Some comments on old magic squares illustrated with postage stamps

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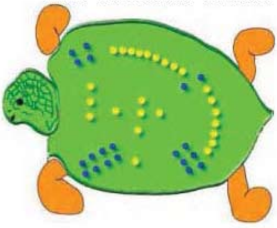


Luoshu: magic turtle?

The magic square known as Luoshu or Lo Shu (Luo River Writing) was apparently found on the back of a turtle on the shores of the Luo River, a tributary of the Yellow River in China, by Emperor Yu the Great (fl. 21st cent. BC).

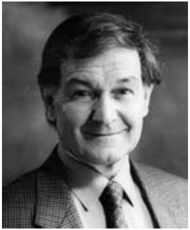
The Luoshu magic square is based on the 3×3 *magic matrix*

$$\mathbf{L} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}.$$

In the matrix $\mathbf{L}$ the numbers in the rows, columns, and two main diagonals all add up to same *magic sum* $m = 15$.

Takao Hayashi (*Historia Mathematica*, 14, 159–166, 1987) observed that

"It is generally accepted that the idea of magic squares was born in ancient China and spread over the world, although we cannot determine the date of the birth. The oldest of all the known documents that refer to magic squares is the *Ta Tai Li Chi* (*Record of Rites*) compiled by Tai the Elder in about 80 AD."



Fox, Moore & Penrose

While there is an enormous body of literature on magic squares, relatively little has been published about magic matrices.

The term "magic matrix" seems to have originated just over 50 years ago with a 1956 paper by Charles Fox (1897–1977), Professor of Mathematics at McGill from 1949–1967.

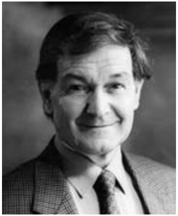
A major interest in our research is to identify old magic squares defined by magic matrices which have a magic Moore–Penrose inverse.

For an $n \times q$ matrix $\mathbf{A}$ the Moore–Penrose inverse $\mathbf{A}^+$ is the unique $q \times n$ matrix $\mathbf{A}^+$ which satisfies the 4 equations:

$$\mathbf{AA}^+\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+,$$
$$(\mathbf{AA}^+)' = \mathbf{AA}^+, \quad (\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A},$$

where $(\cdot)'$ denotes transpose. When $\mathbf{A}$ is square and nonsingular then $\mathbf{A}^+ = \mathbf{A}^{-1}$, the (regular) inverse of $\mathbf{A}$.

The Moore–Penrose inverse is named after Eliakim Hastings Moore (1862–1932) and Sir Roger Penrose (b. 1931).



An important matrix in our research is the "flip matrix".

We define the $n \times n$ *flip matrix* **F** as the identity matrix with its columns reversed.

For example, the 3×3 flip matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The double-flipped Luoshu matrix

$$\mathbf{FLF} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{pmatrix}$$

which defines a magic square in Kazwini's famos 13th century *Cosmography*.

Hence

$$\mathbf{L} + \mathbf{FLF} = \begin{pmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{pmatrix} \propto \mathbf{E},$$

where **E** has every element equal to 1.

We will say that a magic matrix **A** with the property that

$$\mathbf{A} + \mathbf{FAF} \propto \mathbf{E}$$

is **F**-*associated.*

A key result of our research is that the Moore–Penrose inverse of an **F**-associated magic matrix is also **F**-associated and magic.

The 3×3 magic matrix may, in general, be written as

$$\mathbf{T}_{a,b,x} = \begin{pmatrix} x-a & x-b & x+a+b \\ x+2a+b & x & x-2a-b \\ x-a-b & x+b & x+a \end{pmatrix},$$

which is **F**-associated for all values of a, b, x and hence its Moore–Penrose inverse is also **F**-associated and magic for all values of a, b, x.

The magic matrix $\mathbf{T}_{a,b,x}$ has determinant

$$\det \mathbf{T}_{a,b,x} = 9(2a+b)bx$$

and so $\mathbf{T}_{a,b,x}$ is singular if and only if $b = -2a$ or $b = 0$ or $x = 0$.

For example, when $b = 0$, $a \neq 0, x \neq 0$, then the Moore–Penrose inverse

$$\mathbf{T}^{+}_{a,0,x} = \frac{1}{36ax} \begin{pmatrix} 4a-3x & 4a+6x & 4a-3x \\ 4a & 4a & 4a \\ 4a+3x & 4a-6x & 4a+3x \end{pmatrix},$$

which is **F**-associated and magic.

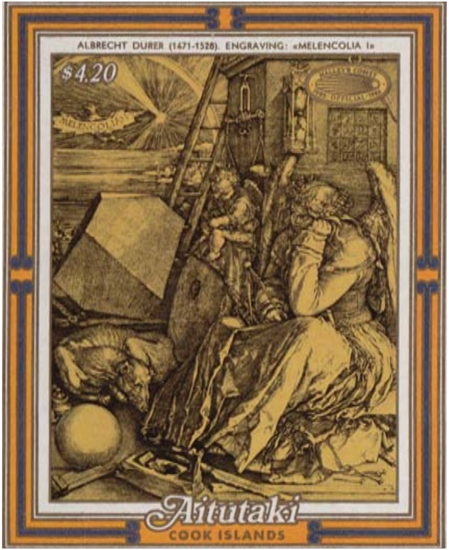
Pacioli & Dürer

The magic square defined by the magic matrix

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

is in the top-right corner of *Melencolia I,* the 1514 copper-plate engraving by the German painter Albrecht Dürer (1471–1528), shown in this 1986 stamp from Aitutaki (Cook Islands).

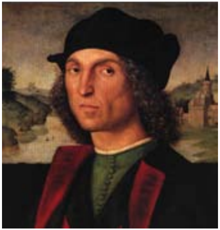
In *A Lifetime of Puzzles: A Collection of Puzzles in Honor of Martin Gardner's 90th Birthday* (AK Peters 2008), David Singmaster observes that the magic square defined by $\mathbf{M}$ is given by the Italian mathematician and Franciscan friar Luca Pacioli (c. 1446–1517) in his *De viribus quantitatis.*



In addition to Singmaster (2008), Neil Mackinnon (*The Mathematical Gazette*, 1993) also noted that the magic square defined by the matrix $\mathbf{M}$ also appears in Pacioli's *De viribus quantitatis*, written 1496–1508.

This well-known portrait of Pacioli with a "student" is attributed to Jacopo de' Barbari (c. 1440–c. 1515) and painted c. 1495.

Mackinnon (1993, p. 140) and Singmaster (2008, p. 83) suggest that the "student" may be Dürer (1471–1528), though R. Emmett Taylor (*No Royal Road*, 1942, p. 203), says he is Guidobaldo da Montefeltro, Duke of Urbino (1472–1508).



The Dürer self-portrait on this stamp from France 1980 was painted c. 1491. The painting (on the right) is of Guidobaldo by Raffaello Sanzio (1483–1520).

Apparently the world's oldest magic text, *De viribus quantitatis* (*On the Powers of Numbers*) was written by Pacioli between 1496 and 1508 and contains the first-ever reference to card tricks as well as guidance on how to juggle, eat fire and make coins dance.

It seems that *De viribus quantitatis* is also the first work to note that da Vinci was left-handed: Pacioli collaborated with, lived with, and taught mathematics to the Italian polymath Leonardo da Vinci (1452 –1519).

