DISTRIBUTION OF COEFFICIENTS OF RANK POLYNOMIALS FOR RANDOM SPARSE GRAPHS

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Abstract. We study the distribution of coefficients of rank polynomials of random sparse graphs. We first discuss the limiting distribution for general graph sequences that converge in the sense of Benjamini-Schramm. Then we compute the limiting distribution and Newton polygons of the coefficients of the rank polynomial of random d-regular graphs.

1. Introduction

Dichromatic polynomial is the most general graph invariant satisfying deletion-contraction properties. It contains important information about G, in particular about its connectivity properties, and about nowhere-zero flows on G. In Statistical Physics, it describes the partition function for the Potts model on G. When restricted to certain curves (or points), the dichromatic polynomial specializes to some well-known graph invariants, including chromatic polynomial, the number of spanning trees, the number of acyclic orientations etc. It is closely related to important invariants in knot theory, including the Jones polynomial. Equivalent forms of dichromatic polynomial include the Tutte and the rank polynomials of the graph. The Tutte polynomial is perhaps better known. (See [Wel] for an excellent survey on the properties of the Tutte polynomial.) In this paper, however, we consider rank polynomials (defined in 2.1), focusing on the coefficients, as it turns out that the behaviour of the coefficients of rank polynomial is more tangible.

We focus on the asymptotic properties of the rank polynomial, focusing primarily on the sparse graphs. The asymptotic behaviour of many graph invariants, including Laplace spectrum, cycle distribution, colouring properties, non-concentration of eigenvectors etc. has been studied extensively before. However, several asymptotic properties of the rank and Tutte polynomials have not been considered before, to our knowledge. In our paper, we focus on the *coefficients* of those polynomials. We first study *Newton polygons* for the rank polynomials for random regular graphs in 2.1. We determine them for d-edge connected d-regular graphs (Theorem 2.3), then describe them for general graphs.

Next, we define probability measures describing the concentration of the (normalized) coefficients of those polynomials. The *coefficient measure* for the rank polynomial is defined in 4.2.

In sections 4 and 5, we discuss the limiting distribution of the coefficients of the rank polynomial associated for Benjamini-Schramm convergent sequences of graphs and compute it exactly for random d-regular graphs.

Date: July 4, 2017.

D. J. was supported by NSERC, FQRNT and Peter Redpath fellowship. S. N. and L. T. were supported by NSERC.

The questions considered in this paper were suggested by numerical experiments in the paper [RT], where some a priori results on the coefficient measures for the Tutte polynomial were also established.

2. Rank polynomial

Let G be a graph without loops or multiple edges, with the vertex set $V = V(G)$ and the edge set $E = E(G)$. We let $|V(G)| = n$ and $|E(G)| = m$. We denote by $G|A$ the subgraph of G whose edge set is A and whose vertex set is $V(G)$, and we denote by $k(H)$ the number of connected components of a graph H.

The *rank polynomial* R_G of a graph G with n vertices and m edges, cf. [Big1, Definition 10.1] is defined by

$$R_G(x,y) := \sum_{A \subset E(G)} x^{r(G|A)} y^{s(G|A)}, \tag{2.1}$$

where $r(G|A) = n - k(G|A)$ denotes the *rank* of $G|A$, and $s(G|A) = |A| - n + k(G|A)$ denotes the *co-rank* of $G|A$.

It is easy to see that $R_G(x,y)$ is indeed a polynomial. Let $G_1, \dots, G_k$ be the connected components of $G|A$; we see that

$$|A| = \sum_{i=1}^{k} |E(G_i)| \geq \sum_{i=1}^{k} (|V(G_i)| - 1) = n - k(G|A),$$

therefore $s(G|A) \geq 0$. The above argument also shows that $s(G|A) = 0$ if and only if $G|A$ is a forest. Also, clearly $r(G|A) \geq 0$, and the equality holds if and only if $A = \emptyset$. Thus $A = \emptyset$ corresponds to the constant term 1 in $R_G(x,y)$.

Denote the coefficients of the rank polynomial by ρ_{rs}:

$$R_G(x,y) = \sum_{r,s} \rho_{rs} x^r y^s. \tag{2.2}$$

Thus ρ_{rs} denotes the number of subgraphs of G with rank r and co-rank s. Those coefficients are the entries of the *rank matrix* $\rho_{rs}(G)$ of the graph G, see [Big1, Ch. 10] or [G-R, Ch. 15].

