An introduction to Ramanujan's magic squares

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[2] This beamer file is for an invited talk presented as a video on Tuesday, 10 January 2012, at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–11 January 2012. This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

This beamer file is for an invited talk presented on Tuesday, 10 January 2012, at the International Workshop and Conference on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–11 January 2012.

I am very grateful to Professor Prasad and the Workshop participants who reminded me of Ramanujan's work on magic squares and to Dr. B. Chaluvaraju for drawing our attention to "Bangalore University's old collections in the library which deal with Yantras and magic squares."

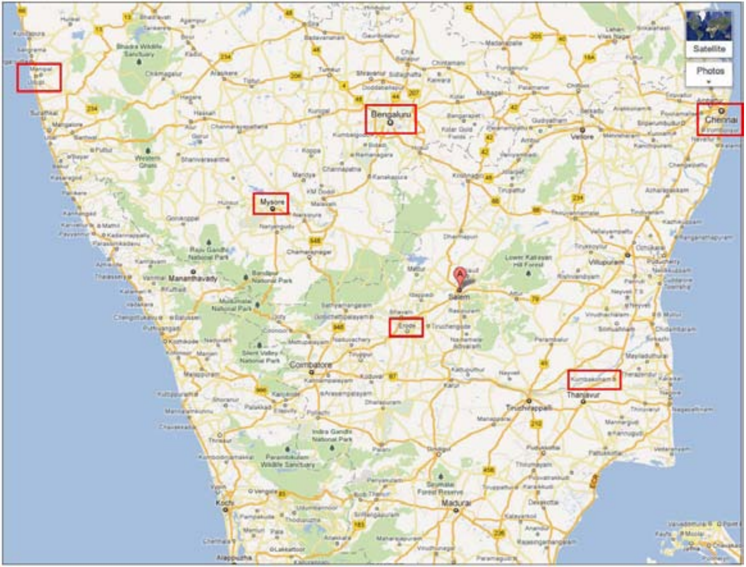
In addition, many thanks go to Pavel Chebotarev and Ka Lok Chu for their help. This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

Srinivasa Aiyangar Ramanujan (1887–1920)

was born in Erode and lived in Kumbakonam

(both then in Madras Presidency, both now in Tamil Nadu),

and died in Chetput (Madras, now Chennai).


Satellite
Photos
Bengaluru
Chennai
Mysore
Salem
Erode
Kumbakonam
Thanjavur
Tiruchirappalli
Coimbatore
Madurai
Vellore
Villupuram
Mananthavady
Kochi

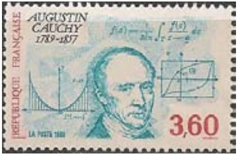
Ramanujan lived most of his life in Kumbakonam, an ancient capital of the Chola Empire. The dozen or so major temples dating from this period made Kumbakonam a magnet to pilgrims from throughout South India.

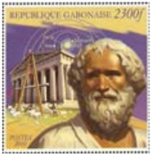
Raja Raja Chola I, popularly known as Raja Raja the Great, ruled the Chola Empire between 985 and 1014 CE.

Srinivasa Aiyangar Ramanujan was born on 22 December 1887, and on 22 December 1962 and on 22 December 2011, India Post issued a postage stamp in his honour.

On 22 December 2011, Prime Minister Dr. Manmohan Singh in Chennai declared 22 December as National Mathematics Day, and declared 2012 as National Mathematical Year. [*The Hindu*, 27 December 2011.]

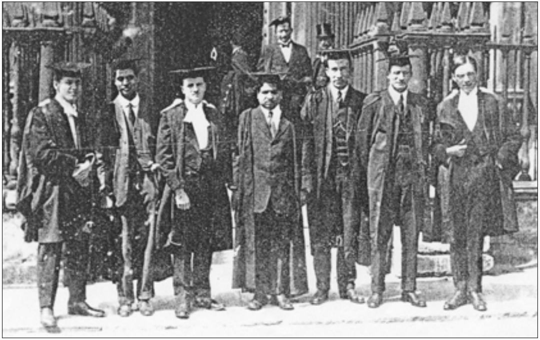
Ramanujan's talent was said[9] by the English mathematician Godfrey Harold "G.H." Hardy (1877–1947) to be in the same league as that of Gauss, Euler, Cauchy, Newton, and Archimedes.