The magic matrix

$$\mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix},$$

which we now call the Pacioli–Dürer matrix, is **F**-associated since

$$\mathbf{M} + \mathbf{FMF} = \begin{pmatrix} 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \\ 17 & 17 & 17 & 17 \end{pmatrix}.$$

The Moore-Penrose inverse

$$\mathbf{M}^+ = \frac{1}{2720} \begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}.$$

is, therefore, also **F**-associated and magic:

$$\mathbf{M}^+ + \mathbf{FM}^+\mathbf{F} = \frac{1}{68} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Magic perfume?

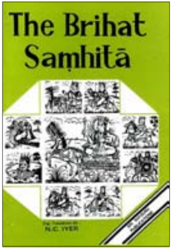
According to Hayashi (*Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, 2nd ed., 2008), the oldest datable magic square in India occurs in the *Bṛhat Saṁhitā*, the Sanskrit encyclopedia on divination by the 6th-century Indian astronomer, mathematician, and astrologer Daivajna Varāhamihira (505–587 AD).

Varāhamihira showed how to make several varieties of perfume with precisely 4 substances selected from 16, including Agar wood (which yields Oud perfume oil).

Varāhamihira also used the magic square defined by the 4×4 magic matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix},$$

with magic sum 18. In the matrix $\mathbf{P}$ the integers $1, 2, \ldots, 8$ each appear twice.



This sheetlet from Qatar 2008 (*Scott* 1035), impregnated with a sandalwood scent, depicts Oud perfume oil and Agar wood (centre two stamps), in addition to (on the far left) Al Marash, "a perfume-pot that is used to spray a nice-smelling liquid (scent) on visitors, which is a sign of hospitality and welcome", and (on the far right) Al Mogbass (TBC).

Qatar, known officially as an emirate, occupies the small Qatar Peninsula on the northeasterly coast of the much larger Arabian Peninsula.

We believe that sandalwood was one of the 16 substances that Varāhamihira used.
This sheetlet from India 2006 (*Scott* 2179) is also impregnated with a sandalwood scent.

Cinnamon is another of the 16 substances that Varāhamihira used, and this sheetlet from India 2009 (*Scott* 2321) features 7 different spices (the stamp in the centre has 3 spices) including cinnamon. This sheetlet is apparently *not* impregnated with any scent.

Hayashi (1987) identifies the 16 substances, transliterated from Sanskrit, as below and as in the table at right, where we have combined the magic matrix **P** with our best guesses for the 16 substances in English.

The Sanskrit text below comes from the English translation by Bhat (1993) of Varāhamihira's *Bṛhat Saṁhitā*. Agar wood (*aguru*) is in the (1,1) cell, sandalwood (*malaya*) in the (4,1) cell and cinnamon (*tvac*) in the (3,2) cell.

षोडशके कच्छपुटे यथा तथा मिश्रिते चतुर्द्रव्ये ।
येऽत्राष्टादश भागास्तेऽस्मिन् गन्धादयो योगाः ॥ २५ ॥

नखतगरतुरुष्कयुता जातीकर्पूरमृगकृतोद्बोधाः ।
गुडनखधूप्या गन्धाः कर्तव्याः सर्वतोभद्राः ॥ २६ ॥

dvitrīndriyāṣṭabhāgair aguruḥ patraṃ turuṣkaśaileyau,
viṣayāṣṭapakṣadahanāḥ priyaṅgumustārasāḥ keśaḥ.
spṛkkātvaktagarāṇāṃ māṃsyāś ca kṛtaikasaptaṣaḍbhāgāḥ,
saptartuvedacandrair malayanakhaśrīkakundrukāḥ.

Two, three, five, and eight parts [respectively of] [3] Aguru, Patra,
l Śaileya [are taken]; five, eight, two, and three [parts respectively of]
ustā, Rasa, and Keśa;
. four, one, seven, and six parts [respectively] of Spṛkkā, Tvac,
Māṃsī; and [finally] seven, six, four, and one [parts respectively of]
:ha, Śrīka, and Kundruka.

aguru **2** *Agar wood ⇒ oud oil*	**patra** **3** *bay leaves*	**turuṣka** **5** *frankincense resin*	**śaileya** **8** *lichen flakes*
priyaṅgu **5** *panic grass oil*	**mustā** **8** *cypriol root oil*	**bola** **2** *myrrh resin*	**keśa** **3** *lemongrass leaves*
spṛkkā **4** *fenugreek oil*	**tvac** **1** *cinnamon sticks*	**tagara** **7** *trangipani flower oil*	**maṁsī** **6** *spikenard leaves*
malaya **7** *sandal-wood*	**nakha** **6** *conch shell powder*	**śrīka** **4** *turpentine resin*	**kundruka** **1** *Deodar cedar resin*

position	ratio 18	Hayashi 1987	My choice: English	stamp plant: English	stamp: country	stamp: year	stamp: *Scott*
11	2	aguru	Agar wood => oud oil	Agar wood	Qatar	2008	1035c
12	3	patra	bay leaves	Bay laurel, Grecian laurel	Greece: Mount Athos	2010	``2nd issue''
13	5	turuska	frankincense resin	frankincense	Oman	1985	285--286
14	8	saileya	lichen flakes	lichen	Liechtenstein	1981	713
21	5	priyangu	panic grass oil	Guinea grass	Bophuthatswana	1984	116
22	8	musta	cypriol root oil	"Sedge in Ninh Binh is mainly Cyperus tojet Jormis specie (white flower)."	Vietnam	1974	740
23	2	rasa	myrrh resin	myrrh	Liechtenstein	1985	822
24	3	kesa	lemongrass leaves	common bulrush, broadleaf cattail	Botswana	1987	427
31	4	sprkka	fenugreek oil	lanceleaf Thermopsis	USSR	1985	5380
32	1	tvac	cinnamon sticks	Indian spices: cinnamon	India	2009	*TBC*
33	7	tagara	frangipani flower oil	frangipani	Niue	1984	421
34	6	mamsi	spikenard leaves	Valerian	Poland	1980	2412
41	7	malaya	sandalwood	sandalwood	India	2007	TBC
42	6	nakha	conch shell powder	conch shell	Travancore & Cochin	1950	16
43	4	srika	turpentine resin	Mount Atlas mastic tree	Syria	2006	TBC
44	1	kundruka	Deodar cedar resin	Deodar cedar	Pakistan	2009	TBC

To obtain a perfume mixture, Varāhamihira mixed together precisely 4 of the 16 substances provided the corresponding numbers add up to 18.

For example, consider the first row of the Varāhamihira-perfume magic matrix **P**

$$\begin{pmatrix} 2 & 3 & 5 & 8 \\ \text{Agar wood} & \text{bay leaves} & \text{frankincense} & \text{lichen} \end{pmatrix}.$$

Then dividing by 18 yields

$$\frac{2}{18}(\text{Agar wood}) + \frac{3}{18}(\text{bay leaves}) + \frac{5}{18}(\text{frankincense}) + \frac{8}{18}(\text{lichen}).$$

From the table on the right we conjecture that Varāhamihira could have made as many as 172 different perfume mixtures in all in c. 500 AD.