2.1. Newton polygon of the rank polynomial.

In this section we consider the *Newton polygon* $\Pi(R_G)$ for the rank polynomial R_G. is the convex hull of the set of all lattice points $\{(r,s) \in \mathbb{Z}^2 : \rho_{rs} \neq 0\}$.

We start by making some simple observations about $\Pi(R_G)$. We first remark that all the points lying on the line segment $I_0(G) := \{(a,0) : 0 \leq a \leq m-1\}$ belong to $\Pi(R_G)$: They correspond to forest subgraphs of G with a edges. This segment coincides with $\Pi(R_G)$ if and only if G is a forest. From now on, we assume that G is connected and has at least one cycle; in that case $I_0(G)$ is a proper subset of $\Pi(R_G)$.

If the graph $G|A$ has one connected component and contains all the vertices of G, then $r(G|A) = n-1$; its co-rank varies between $s = 0$ (a spanning tree) and $s = m - n + 1$ ($A = E$); accordingly all the points $\{(n-1, b) : 0 \leq b \leq m - n + 1\}$ belong to $\Pi(R_G)$. The tangent of the angle $\alpha(m,n)$ that the line from $(0,0)$ to $(n-1, m-n+1)$ forms with the x-axis satisfies $\tan \alpha(m,n) = \frac{m-n+1}{n-1} = \frac{m}{n-1} - 1$.

Accordingly, if we denote by α_0 the angle formed by the sides of $\Pi(R_G)$ at the origin $(0,0)$, it will satisfy

$$\tan\alpha_0 = \sup_{A\subseteq E:(G|A)\ has\ cycles} \frac{|E(G|A)|}{|V(G|A)|-1} - 1.$$

In general, the supremum need not be attained for $A = E$.

Example 2.1. Consider a graph $G(n,s)$ consisting of complete graph K_{n+1} on $(n+1)$ vertices; and a path of length s starting at one of the vertices of K_{n+1}. If we take A to be all the edges of K_{n+1}, then $\tan\alpha(A) = \frac{(n+1)n}{2n} - 1 = \frac{n-1}{2}$. If we take $A = E$, then we get $\tan\alpha(E) = \frac{(n+1)n/2+s}{n+s} - 1 < \tan\alpha(A)$ by an easy calculation.

For regular graphs, $\Pi(R_G)$ also need not be a triangle.

Example 2.2. Consider a graph $G(k,q)$ constructed as follows: take $q \geq 2$ disjoint copies of the complete graph K_k; then connect those graphs by edge switching along a cycle of length q. Clearly, after removing $2 \leq p \leq q$ "cycle" edges from that graph, we will get a graph $G_p(k,q)$ with p connected components. It is easy to show that for $k \geq 5$, we shall have

$$\tan\alpha(G_p(k,q)) = \frac{qk(k-3)}{2(kq-p)} > \frac{qk(k-3)+2}{2(kq-1)} = \tan\alpha(G(k,q)).$$

It follows that $\Pi(R_{G(k,q)})$ is not a triangle.

The situation is simpler for d-edge connected d-regular graphs.

Theorem 2.3. *Let G be a d-edge connected d-regular graph $(d \geq 3)$ on n vertices. Then the Newton polygon $\Pi(R_G)$ of the rank polynomial R_G is the triangle $T(n,d)$ with the vertices at $(0,0),(0,n-1)$ and $(n-1,m-n+1)$.*

Proof: Let $A_n = (0,0), B_n = (0,n-1)$ and $C_n = (n-1,m-n+1)$ be the vertices of $T(n,d)$; since d is fixed, we suppress the dependence on d.

We have shown previously that the line segments $[A_n,B_n]$ and $[B_n,C_n]$ belong to the boundary of $\Pi(R_G)$; it remains to be shown that $\Pi(R_G) \subseteq T(n,d)$. Equivalently, all points in $\Pi(R_G)$ lie below the line segment $[A_n,C_n]$. The tangent of the angle $\alpha(G)$ formed by the sides C_n,A_n and B_n,A_n of T_n is equal to

$$\tan\alpha(G) = \frac{m-n+1}{n-1} = \frac{dn/2-n+1}{n-1}.$$

Let $A \subseteq E$; suppose the graph $G|A$ has connected components. The number of edges in $E \setminus A$ satisfies

$$|E \setminus A| \geq \frac{d\cdot k}{2}, \tag{2.3}$$

by the d-edge connectivity of G: for a fixed component C of $G|A$, there are at least d edges with exactly one endpoint in C.