9 "Srinivasa Ramanujan", *Wikipedia*, 7 January 2012, p. 1.

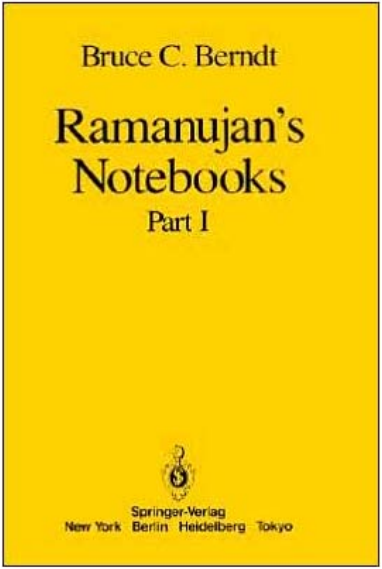
From 1914–1919 Ramanujan worked
with G. H. Hardy at Trinity College, Cambridge[11].



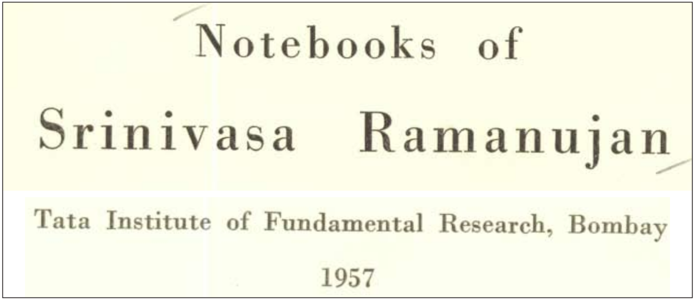
[11] Photographs (left panel: Ramanujan, centre) from "Srinivasa Ramanujan", *Wikipedia*, 7 January 2012, p. 8.

Ramanujan's work on magic squares is presented, in some detail, in Chapter 1 (pp. 16–24) of *Ramanujan's Notebooks, Part I,* by Bruce C. Berndt (Springer 1985)

"The origin of Chapter 1 probably is found in Ramanujan's early school days and is therefore much earlier than the remainder of the notebooks."



Ramanujan's work on magic squares was also presented, photographed from its original form, in *Notebooks of Srinivasa Ramanujan,* Volume I, Notebook 1, and Volume II, Notebook 2, pub. Tata Institute of Fundamental Research, Bombay, 1957.


Notebooks of

Srinivasa Ramanujan

Tata Institute of Fundamental Research, Bombay

1957

In Berndt 1985, Corollary 1, p. 17, we find:

In a 3×3 magic square,
the elements in the
middle row, middle column,
and each [main] diagonal
are in arithmetic progression.

And so we have the general form for a 3×3 magic matrix

$$\mathbf{R}_3 = \begin{pmatrix} h+u & h-u+v & h-v \\ h-u-v & h & h+u+v \\ h+v & h+u-v & h-u \end{pmatrix} = h\mathbf{E}_3 + u\mathbf{U}_3 + v\mathbf{V}_3$$

$$= h\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + u\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} + v\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

.

Berndt (*op. cit.*, p. 21) presents

$$\mathbf{R}_4 = \begin{pmatrix} a+p & d+s & c+q & b+r \\ c+r & b+q & a+s & d+p \\ b+s & c+p & d+r & a+q \\ d+q & a+r & b+p & c+s \end{pmatrix}$$

$$= \begin{pmatrix} a & d & c & b \\ c & b & a & d \\ b & c & d & a \\ d & a & b & c \end{pmatrix} + \begin{pmatrix} p & s & q & r \\ r & q & s & p \\ s & p & r & q \\ q & r & p & s \end{pmatrix},$$

the sum of two orthogonal Latin squares (Graeco-Latin square).

Ramanujan (Tata Institute 1957, Volume II, Notebook 2, p. 12 = original p. 8) gives this 5 × 5 magic square, which is also the sum of two orthogonal Latin squares (Graeco-Latin square).