1st substance	2nd substance	3rd substance	4th substance	Combinations
1	1	8	8	1
1	2	7	8	16
1	3	6	8	16
1	3	7	7	4
1	4	5	8	16
1	4	6	7	16
1	5	5	7	4
1	5	6	6	4
2	2	6	8	4
2	2	7	7	1
2	3	5	8	16
2	3	6	7	16
2	4	4	8	4
2	4	5	7	16
2	4	6	6	4
2	5	5	6	4
3	3	4	8	4
3	3	5	7	4
3	3	6	6	1
3	4	4	7	4
3	4	5	6	16
4	4	5	5	1
			GRAND TOTAL	172

The Varāhamihira-perfume magic matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix},$$

is not **F**-associated but it is "**H**-associated".

The $n \times n$ magic matrix **A** with $n = 2h$ and magic sum $m_{\mathbf{A}}$ is said to be **H**-*associated* whenever

$$\mathbf{A} + \mathbf{HAH} \propto \mathbf{E},$$

where

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_h \\ \mathbf{I}_h & \mathbf{0} \end{pmatrix}$$

and **E** has every element equal to 1.

Equivalently, the magic matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

is **H**-associated whenever

$$\mathbf{A}_{11} + \mathbf{A}_{22} = \mathbf{A}_{12} + \mathbf{A}_{21} \propto \mathbf{E},$$

where these matrices are all $h \times h = \frac{n}{2} \times \frac{n}{2}$.

The Moore–Penrose inverse

$$\mathbf{P}^+ = \frac{1}{2448} \begin{pmatrix} -173 & -65 & 133 & 241 \\ 97 & 277 & -209 & -29 \\ -65 & -173 & 241 & 133 \\ 277 & 97 & -29 & -209 \end{pmatrix}$$

is **H**-associated and hence magic.

$$\mathbf{A} \text{ is } \mathbf{V}\text{-associated} \Longrightarrow \mathbf{A}^+ \text{ is } \mathbf{V}\text{-associated}$$

The key result of our recent research is:

THEOREM 1. *Suppose that the magic matrix* **A** *is* **V***-associated for some* centrosymmetric involutory *matrix* **V**. *Then the Moore–Penrose inverse* $\mathbf{A}^+$ *is also* **V***-associated and hence magic.*

The matrix **V** is *centrosymmetric* whenever

$$\mathbf{V} = \mathbf{FVF},$$

where **F** is the flip matrix.

The matrix **V** is *involutory* whenever it is symmetric, $\mathbf{V}^2 = \mathbf{I}$, and all row totals are 1.

We have not found a magic matrix **A** with a magic Moore–Penrose inverse that is not **V**-associated for some centrosymmetric involutory matrix **V**. Maybe none exists? TBC

The Varāhamihira-perfume magic matrix

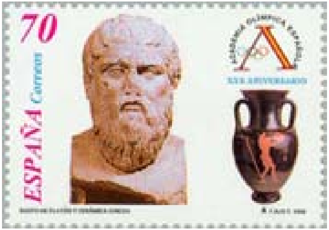
$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix}$$

is not **F**-associated but it is **H**-associated, with $\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{pmatrix}$, centrosymmetric involutory. It follows at once from our Theorem 1 that the Moore–Penrose inverse

$$\mathbf{P}^+ = \frac{1}{2448} \begin{pmatrix} -173 & -65 & 133 & 241 \\ 97 & 277 & -209 & -29 \\ -65 & -173 & 241 & 133 \\ 277 & 97 & -29 & -209 \end{pmatrix}.$$

is **H**-associated and hence magic.

The flip matrix **F** is also centrosymmetric involutory.



Most-perfect & Plato

The Varāhamihira-perfume magic matrix $\mathbf{P}$ and its Moore–Penrose inverse $\mathbf{P}^+$ are both "most-perfect".

A magic matrix is most-perfect whenever it is

1. $\mathbf{H}$-associated,
2. "pandiagonal", and
3. enjoys the "blocks of 4" property.

A magic matrix $\mathbf{A}$ is said to be pandiagonal (diabolical or Nasik) whenever the numbers in all the broken diagonals (parallel to the two main diagonals) as well as in the two main diagonals add up to the magic sum $m_{\mathbf{A}}$.

Following Emory McClintock (*American Journal of Mathematics*, 19, 99–120, 1897), the $n \times n$ magic matrix $\mathbf{A} = \{a_{i,j}\}$ with $n = 4p$ and magic sum $m_{\mathbf{A}}$ has the blocks of 4 property whenever

$$a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} = m_{\mathbf{A}}$$

for all $i, j = 1, \dots, n = 4p$ with the subscripts taken modulo n.

The Varāhamihira-perfume matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix}$$

is 4×4 and so $p = 1$ and there are 16 blocks of 4 with entries adding to $m_{\mathbf{P}} = 18$.

All 4×4 magic matrices that are **H**-associated are pandiagonal and enjoy the blocks of 4 property, and so are most-perfect.

And all 4×4 magic matrices that are **H**-associated have a Moore–Penrose inverse that is **H**-associated and hence is most-perfect.

And so both the Varāhamihira-perfume magic matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix},$$

and its Moore–Penrose inverse

$$\mathbf{P}^{+} = \frac{1}{2448} \begin{pmatrix} -173 & -65 & 133 & 241 \\ 97 & 277 & -209 & -29 \\ -65 & -173 & 241 & 133 \\ 277 & 97 & -29 & -209 \end{pmatrix}$$

are most-perfect, i.e., **H**-associated, pandiagonal and enjoy the blocks of 4 property.

We agree with Hayashi (1987) that Varāhamihira probably created the magic matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 5 & 8 & 2 & 3 \\ 4 & 1 & 7 & 6 \\ 7 & 6 & 4 & 1 \end{pmatrix}$$

from a *classic* magic matrix $\mathbf{Q}$, say, with entries $1, 2, \ldots, 16$.

Hayashi (1987) gives four choices for such a classic magic matrix $\mathbf{Q}$ but suggests that the most likely candidate is the most-perfect

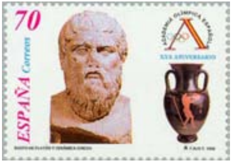
$$\mathbf{Q}_0 = \begin{pmatrix} \underline{10} & 3 & \underline{13} & 8 \\ 5 & \underline{16} & 2 & \underline{11} \\ 4 & \underline{9} & 7 & \underline{14} \\ \underline{15} & 6 & \underline{12} & 1 \end{pmatrix}.$$

Subtracting 8 from the underlined entries in $\mathbf{Q}_0$ yields $\mathbf{P}$.

Rotating $\mathbf{Q}_0$ by 90° counter-clockwise yields the so-called "famous pandiagonal magic square of order 4 in Islam", the most-perfect

$$\mathbf{Q}_1 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix} = \mathbf{F}\mathbf{Q}_0'.$$

The well-known Sufi writer on topics relating to mathematics, Ahmad ibn 'Ali ibn Yusuf al-Buni (d. 1225), ascribed $\mathbf{Q}_1$ to the Greek philosopher and mathematician Plato (c. 428–348 BC) and we will call $\mathbf{Q}_1$ the "Plato magic matrix".



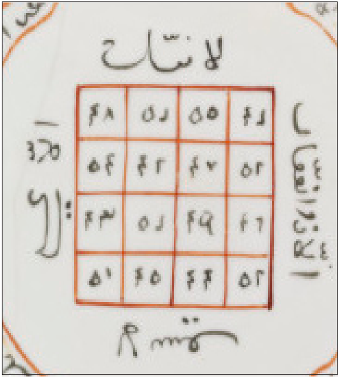
As the anthropologist Schuyler Van Rensselaer Cammann (1912–1991) indicated in Part 1 of "Islamic and Indian magic squares" (*History of Religions*, 8, 1969, p. 189), however, "this merely shows al-Buni's contemporary reverence for antiquity. We have no evidence to show that the classical Greeks had magic squares with numbers".