The monomial corresponding to the graph $G|A$ corresponds to the point $C(A)$ with the coordinates $(n-k(A),|A|-n+k(A))$, hence the tangent of the angle $\alpha(A)$ formed by the lines A_nB_n and A_nC_n is given by

$$\tan\alpha(A) = \frac{|A|-n+k(A)}{n-k(A)}.$$

To prove the theorem, it suffices to show that $\tan\alpha(S) \leq \tan\alpha(G)$, which is equivalent to

$$(dn/2 - n + 1)(n - k(A)) \geq (|A| - n + k(A))(n-1).$$

After cancellations and multiplication by 2, we get $dn(n-k(A)) \geq 2(n-1)|A|$ or $dn^2 - 2(n-1)|A| \geq k(A)dn$. This can be rewritten as

$$I_n := dn + 2(n-1)\left[\frac{dn}{2} - k(A)\right] = dn + 2(n-1)|E \setminus A| \geq dn \cdot k(A).$$

Substituting (2.3), we find that the left-hand side I_n satisfies

$$I_n \geq dn + 2(n-1)\frac{d \cdot k(A)}{2}.$$

Accordingly, it suffices to show that

$$dn + (n-1)d \cdot k(A) \geq dn \cdot k(A).$$

But the latter is equivalent to

$$dn \geq d \cdot k(A),$$

which is clearly true. □

We remark that random d-regular graphs are a.a.s. d-edge connected [Bol].

Corollary 2.4. *The conclusion of Theorem 2.3 holds for random regular graphs with probability tending to* 1 *as* $n \to \infty$.

3. Newton polygon of the rank polynomial for general graphs

In this section we describe the Newton polygon of the rank polynomial for general graphs. The key relevant notion turns out to be the so-called *k-edge-connectivity*, defined in [BC] and also [Gol80, Gol81].

Let G be a connected graph with n vertices. For $k \leq n$, the *k-edge-connectivity* $\lambda_k(G)$ is defined to be the minimal number of edges that need to be removed from G to separate G into k connected components.

We remark that the notation in [BC] is slightly different from ours: λ_i in [BC] equals to our λ_{i+1}; our notation is consistent with the notation in [ZHLS].

Consider the vertical line $\{r = const\} \subset \mathbb{Z}^2$; we would like to determine the largest s such that the point (r,s) lies in $\Pi(R_G)$. Let $A \subset E(G)$, and let $x^{r(G|A)}y^{s(G|A)}$ be the corresponding monomial in R_G. We have $r(G|A) = n - k(G|A)$ (from 2.2), so fixing r is equivalent to fixing the number $k = k(G|A)$ of the connected components of the graph $(G|A)$. For a fixed r, we would like to find the largest possible $s = s(G|A)$. We remark that $r(G|A) + s(G|A) = |A|$, so finding the largest s is equivalent to finding the largest $|A|$ such that removing $E \setminus A$ from the graph G disconnects it into k connected components. By the definition of k-edge-connectivity, we have $m - |A| = \lambda_k(G)$ and therefore $|A| = m - \lambda_k(G)$.

We summarize the above discussion.

Theorem 3.1. *Let G be a connected graph with n vertices and m edges, and let $\Pi = \Pi(R_G)$ denote the Newton polygon of the rank polynomial R_G.*

(a) *The lower part of the boundary $\partial\Pi$ contains the horizontal line segment connecting the points $(0,0)$ and $(n-1,0)$;*
(b) *The right part of the boundary $\partial\Pi$ contains the vertical line segment connecting the points $(n-1,0)$ and $(n-1, m-n+1)$;*

(c) *The upper part of the boundary* $\partial\Pi$ *coincides with the ("upper") convex hull of the set of points*

$$\{(n-k, m-\lambda_k(G)) : 1 \le k \le n\},$$

where $\lambda_k(G)$ *denotes the* k*-edge-connectivity of* G*.*

It is clear from (2.1) that if the graph G_2 is obtained from the graph G_1 by adding several new edges ($G_1 \subseteq G_2$), then any monomial appearing in R_{G_1} also appears in R_{G_2}, since any subset A of $E(G_1)$ is also a subset of $E(G_2)$. It follows that $\Pi(R_{G_1}) \subseteq \Pi(R_{G_2})$. It was shown in [BC, Corollary 2] that $\lambda_k(K_n) = (k-1)(2n-k)/2$. Accordingly, for any graph G on n vertices, $\Pi(R_G)$ lies below the "upper" convex hull of the set

$$\{(n-k, n(n-1)/2 - (k-1)(2n-k)/2) : 1 \le k \le n\}.$$

Below, we summarize some of the known results about $\lambda_k(G)$. It was shown in [BC, Theorem 1] that for $1 \le k < j < n$, we have

