge. The second converges absolutely

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

♣

4. Consider the function $f$ defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that $f$ is infinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0, \forall n \in \mathbb{N}$. Conclude that $f$ does not have a converging power series expansion near the origin.
*Solution:*
We prove using mathematical induction that

$$f^{(n)}(x) = r(x) e^{-1/x^2}, \quad \forall n \in \mathbb{N}$$

where $r(x)$ is some rational function. This is clear for $n = 1$, and assuming true for $n$ it is clearly true for $n+1$. Now the exponential term gurantees that all derivatives vanish at the origin $f^{(n)}(0) = 0, \forall n \geq 0$. Whence, we cannot write $f$ as a power series near the origin.

♣

Let $\gamma$ be a circle centered at the origin with positive orientation. Evaluate

$$\int_\gamma z^n dz$$

for $n \in \mathbb{Z}$. Now, do this for a circle not containing the origin.
*Solution:*
Parametrize $\gamma$ as

$$z(t) = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

Then, $dz = ire^{i\theta}$ and suppose $n \neq -1$

$$\begin{aligned}
\int_\gamma z^n dz &= \int_0^{2\pi} ir^{n+1} e^{i(n+1)\theta} d\theta && (11)\\
&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta && (12)\\
&= \frac{r^{n+1}}{n+1} \left[ e^{2\pi i(n+1)} - 1 \right] && (13)\\
&= 0 && (14)
\end{aligned}$$

When $n = -1$, we have

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i$$

Now suppose $\gamma$ does not contain the origin. Then we may parametrize it by

$$z(t) = z_0 + re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

For $n \neq -1$, we again obtain

$$\int_\gamma z^n dz = 0$$

For $n = -1$, we have

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{i\theta}}{z_0 + re^{i\theta}} d\theta = \ln(z_0 + re^{i\theta}) \Big|_0^{2\pi} = 0$$

♣