Hyderabad, Queen Mary & Plato

The Plato magic matrix $\mathbf{Q}_1$ is also connected to the beautiful "Hyderabad (Queen Mary) magic square plate", which is now in the Victoria & Albert Museum in London.

The plate was made c. 1775 in the "Porcelain Capital" Jingdezhen (Jiangxi Province, China) and kept in the Treasure Room at the Golconda Fort in Hyderabad (India).

It was presented to Queen Mary (1867–1953) in 1906, when she, as Princess of Wales, toured India with the Prince of Wales, later King George V (1865–1936), grandfather of Queen Elizabeth II (b. 1926).



The plate is filled with quotations in Arabic from the Koran and in the centre is the “magic square”,

$$\begin{pmatrix} 48 & 56 & 55 & 46 \\ 54 & 42 & 47 & 52 \\ 43 & 56 & 49 & 46 \\ 50 & 45 & 44 & 52 \end{pmatrix}.$$

The row and column totals are, however, not all equal. Moreover, the numbers 41, 51 & 53 are missing while 46, 52 & 56 are duplicated.

It has been suggested that a few mistakes may have been made because the engraver may have had difficulty interpreting the Arabic characters accurately.

Cheng Te-k'un (1974) observes that Professor Ho Peng Yoke (University of Malaya) has suggested that the "magic square" on the plate could have come from the classic magic square

$$\mathbf{R} = \begin{pmatrix} 48 & 51 & 54 & 41 \\ 53 & 42 & 47 & 52 \\ 43 & 56 & 49 & 46 \\ 50 & 45 & 44 & 55 \end{pmatrix},$$

contained in the *Rasa'il: Encyclopedia of the Brethren of Purity,* an encyclopaedic work of 3rd or 4th century Islam.

The matrix **R** is magic and, in fact, **H**-associated and hence most-perfect.

We note that

$$\mathbf{R} - 40\mathbf{E} = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 2 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix} = \mathbf{Q}_1,$$

the **H**-associated most-perfect Plato magic matrix.

Here, as before, the matrix **E** has every element equal to 1 and so subtracting 40 from each element of **R** yields $\mathbf{Q}_1$.

Pherū magic & *Moonlight*

We have commented on magic matrices that are **F**- or **H**-associated and we now look at a magic matrix that is **G**-associated.

We define

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In his *Gaṇitasārakaumudī,* the Jain polymath Ṭhakkura Pherū (fl. 1291–1323) gives the magic square defined by the magic matrix

$$\mathbf{Q}_2 = \begin{pmatrix} 12 & 3 & 6 & 13 \\ 14 & 5 & 4 & 11 \\ 7 & 16 & 9 & 2 \\ 1 & 10 & 15 & 8 \end{pmatrix},$$

which is **G**-associated:

$$\mathbf{Q}_2 + \mathbf{G}\mathbf{Q}_2\mathbf{G} \propto \mathbf{E}.$$

Since **G** is centrosymmetric and involutory our Theorem 1 applies and hence the Moore–Penrose inverse $\mathbf{Q}_2^+$ is **G**-associated and magic.

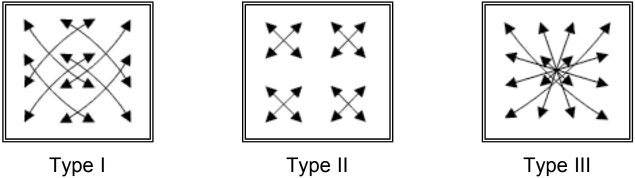
We note the "alphabet string"

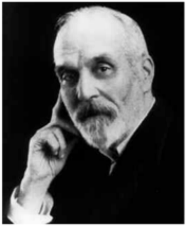
$$\mathbf{F} + \mathbf{G} + \mathbf{H} + \mathbf{I} = \mathbf{E}.$$

The *Gaṇitasārakaumudī: The Moonlight of the Essence of Mathematics*, first published in the early 14th century in Sanskrit, was recently republished in English and typeset in LaTeX (Manohar, New Delhi 2009).

Edited with Introduction, Translation & Mathematical Commentary by SaKHYa = Sreeramula Rajeswara Sarma, Takanori Kusuba, Takao Hayashi and Michio Yano.

Niuafo'ou (= new coconut) is the most northerly island in the kingdom of Tonga, in the southern Pacific Ocean between Fiji and Samoa. It is a volcanic rim island of 15 km^2 with a population of 650 in 2006. It is a still active volcano. According to *Wikipedia*, the barn owl (*Tyto alba*) is the most widely distributed species of owl, and one of the most widespread of all birds.

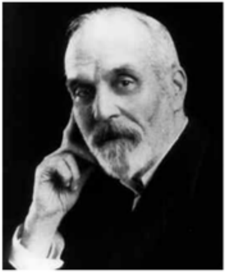




Frénicle & Dudeney

The French mathematician Bernard Frénicle de Bessy (c. 1605–1675) identified the 880 (or $7040 = 8 \times 880$ to include reflections and rotations) classic 4×4 fully-magic squares.

The English author and mathematician Henry Ernest Dudeney (1857–1930)



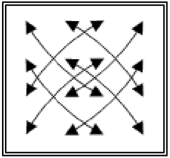
classified them into 12 types. Dudeney's types I, II & III correspond to our **H**-, **G**- and **F**-associatedness.

1. Dudeney Type I = **H**-associated = most-perfect,
2. Dudeney Type II = **G**-associated,
3. Dudeney Type III = **F**-associated.

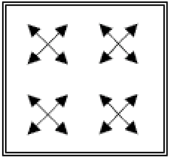
We find that the 48 (out of 880) of Dudeney Type I are **H**-associated (and hence most-perfect), the 48 of Dudeney Type II are **G**-associated & the 48 of Dudeney Type III are **F**-associated, and hence these 144 ($= 3 \times 48$) all have fully-magic Moore–Penrose inverses.

None of the other $880 - 144 = 736$ has a magic Moore–Penrose inverse and indeed none of these is **V**-associated.

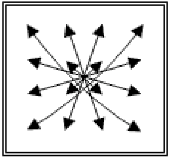
Moreover, of the 144 that are **V**-associated, precisely 24 are EP: 8 each of Dudeney types I, II, and III.



Type I



Type II



Type III

$$\mathbf{Q}_1 = \begin{pmatrix} 8 & 11 & 14 & 1 \\ 13 & 2 & 7 & 12 \\ 3 & 16 & 9 & 6 \\ 10 & 5 & 4 & 15 \end{pmatrix}, \quad \mathbf{Q}_2 = \begin{pmatrix} 12 & 3 & 6 & 13 \\ 14 & 5 & 4 & 11 \\ 7 & 16 & 9 & 2 \\ 1 & 10 & 15 & 8 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix},$$

Plato *Pherū* *Dürer–Pacioli*

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

EP magic & Schwerdtfeger

Another goal in our research is to identify singular magic matrices with a magic Moore–Penrose inverse that are also EP.

The square matrix $\mathbf{A}$ with index 1 is said to be EP whenever

$$\text{rank}(\mathbf{A} : \mathbf{A}') = \text{rank}(\mathbf{A}),$$

i.e., whenever the column spaces of $\mathbf{A}$ and its transpose $\mathbf{A}'$ coincide.