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

Berndt (*op. cit.*) reports two 7 × 7 (p. 24) and two 8 × 8 (p. 22) magic squares (but apparently no 6 × 6) by Ramanujan, including

$$\mathbf{R}_7 = \begin{pmatrix} 1 & 49 & 41 & 33 & 25 & 17 & 9 \\ 18 & 10 & 2 & 43 & 42 & 34 & 26 \\ 35 & 27 & 19 & 11 & 3 & 44 & 36 \\ 45 & 37 & 29 & 28 & 20 & 12 & 4 \\ 13 & 5 & 46 & 38 & 30 & 22 & 21 \\ 23 & 15 & 14 & 6 & 47 & 39 & 31 \\ 40 & 32 & 24 & 16 & 8 & 7 & 48 \end{pmatrix}, \quad \mathbf{R}_8 = \begin{pmatrix} 1 & 62 & 59 & 8 & 9 & 54 & 51 & 16 \\ 60 & 7 & 2 & 61 & 52 & 15 & 10 & 53 \\ 6 & 57 & 64 & 3 & 14 & 49 & 56 & 11 \\ 63 & 4 & 5 & 58 & 55 & 12 & 13 & 50 \\ 17 & 46 & 43 & 24 & 25 & 38 & 35 & 32 \\ 44 & 23 & 18 & 45 & 36 & 31 & 26 & 37 \\ 22 & 41 & 48 & 19 & 30 & 33 & 40 & 27 \\ 47 & 20 & 21 & 42 & 39 & 28 & 29 & 34 \end{pmatrix},$$

and says that $\mathbf{R}_8$ is "constructed from four 4 × 4 magic squares".

We find that $\mathbf{R}_8$ may be constructed from <u>two</u> 4 × 4 magic squares.

The classic fully-magic Nasik (pandiagonal) matrix with magic sum $m(\mathbf{R}_8) = 260$

$$\mathbf{R}_8 = \begin{pmatrix} 1 & 62 & 59 & 8 & 9 & 54 & 51 & 16 \\ 60 & 7 & 2 & 61 & 52 & 15 & 10 & 53 \\ 6 & 57 & 64 & 3 & 14 & 49 & 56 & 11 \\ 63 & 4 & 5 & 58 & 55 & 12 & 13 & 50 \\ 17 & 46 & 43 & 24 & 25 & 38 & 35 & 32 \\ 44 & 23 & 18 & 45 & 36 & 31 & 26 & 37 \\ 22 & 41 & 48 & 19 & 30 & 33 & 40 & 27 \\ 47 & 20 & 21 & 42 & 39 & 28 & 29 & 34 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{R}_8^{(11)} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{R}_8^{(12)} \\ \mathbf{0}_4 & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{0}_4 \\ \mathbf{R}_8^{(21)} & \mathbf{0}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{0}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{R}_8^{(22)} \end{pmatrix},$$

where $\mathbf{R}_8^{(11)}, \mathbf{R}_8^{(12)}, \mathbf{R}_8^{(21)}, \mathbf{R}_8^{(22)}$ are 4×4 fully-magic Nasik (pandiagonal) matrices each with magic sum $130 = \frac{1}{2} m(\mathbf{R}_8)$.

Moreover, the magic matrices $\mathbf{R}_8^{(11)}, \mathbf{R}_8^{(12)}, \mathbf{R}_8^{(21)}, \mathbf{R}_8^{(22)}$ are interchangeable and so there are $4! = 24$ fully-magic Nasik (pandiagonal) 8×8 matrices like $\mathbf{R}_8$.

Furthermore,

$$\mathbf{R}_8^{(12)} = \mathbf{R}_8^{(11)} + 8\mathbf{X}, \qquad \mathbf{R}_8^{(21)} = \mathbf{R}_8^{(11)} + 16\mathbf{X}, \qquad \mathbf{R}_8^{(22)} = \mathbf{R}_8^{(11)} + 24\mathbf{X},$$

where the fully-magic Nasik (pandiagonal) 4×4 matrices

$$\mathbf{R}_8^{(11)} = \begin{pmatrix} 1 & 62 & 59 & 8 \\ 60 & 7 & 2 & 61 \\ 6 & 57 & 64 & 3 \\ 63 & 4 & 5 & 58 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix},$$

with magic sums $m(\mathbf{R}_8^{(11)}) = \frac{1}{2} m(\mathbf{R}_8) = 130$ and $m(\mathbf{X}) = 0$. And so we may write Ramanujan's 8×8 magic matrix as the sum of two Kronecker products:

$$\mathbf{R}_8 = \begin{pmatrix} 1 & 62 & 59 & 8 & 9 & 54 & 51 & 16 \\ 60 & 7 & 2 & 61 & 52 & 15 & 10 & 53 \\ 6 & 57 & 64 & 3 & 14 & 49 & 56 & 11 \\ 63 & 4 & 5 & 58 & 55 & 12 & 13 & 50 \\ 17 & 46 & 43 & 24 & 25 & 38 & 35 & 32 \\ 44 & 23 & 18 & 45 & 36 & 31 & 26 & 37 \\ 22 & 41 & 48 & 19 & 30 & 33 & 40 & 27 \\ 47 & 20 & 21 & 42 & 39 & 28 & 29 & 34 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \mathbf{R}_8^{(11)} + 8 \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \otimes \mathbf{X}.$$

The 4×4 magic-basis matrix

$$\mathbf{B}_4 = \begin{pmatrix} -3 & 1 & 1 & -3 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{pmatrix} = -\mathbf{DX},$$

where

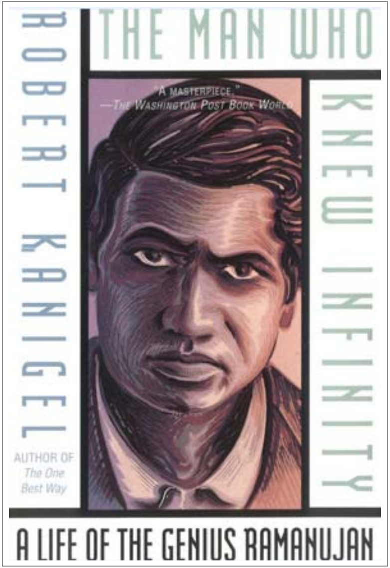
$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \text{while} \quad \mathbf{X} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

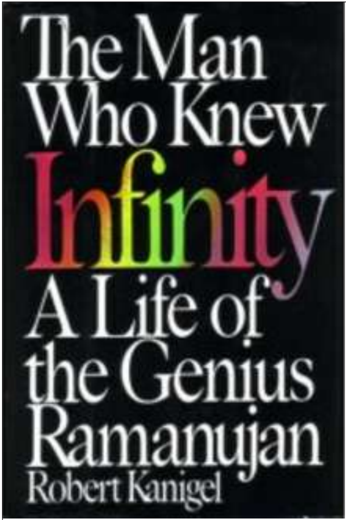
is the doubly-balanced 4×4 magic matrix used in our "two-Kronecker-products construction" of Ramanujan's 8×8 magic matrix $\mathbf{R}_8$. The 4×4 Agrippa–Cardano magic matrix

$$\mathbf{A}_4 = \tfrac{1}{2}\big(4\mathbf{B}_4 - \mathbf{B}_4' + (4^2+1)\mathbf{E}_4\big) = \begin{pmatrix} 4 & 14 & 15 & 1 \\ 9 & 7 & 6 & 12 \\ 5 & 11 & 10 & 8 \\ 16 & 2 & 3 & 13 \end{pmatrix}.$$

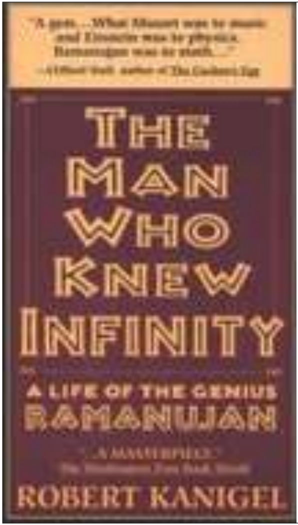
For a
"model of the biographer's art"
we recommend

The Man who Knew Infinity: A Life of the Genius Ramanujan,
by Robert Kanigel,
pub. Charles Scribner's Sons 1991;
Washington Square Press 1992



