Mathematics 264 Midterm solutions

Each of the following questions is worth 10 points. Please show all your work.

1) Suppose a wire is bent into the shape of the parametrized curve

$$x(t) = 5t^3 \qquad y(t) = 2 + t^4 \qquad z(t) = 4t + 3,$$

from point $(0, 2, 3)$ to $(5, 3, 7)$. Suppose the mass density of the wire is given by $\rho(x, y, z) = xyz$. Write down an integral with respect to t whose value gives the mass of the wire (do **not** evaluate the integral).

Solution:

$$\begin{aligned} \text{mass} &= \int_0^1 \rho(x(t), y(t), z(t)) \sqrt{|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2}\, dt \\ &= \int_0^1 5t^3(4t+3)(2+t^4)\sqrt{225t^4 + 16t^6 + 16}\, dt. \end{aligned}$$

2) Find the surface area of the part $z = xy$ that lies inside the cylinder given by $x^2 + y^2 = a^2$.

Solution:
With domain of integration $D = \{(r, \theta); r \leq a, 0 \leq \theta \leq 2\pi\}$, the surface area equals

$$\begin{aligned}
\int\int_D dS &= \int\int_D \sqrt{1 + (\partial_x z)^2 + (\partial_y z)^2}\, dxdy \\
&= \int_0^a \int_0^{2\pi} \sqrt{1 + r^2}\, r dr d\theta \\
&= 2\pi \int_0^a \sqrt{1 + r^2}\, r dr \\
&= \frac{2\pi}{3} \Big((1 + a^2)^{3/2} - 1 \Big).
\end{aligned}$$

3) Let $\mathbf{F}(x,y) = e^{x+y}\sin(y)\,\mathbf{i} + e^{x+y}\cos(y)\,\mathbf{j}$.
Evaluate

$$\int_C F_1 dx + F_2 dy,$$

along the straight line segment from $(0,0)$ to $(1,\pi/2)$.
Hint: First compute $\nabla(e^{x+y}\sin(y))$ and compare it to $\mathbf{F}(x,y)$.

Solution: Since $F = (e^{x+y}\sin y, e^{x+y}\cos y)$, one computes that

$$\nabla(e^{x+y}\sin y) = (e^{x+y}\sin y, e^{x+y}\sin y + e^{x+y}\cos y) = \mathbf{F}(x,y) + (0, e^{x+y}\sin y).$$

So,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla\phi\cdot d\mathbf{r} - \int_C (0, e^{x+y}\sin y)\cdot d\mathbf{r}$$
$$= \phi(1,\pi/2) - \phi(0,0) - \int_C (0, e^{x+y}\sin y)\cdot d\mathbf{r}.\ (*)$$

We parametrize $C = \{(x(t),y(t)); x(t) = t, y(t) = \frac{\pi}{2}t, 0 \le t \le 1\}$ and so the last line in $(*)$ equals

$$e^{1+\pi/2} - \int_0^1 (0, e^{(1+\pi/2)t}\sin(\frac{\pi}{2}t))\cdot(1,\frac{\pi}{2})\,dt$$
$$= e^{1+\pi/2} - \pi/2\int_0^1 e^{(1+\pi/2)t}\sin(\frac{\pi}{2}t)dt$$

Can compute this last integral by two integrations by parts.

4) Volume of an ice cream cone.

a) For the "ice cream cone" bounded bellow by the cone $z^2 = 3(x^2 + y^2)$ with $z > 0$ and above by the spherical cap $x^2 + y^2 + z^2 = 4$ compute the volume using cylindrical coordinates.

b) Repeat the exercise now using spherical polar co-ordinates.

Solution (a): To determine domain of integration, substitute cone equation into equation for the sphere to get $x^2 + y^2 = 1$. So with region of integration

$$R = \{(r, \theta, z); r \leq 1, 0 \leq \theta \leq 2\pi, \sqrt{3}r \leq z \leq \sqrt{4 - r^2}\},$$

$$\begin{aligned} \text{volume} &= \int\int\int_R dV = \int_0^1 \int_0^{2\pi} \int_{\sqrt{3r^2}}^{\sqrt{4-r^2}} r dz dr d\theta \\ &= 2\pi \int_0^1 r\sqrt{4 - r^2} dr - 2\pi\sqrt{3} \int_0^1 r^2 dr \\ &= \frac{2\pi}{3}(8 - 3^{3/2}) - \frac{2\pi}{\sqrt{3}}. \end{aligned}$$

Solution (b): drop angle of cone is $\pi/6$. So, region of integration is given by $R = \{(R, \theta, \phi); 0 \leq \phi \leq \pi/6, 0 \leq \theta \leq 2\pi, 0 \leq R \leq 2\}$.

$$\begin{aligned} \text{volume} &= \int\int\int_R dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 R^2 \sin\phi \, dR d\phi d\theta \\ &= 2\pi(8/3) \int_0^{\pi/6} \sin\phi d\phi = 2\pi(8/3)[1 - \frac{\sqrt{3}}{2}]. \end{aligned}$$

5) Find the centre of mass of an object occupying the cube $0 \leq x, y, z \leq a$ with density given by $\rho = x^2 + y^2 + z^2$.

Solution: coordinates of center of mass are all equal by symmetry, so enough to compute x-component. Let C be the cube. Then, with total mass $M = \int\int\int_C \rho dV = \int_0^a \int_0^a \int_0^a \rho dx dy dz = a^5$,

$$\bar{x} = \frac{1}{M} \cdot \int_0^a \int_0^a \int_0^a x(x^2 + y^2 + z^2)\, dx dy dz = \frac{7}{12} a$$

Center of mass is $(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a)$.