The singular matrix $\mathbf{A}$ has index 1 whenever $\text{rank}(\mathbf{A}^2) = \text{rank}(\mathbf{A})$.

To see that an EP matrix must have index 1, we first use a formula from Marsaglia & Styan (1974) to show that $\mathbf{A}$ is EP if and only if

$$\mathbf{A}\mathbf{A}^+\mathbf{A}' = \mathbf{A}'. \tag{1}$$

Premultiplying (1) by $\mathbf{A}$ yields

$$\mathbf{A}^2\mathbf{A}^+\mathbf{A}' = \mathbf{A}\mathbf{A}'$$

and so

$$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}') \leq \text{rank}(\mathbf{A}^2) \leq \text{rank}(\mathbf{A})$$

and so $\mathbf{A}$ has index 1.

The *group-inverse* $\mathbf{A}^\#$ of the index 1 matrix $\mathbf{A}$ coincides with the Moore–Penrose inverse $\mathbf{A}^+$ if and only if $\mathbf{A}$ is EP [Ben-Israel & Greville: *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, 2003 (paperback 2010), p. 157.]

The square matrix $\mathbf{A}$ with index 1 has a *group-inverse* $\mathbf{A}^\#$ that satisfies the equations

$$\mathbf{A}\mathbf{A}^\#\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^\#\mathbf{A}\mathbf{A}^\# = \mathbf{A}^\#, \quad \mathbf{A}\mathbf{A}^\# = \mathbf{A}^\#\mathbf{A}.$$

We believe that the term EP was introduced in 1950 in the First Edition of Schwerdtfeger's *Introduction to Linear Algebra and the Theory of Matrices* [2nd edition (Noordhoff, 1961, pp. 130–131)].

Hans Wilhelm Eduard Schwerdtfeger (1902–1990) was Professor of Mathematics at McGill University from 1960–1983.

Captain Shortreede, Major-General Shortrede,
Gwalior & the “rhomboid” property

Journal of the Asiatic Society of Bengal, New Series, vol. 11, no. 124 (1842), pp. 292–293.

On an Ancient Magic Square, cut in a Temple at Gwalior. By Captain SHORTREEDE.

As every thing tending to throw any certain light on the antiquities of India has an interest, I send you the following inscription of a Magic Square, which I copied last year from an old temple in the hill fort of Gwalior. It bears the date समवत १५४० = A. D. 1483.

The temple is on the northern side of the hill, and at one time it has been a very magnificent edifice, though now it be sorely dilapidated.

There is another and larger ancient temple in the fort, of a peculiar form, which the Musalmans have converted into a Musjid.

If I remember rightly, the Magic Square is cut on the inner side of the northern wall, close to where the excavation has been made. I did not measure the dimensions; but the form is as follows :—

We define the 13th century magic square

१६	९	४	५
३	६	१५	१०
१३	१२	१	८
२	७	१४	११

by the Shortrede–Gwalior magic matrix

$$\mathbf{W} = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix}.$$

Almost surely "Captain Shortreede" = Major-General Robert Shortrede (1800–1868), with the extra "e" a typo.

Gwalior is in Madhya Pradesh, about 120 km south of Agra.

From his obituary [*Monthly Notices of the Royal Astronomical Society*, vol. 29, pp. 120–121, 1868] we find that Robert Shortrede was born on 19 July 1800 in Jedburgh in Scotland, about halfway between Edinburgh and Newcastle Upon Tyne.

Having "early evinced unusual aptitude for mathematics ... thinking that India presented ample scope for his talents in that direction, he obtained an appointment to that country".

Shortrede went to India in 1822 and was appointed to the Deccan Survey. The Deccan Plateau extends over eight Indian states and encompasses a wide range of habitats, covering most of central and southern India.

Shortrede was appointed to the Great Trigonometric Survey (GTS) in which he remained until 1845. The GTS was piloted in its initial stages by William Lambton (c. 1753–1823), and later by Sir George Everest (1790–1866).

Among the many accomplishments of the GTS was the measurement of the height of the Himalayan giants: Everest, K2, and Kanchenjunga. In 1865, Mount Everest was named in Sir George Everest's honour, despite his objections.

The Scottish historical novelist and poet Sir Walter Scott, 1st Baronet (1771–1832), was an "old and intimate friend of the Shortrede family" and both Shortrede and Scott studied at the University of Edinburgh.

> It will be observed, that the places of the numbers 1, 2, 3, 4, form a rhomboid, as do also 5, 6, 7, 8; 9, 10, 11, 12; 13, 14, 15, 16. It may be remarked also, that the sum of every two alternate numbers taken diagonally is 17: and that all these properties will hold good if the lines be transposed vertically or horizontally in the same order; that is, if the top line be brought to the bottom; or if the left hand vertical line be carried over to the right.

Shortrede observed that the matrix **W** is **H**-associated, from which it follows that **W** is most-perfect.

We observe that **W** is EP and may be one of the oldest EP magic matrices with a magic Moore–Penrose inverse.

Shortrede observed that the (underlined) numbers $1, 2, 3, 4$ in **W** form a "rhomboid"

$$\mathbf{W} = \begin{pmatrix} 16 & 9 & \underline{4} & 5 \\ \underline{3} & 6 & 15 & 10 \\ 13 & 12 & \underline{1} & 8 \\ \underline{2} & 7 & 14 & 11 \end{pmatrix}$$

as do the numbers $5, 6, 7, 8$; $9, 10, 11, 12$ and $13, 14, 15, 16$.

We refer to this as the "rhomboid" property.

Euclid of Alexandria (fl. 300 BC) apparently introduced the term rhomboid in his *Elements* (Book I, Def. 22):

"Of quadrilateral figures, ... an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled."

Wikipedia says "The term rhomboid is now more often used for a parallelepiped, a solid figure with six faces in which each face is a parallelogram and pairs of opposite faces lie in parallel planes."

Collegium Maius (Latin for "Great College") is the Jagiellonian University's oldest building, dating back to the 15th century.

Collegium Maius, Cracow — A550

40g, Copernicus House, Torun, vert. 2.50z, Olsztyn Castle. 4z, Frombork Cathedral, vert.

Nicolaus Copernicus (1473-1543), astronomer. Printed in sheets of 15 with labels showing portrait of Copernicus, page from "Euclid's Geometry," astrolabe or drawing of heliocentric system, respectively.

1971, June		**Litho.**	***Perf. 11***	
1818	A550	40g multicolored	.20	.20
1819	A550	60g blk, red brn & sep	.20	.20
1820	A550	2.50z multicolored	.25	.20
1821	A550	4z multicolored	.45	.20
		Nos. 1818-1821 (4)	1.10	.80

Hayashi (2008) reports several 4×4 magic squares from the *Gaṇita Kaumudī* (1356) by the Kerala mathematician Nārāyaṇa Paṇḍita (fl. 1340–1400) which are EP. Moreover, most of these, like **W**, have the rhomboid property.

In Part II of "Islamic and Indian magic squares", Cammann [*History of Religions*, 8, 271–299, 1969] gives Nārāyaṇa's favourite magic square defined by the magic matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix} = \mathbf{HWH},$$

the matrix **W** with its main 2×2 diagonal and off-diagonal blocks switched.

Moreover, **X** is the "complement" of **W** in that, since **W** is **H**-associated:

$$\mathbf{W} + \mathbf{X} \propto \mathbf{E}$$

and so **X** also has the rhomboid property.