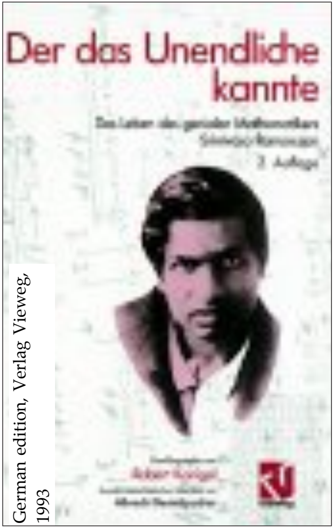


Original Scribners hardcover edition, 1991

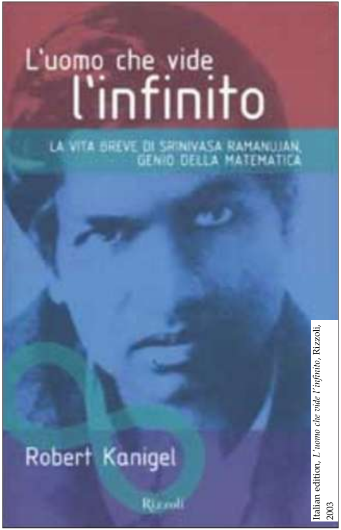


Original Washington Square Press paperback edition, 1992

1. Scribner's hardcover, 1991
2. U.K. hardcover, Scribner's, 1991
3. Washington Square Press paperback, 1992
4. U.K. paperback, Abacus, 1992
5. Indian edition, Rupa, 1992
6. German edition, Vieweg Verlag, 1993
7. Cassette book, National Library for the Blind, 1993
8. Japanese edition, Kousakusha, 1994
9. Korean edition, Science Books, 2000
10. Chinese editions, Shanghai Scientific, 2002, 2008
11. Italian edition, Rizzoli, 2003
12. Thai edition, Matichon, 2007
13. Audio edition, Blackstone Audio, 2007
14. Greek edition, Travlos, 2008



German edition, Verlag Vieweg, 1993



Italian edition, *L'uomo che vide l'infinito*, Rizzoli, 2003

In *The Man who Knew Infinity* we learn that (Prologue, pp. 1–2):

One day in the summer of 1913, a twenty-year-old Bengali from an old and prosperous Calcutta family stood in the chapel of King's College, Cambridge, England.

Prasantha Chandra Mahalanobis (1893–1972) was smitten.

Scarcely off the boat from India and planning to study in London,
Mahalanobis had come up to Cambridge
on the train for the day to sightsee.

The next day he met with the provost, and soon,
to his astonishment and delight, he was a student at
King's College, Cambridge.

He had been at Cambridge for about six months
when his mathematics tutor asked him,

"Have you met your wonderful countryman Ramanujan?"

He had not yet met him, but he had heard of him.

Soon Mahalanobis did meet Ramanujan, and the two became friends;
on Sunday mornings, after breakfast, they'd go for long walks,
talk about life, philosophy, mathematics.

Later, looking back, Mahalanobis would date the flowering of their friendship to one day in the fall following Ramanujan's arrival.

He'd gone to see him at his place in Whewell's Court.
Cambridge was deserted. And cold.

Are you warm at night? asked Mahalanobis,
seeing Ramanujan beside the fire.

No, replied the mathematician from always-warm Madras,
he slept with his overcoat on, wrapped in a shawl.

Until early 1914 Ramanujan lived in a traditional home on Sarangapani Street in Kumbakonam. The family home is now a museum.

From 1914–1919 Ramanujan lived in Whewell's Court, a 5-minute walk from Hardy's rooms. Whewell's Court was a 3-story stone warren of rooms laced with arched Gothic windows and pierced at intervals by staircases leading to rooms.

Figuring his friend hadn't enough blankets,
Mahalanobis stepped back into the little sleeping alcove
on the other side of the fireplace.

The bedspread was loose, as if Ramanujan had just gotten up.
Yet the blankets lay perfectly undisturbed,
tucked neatly under the mattress.

Yes, Ramanujan had enough blankets;
he just didn't know what to do with them.

Gently, patiently, Mahalanobis showed him how you peeled them back,
made a little hollow for yourself, slipped inside ...

For five years, walled off from India by World War One (1914–1918), Ramanujan would remain in strange, cold, distant England, fashioning, through 21 major papers, an enduring mathematical legacy.

Then, he would go home to India to a hero's welcome.

"Srinivasa Ramanujan", an Englishman would later say of him, "was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years."