We observe that 24 of the 880 magic 4×4 matrices found by Frénicle de Bessy (1693) are EP: 8 each of Dudeney Type I (**H**-associated, pandiagonal, most-perfect), Dudeney Type II (**G**-associated), and Dudeney Type III (**F**-associated). And so all 24 have a magic Moore–Penrose inverse.

But of these 24 we find only 2 with the rhomboid property: Nārāyaṇa's favourite magic matrix $\mathbf{X} = \mathbf{HWH}$, and the magic matrix

$$\begin{pmatrix} 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \\ 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \end{pmatrix} = \mathbf{HX} = \mathbf{WH}.$$

Shortrede ends his 1842 article by extending the matrix **W** with an arrangement of numbers, which also includes Nārāyaṇa's favourite magic matrix **X**. Also embedded are three more 4×4 magic matrices: one EP but not rhomboid, one not EP but rhomboid, and the third neither EP nor rhomboid.

I add a copy of the inscription in our common numerals, in case it may be wanted, as also a sample of the way in which it may be extended, which probably is similar to that in Dr. Franklin's Magical Square of Squares, but on this point I cannot speak positively, as I do not distinctly remember the particulars of Dr. Franklin's Square of Squares, and have at present no means of reference.

W =

16	9	4	5	16			
3	6	15	10	3			
13	12	1	8	13	12		
2	7	14	11	2	7	14	11
	9	4	5	16	9	4	5
X =		15	10	3	6	15	10
				13	12	1	8
				2			

Benjamin Franklin & Peter Collinson

We believe that Shortrede's reference to "Dr. Franklin's Magical Square of Squares" is to the letter written c. 1750 by Benjamin Franklin (1705–1790) to Peter Collinson (1694–1768).

The letter was apparently first published in the 4th edition (1769) of Franklin's *Experiments and Observations on Electricity* [reprinted in *The Works of Benjamin Franklin* (Jared Sparks, ed.), Boston, 1840, pp. 100–103, and digitized by Google].

Benjamin Franklin was one of the Founding Fathers of the United States and a leading scientist, while Peter Collinson was an avid gardener, and the middleman for an international exchange of scientific ideas in mid-18th century London.

The letter included an 8×8 and a 16×16 magic square in which the numbers in the "bent diagonals" (but not the main diagonals) add up to the magic sum. The 8×8 Franklin magic square is

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

... defined by the Franklin magic matrix **B**

$$\begin{pmatrix} 52 & 61 & 4 & 13 & 20 & 29 & 36 & 45 \\ 14 & 3 & 62 & 51 & 46 & 35 & 30 & 19 \\ 53 & 60 & 5 & 12 & 21 & 28 & 37 & 44 \\ 11 & 6 & 59 & 54 & 43 & 38 & 27 & 22 \\ 55 & 58 & 7 & 10 & 23 & 26 & 39 & 42 \\ 9 & 8 & 57 & 56 & 41 & 40 & 25 & 24 \\ 50 & 63 & 2 & 15 & 18 & 31 & 34 & 47 \\ 16 & 1 & 64 & 49 & 48 & 33 & 32 & 17 \end{pmatrix}.$$

The row and column sums of **B** are all 260 but the forwards diagonal sum (the trace) is 228, while the backwards diagonal sum is 292. We note that

$$228 + 292 = 2 \times 260.$$

The Franklin matrix **B** is only "semi-magic".

The 8×8 Franklin matrix **B** is singular with rank 3 and index 1, but is not EP.

The row and column sums of the Moore–Penrose inverse $\mathbf{B}^+$ are all equal (as expected) to $\frac{1}{260}$, while the forwards diagonal and backwards diagonal sums coincide (surprisingly) being both equal to

$$\frac{17}{2730} \neq \frac{1}{260}.$$

Since **B** has index 1 it has a group inverse $\mathbf{B}^\#$. The row and column sums of $\mathbf{B}^\#$ are all equal (again as expected) to $\frac{1}{260}$, while the forwards diagonal and backwards diagonal sums coincide again surprisingly) being both equal to

$$\frac{69}{1040} \neq \frac{1}{260}.$$

And so the group inverse $\mathbf{B}^\#$ and the Moore–Penrose inverse $\mathbf{B}^+$ do not coincide confirming that **B** is not EP.

We have not found any similarity between Shortrede's "extension" and the 8 × 8 Franklin magic square, but it is the only magic square other than the Pacioli–Dürer magic square that we have found on a postage stamp (USA 2006, *Scott* 4022).

The stamp also includes a whimsical 19th-century Currier and Ives lithograph depicting Franklin and his son performing the legendary electricity experiment with a kite (see also the very nice sheetlet from Sierra Leone 2002, *Scott* 2520).

Also shown is a schematic drawing of Franklin's "three-wheeled clock" from *Select Mechanical Exercises* by James Ferguson (1710–1776) and Franklin at a writing desk from a mural by Charles Elliott Mills.

Raghunandana, Zlobec & Khajuraho–Dudhai

An early method for making magic squares of order 4 having any optional magic sum is given in *Smṛtitattva*, the 16th-century encyclopaedia on Hindu Law by the Bengali scholar Raghunandana Bhaṭṭācārya (fl. c. 1520/1570).

The magic matrix

$$\mathbf{K}^{(a)} = \begin{pmatrix} 1 & 8 & a-7 & a-2 \\ a-5 & a-4 & 3 & 6 \\ 7 & 2 & a-1 & a-8 \\ a-3 & a-6 & 5 & 4 \end{pmatrix}$$

has magic sum $2a$, and is $\mathbf{H}$-associated and hence most-perfect for all values of a, and therefore so is its Moore–Penrose inverse.

The matrix $\mathbf{K}^{(a)}$ is classic only when $a = 17$

$$\mathbf{K}^{(17)} = \begin{pmatrix} 1 & 8 & 10 & 15 \\ 12 & 13 & 3 & 6 \\ 7 & 2 & 16 & 9 \\ 14 & 11 & 5 & 4 \end{pmatrix},$$

and EP only when $a = 11$

$$\mathbf{K}^{(11)} = \begin{pmatrix} 1 & 8 & 4 & 9 \\ 6 & 7 & 3 & 6 \\ 7 & 2 & 10 & 3 \\ 8 & 5 & 5 & 4 \end{pmatrix}.$$

According to the *Banglapedia: National Encyclopedia of Bangladesh*, "Bhattacharya, Raghunandan (c. 15th-16th century)" was a famous scholar of Hindu law (*Smriti*), born in Nadia (now in West Bengal, India, very close to the border with Bangladesh). He is central to Hindu law in Bengal which is described as pre-Raghunandan, Raghunandan, and post-Raghunandan.

Hayashi (2008, Fig. 24) gives the matrix $\mathbf{K}^{(a)}$ and attributes it to "Laghunandana" which we believe is a typo for "Raghunandana".

In September 2010 we constructed a philatelic magic square (PMS) for the 70th birthday celebration (in Split, Croatia) of my McGill colleague Sanjo Zlobec based on the magic matrix $\mathbf{S} =$

$$\begin{pmatrix} 70 & 65 & 15 & 40 \\ 45 & 10 & 60 & 75 \\ 20 & 35 & 85 & 50 \\ 55 & 80 & 30 & 25 \end{pmatrix} = 5 \begin{pmatrix} 14 & 13 & 3 & 8 \\ 9 & 2 & 12 & 15 \\ 4 & 7 & 17 & 10 \\ 11 & 16 & 6 & 5 \end{pmatrix},$$

which is $\mathbf{F}$-associated and so its Moore–Penrose inverse is $\mathbf{F}$-associated and magic.

Moreover, $\mathbf{S}$ is EP and so the group inverse $\mathbf{S}^{\#}$ and Moore–Penrose inverse $\mathbf{S}^{+}$ coincide:

$$\mathbf{S}^{\#} = \mathbf{S}^{+}$$

$$= \frac{1}{25840} \begin{pmatrix} 205 & 167 & -213 & -23 \\ 15 & -251 & 129 & 243 \\ -175 & -61 & 319 & 53 \\ 91 & 281 & -99 & -137 \end{pmatrix}.$$

But $\mathbf{S}$ does not enjoy the rhomboid property.

The Khajuraho–Dudhai magic matrix

$$\mathbf{K} = \begin{pmatrix} 7 & 12 & 1 & 14 \\ 2 & 13 & 8 & 11 \\ 16 & 3 & 10 & 5 \\ 9 & 6 & 15 & 4 \end{pmatrix}$$

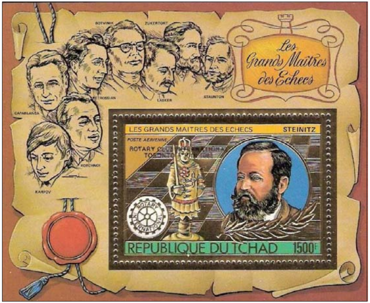
corresponds to a magic square found in Khajuraho and in Dudhai (Jhansi district), both also in India (and both fairly near Gwalior).

The Khajuraho magic square, which dates back to the 12th or 13th century, was found "in a Jaina inscription in the Parshvanath Jain temple" and/or on the "front wall of the Jain temple Jinanatha".

The matrix $\mathbf{K}$ is not EP, but it is $\mathbf{H}$-associated and hence most-perfect.

The Dudhai magic square was carved on the "underside of a lintel from the collapsed doorway of a shrine known as the Chota Surang (small tunnel)" apparently built in the 11th century AD.

The stamp (upper right) depicts Laxmi Bai transformed from a queen into a warrior fighting for justice. Laxmi Bai, The Rani of Jhansi (c. 1828–1858), was the queen of the Maratha-ruled princely state of Jhansi and a symbol of resistance to British rule in India.



Firth & Zukertort

At the end of the rare book entitled *The Magic Square* by W. A. Firth [*sic*] (1887) is the one-page "article" entitled "The Magic Chess Board, invented by W. Firth [*sic*], dedicated to Dr. Zukertort".

"The Magic Chess Board" is an 8×8 magic square with "Firth–Zukertort magic matrix"

$$\mathbf{Z} = \begin{pmatrix} 64 & 21 & 42 & \underline{3} & 37 & 16 & 51 & 26 \\ 38 & 15 & 52 & 25 & 63 & 22 & 41 & \underline{4} \\ 11 & 34 & 29 & 56 & 18 & 59 & 8 & 45 \\ 17 & 60 & 7 & 46 & 12 & 33 & 30 & 55 \\ 10 & 35 & 32 & 53 & 19 & 58 & 5 & 48 \\ 20 & 57 & 6 & 47 & 9 & 36 & 31 & 54 \\ 61 & 24 & 43 & \underline{2} & 40 & 13 & 50 & 27 \\ 39 & 14 & 49 & 28 & 62 & 23 & 44 & \underline{1} \end{pmatrix}.$$

Firth showed that if the top left, top right, lower left and lower right 4×4 squares are stacked then they form a "magic cube".

An $n \times n \times n$ magic cube is the 3-dimensional equivalent of an $n \times n$ magic square, with the sum of the numbers in each row, each column, each "pillar" and the four main space diagonals equal to the same magic number.

The Firth–Zukertort magic matrix $\mathbf{Z}$ is $\mathbf{F}$-associated and hence so is its Moore–Penrose inverse. Moreover, $\mathbf{Z}$ has index 1 and so has a group-inverse.

Furthermore, $\mathbf{Z}$ has index 1 and is EP, and hence the Moore–Penrose inverse and group-inverse coincide. In addition $\mathbf{Z}$ enjoys the rhomboid property: we have underlined the entries $1, 2, 3, 4$.

THE

MAGIC CHESS BOARD,

INVENTED BY W. FIRTH.

Dedicated to Dr. ZUKERTORT.

THE numbers 1...64 placed thus on the squares of a Chess Board, when added in rows, columns, and diagonals, have the same sum, 260, forming a Magic Square of 8. Cut the Square into four quarters and imagine them held in the air in the form of a cube, the numbers will then form a Magic Cube, any line of numbers parallel to any edge having the same sum, 130, as also the four diagonals of the Cube.

64	21	42	3	37	16	51	26
38	15	52	25	63	22	41	4
11	34	29	56	18	59	8	45
17	60	7	46	12	33	30	55
10	35	32	53	19	58	5	48
20	57	6	47	9	36	31	54
61	24	43	2	40	13	50	27
39	14	49	28	62	23	44	1

64	21	42	3
38	15	52	25
11	34	29	56
17	60	7	46

37	16	51	26
63	22	41	4
18	59	8	45
12	33	30	55

10	35	32	53
20	57	6	47
61	24	43	2
39	14	49	28

19	58	5	48
9	36	31	54
40	13	50	27
62	23	44	1

PUBLISHED BY

JAS. WADE, 18, TAVISTOCK, STREET,

COVENT GARDEN, W.C.

Firth dedicated his "Magic Chess Board" to "Dr. Zukertort", almost certainly the chess master Johannes Hermann Zukertort (1842–1888), who was one of the leading world players for most of the 1870s and 1880s, and who lost $7\frac{1}{2}$–$12\frac{1}{2}$ to Wilhelm Steinitz (1836–1900) in the 1886 inaugural World Chess Championship.

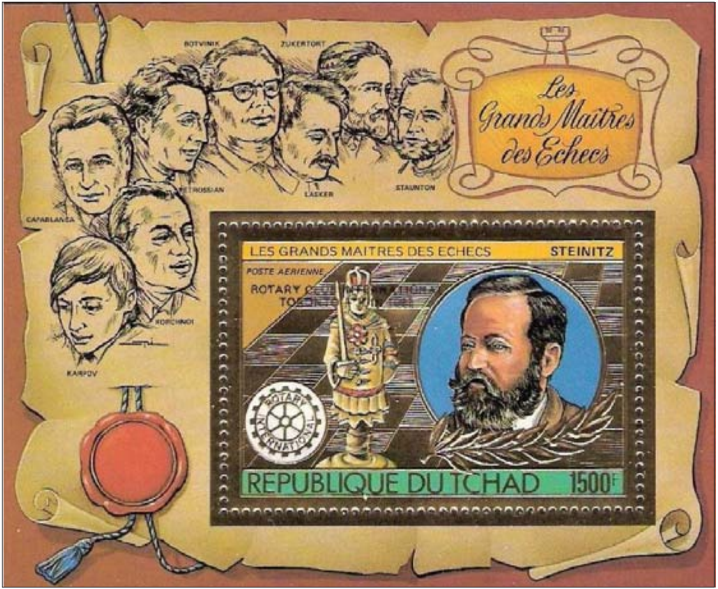
We know of no other distinguished chess player associated with a magic square (or magic cube or magic chessboard).

The only stamps that we have found that are connected with Zukertort are a souvenir sheet for "Les grands maîtres des échecs" from Chad 1982 and a pair of stamps from the Central African Republic 1983.

There is a portrait of Zukertort in the selvage of the souvenir sheet from Chad, which shows a single stamp with a portrait of Steinitz. Zukertort is shown in the selvage between Emanuel and Staunton at the top centre.

Proceeding anti-clockwise from Lasker there are portraits of Botvinnik, Petrosian, Capablanca, Korchnoi and Karpov.

The German mathematician Emanuel Lasker (1868–1941) was World Chess Champion for 27 years (1894–1921) and was known for his contributions to commutative algebra, as he defined the primary decomposition property of the ideals of some commutative rings when he proved that polynomial rings have the primary decomposition property.



BOTVINIK
ZUKERTORT
Les Grands Maîtres des Echecs
PETROSSIAN
LASKER
STAUNTON
CAPABLANCA
KORCHNOI
KARPOV
LES GRANDS MAITRES DES ECHECS
STEINITZ
POSTE AERIENNE
ROTARY INTERNATIONAL
REPUBLIQUE DU TCHAD
1500F

Zukertort, Steinitz & Spassky

There are many stamps which honour Steinitz, including the one shown in the souvenir sheet from Chad. The only stamps *per se* that we have found associated with Zukertort are two from a set from the Central African Republic in 1983.

According to the website "Chess positions on stamps", the chess position shown on the 300F stamp is from the 9th game of the 1886 Steinitz–Zukertort championship match though the portrait there is of Boris Spassky; this game (with Zukertort white) was played on 10 February 1886.

The 5F stamp shows a portrait of Steinitz together with a position which derives from a game Spassky played in 1953, nearly 50 years after Steinitz died! Presumably the intent was to put the Steinitz–Zukertort chess position on the 5F stamp and the Spassky chess position on the 300F stamp!

We know very little about "W. A. Firth" but note that the title page of *The Magic Square* says "W. A. Firth, B. A., Cantab., late scholar of Emmanuel College, Cambridge, and mathematical master of St. Malachy's College, Belfast."

The preface is signed with the address "31 Hamilton Street, Belfast" and it seems to us, therefore, almost certain that our "W. A. Firth" is the "William A. Firth" who died on 12 September 1890.

From the Public Record Office of Northern Ireland, we found that "The Will of William A. Firth, late of Belfast, Mathematical Professor, who died on 12 September 1890 at same place was proved at Belfast by Margaret Firth of Hamilton-street Belfast, Widow, one of the Executors."

We believe that our "W. A. Firth" is the "Mr. William A. Firth (Whiterock, Belfast)" who with "Mr. Henry Perigal were balloted for and duly elected" members of the Quekett Microscopical Club on 22 July 1881.

Henry Perigal Junior (1801–1898) was an English amateur mathematician best known for his elegant dissection proof of Pythagoras's theorem, a diagram of which is carved on his gravestone in the churchyard of the Church of St. Mary and St. Peter in Wennington, Essex, England.

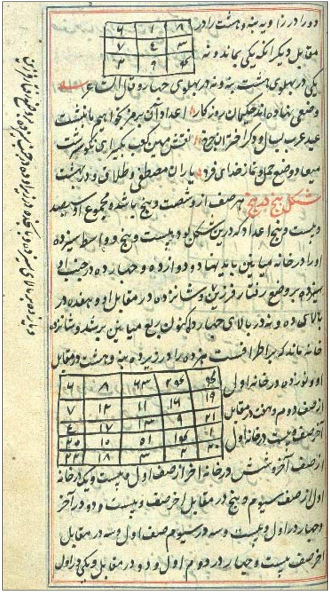
The Quekett Microscopical Club was named after the English microscopist and histologist John Thomas Quekett (1815–1861). Founded in 1865 the Club is dedicated to optical microscopy, amateur and professional.

Kazwini's *Cosmography*

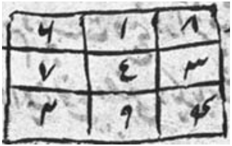
We end this talk with comments on three magic squares in Kazwini's *Cosmography* = *Book of the Marvels of Nature and the Singularities of Created Things* by Zakariyyā' ibn Muḥammad ibn Maḥmūd Abū Yaḥyā al-Ḳazwīnī (c. 1203–1283), first published in the 13th century.

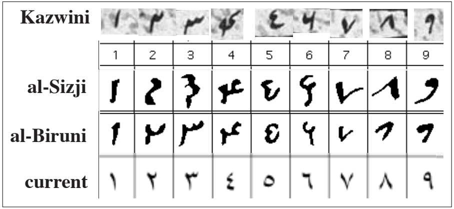
Four Persian translations of this book have been digitized and copies are available open-access online at the US National Library of Medicine, which is operated by the US federal government and is the world's largest medical library.

Shown here is Folio 310a of copy #MS P 3 from the National Library of Medicine with two magic squares: a 3×3 at the top and a 5×5 near the bottom. A third magic square, which is 4×4, is partially visible (in the middle), possibly from the verso.–TBC



We define the 3×3 "Kazwini magic square"





by the magic matrix

$$\mathbf{K}_3 = \begin{pmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{pmatrix} = \mathbf{F} \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix} \mathbf{F} = \mathbf{FLF},$$

the Luoshu magic matrix $\mathbf{L}$ double-flipped.

The "Hindu-Arabic numerals" used here seem to be very similar to those used by the Persian astronomer and mathematician Abu Sa'id Ahmed ibn Mohammed ibn Abd Jalil Sijzi (c. 945–c. 1020) and/or by the Persian scholar Abu Arrayhan Muhammad ibn Ahmad al-Biruni (973–1048).

We define the 5 × 5 Kazwini magic square

by the magic matrix

$$\mathbf{K}_5 = \begin{pmatrix} 6 & 8 & 23 & 24 & 4 \\ 7 & 12 & 11 & 16 & 19 \\ 5 & 17 & 13 & 9 & 21 \\ 25 & 10 & 15 & 14 & 1 \\ 22 & 18 & 3 & 2 & 20 \end{pmatrix},$$

which is classic with magic sum 65. Moreover, $\mathbf{K}_5$ is nonsingular but its inverse is <u>not</u> magic.

However, $\mathbf{K}_5$ is almost $\mathbf{F}$-associated: 17 out of 25 elements of the matrix

$$\mathbf{K}_5 + \mathbf{F}\mathbf{K}_5\mathbf{F} = \begin{pmatrix} 26 & 10 & 26 & 42 & 26 \\ 8 & 26 & 26 & 26 & 44 \\ 26 & 26 & 26 & 26 & 26 \\ 44 & 26 & 26 & 26 & 8 \\ 26 & 42 & 26 & 10 & 26 \end{pmatrix},$$

are equal to 26.

The inverse matrix $\mathbf{K}_5^{-1} = \frac{1}{83200} \times$

$$\begin{pmatrix} -22208 & 12632 & 10032 & 19912 & -19088 \\ 83872 & -50288 & -45088 & -69008 & 81792 \\ -51328 & 22512 & 38112 & 45392 & -53408 \\ 38112 & -12848 & -28448 & -31568 & 36032 \\ -47168 & 29272 & 26672 & 36552 & -44048 \end{pmatrix}$$

is almost magic: the row and column totals are all equal and the two traces are equal. The rows and columns add to to $\frac{1}{65}$ but the two traces are $\frac{275}{208} \neq \frac{1}{65}$.