$$\lambda_j \ge \frac{(j-1)(j-k+1)}{(j+1)(j-k-1)}\lambda_{j-k} \tag{3.1}$$

A special case is $\lambda_j \ge \lambda_{j-1} j(j-1)/[(j+1)(j-2)]$. Also, it was shown in [BC, Theorem 2] that if G has minimum degree δ, $1 \le k < n$ and $\lambda_2(G) \ge \lfloor n/k \rfloor$, then $\lambda_k(G) \ge \delta$. It was also shown in [BC, Corollary 4] that for a d-regular graph containing an i-clique K_i, for $d \ge in/(i+1)$ we have $\lambda_{i+1} = di - \frac{i(i-1)}{2}$. It was also shown in [ZHLS, Theorem 2.5] that for $n \ge l > 1$, we have

$$\lambda_l(G) \ge \frac{l \cdot \lambda_2(G)}{2}. \tag{3.2}$$

We next discuss the values of $\lambda_k(G)$ for small k and small $n-k$. Clearly, $\lambda_1(G) = 0$ for connected graphs; $\lambda_2(G)$ is the edge connectivity of G. Clearly, $\lambda_2(G) \le \delta(G)$, the minimal degree of G. Next, $\lambda_n(G) = m$ for connected graphs.

Let $A \subset E(G)$; the graph (G, A) has the same vertex set as G, and its edge set coincides with A; note that (G, A) is in general different from the graph $(G|A)$ For simple graphs G, if the set $A \subset E$ contains at least 2 edges, then (G, A) has at most $(n-2)$ components, so $\lambda_{n-1}(G) = m-1$, hence the point $(1,1) \in \Pi(R_G)$. Also, if G has girth 3 (contains triangles), then $\lambda_{n-2}(G) = m-3$, hence the point $(2,3) \in \Pi(R_G)$.

4. Benjamini-Schramm convergence and the coefficient measures.

Below, we define a probability measure that describes the *relative size* of the coefficients of the rank polynomial; we call it *the coefficient measure.* Recall that the coefficients ρ_{rs} of the rank polynomial R_G were defined in 2.2. The monomials of R_G are in bijection with subsets of $E(G)$, where $|E(G)| = m$, hence

$$\sum_{r,s} \rho_{rs} = 2^m. \tag{4.1}$$

Also, for any $A \subset E$ we have $r(G|A) \le n-1$, and $s(G|A) \le |A| \le m$. It seems natural to consider a probability measure associated to R_G defined as follows

$$\mu_R(G) \ := \ \frac{1}{2^m} \sum_{rs} \rho_G(r,s) \cdot \delta(r/n, s/m), \tag{4.2}$$

By the previous remarks, $\mu_R(G)$ is a probability measure supported in the triangle with vertices at $(0,0),(1,0)$ and $(0,1)$.

Consider a sequence G_i of graphs such that $|V(G_i)| \to \infty$ and $G_i \to \Gamma$ in the sense of Benjamini-Schramm. In this section we study convergence of the coefficient measures $\mu_R(G_i)$. Let us first recall the definition of Benjamini-Schramm convergence. For a graph G and rooted graph g, an integer r, let $\mathbb{P}(G,g,r)$ denote the probability that the ball of radius r centered at a uniformly random vertex of G is isomorphic to G. We say that a sequence of graphs G_i with $|V(G_i)| \to \infty$ and a uniformly bounded maximum degree is *Benjamini-Schramm convergent* if $\mathbb{P}(G_i,g,r)$ converges for all g and r. In the remainder of the paper we study the behavior of coefficients of rank polynomials of Benjamini-Schramm convergent sequences of graphs. First we show that the coefficient measures converge.

Theorem 4.1. *Let G_i be a Benjamini-Schramm convergent sequence of connected graphs. Then $\mu_R(G_i)$ converges to a point mass.*

Proof. Recall that we need to show almost sure convergence of $\frac{|V(G_i)-k(G_i|A_i)|}{|V(G_i)|}$ and of $\frac{|A_i|-|V(G_i)+k(G_i|A_i)|}{|E(G_i)|}$, where A_i is a subset of $E(G_i)$ chosen uniformly at random.

It is well known (see e.g. [,]) that the average degree of G_i converges to some constant d.

Next let us consider the convergence of $\frac{|A_i|}{|E(G_i)|}$. For each i, $|A_i|$ is a binomial random variable of mean $\frac{|E(G_i)|}{2}$ and variance $\frac{|E(G_i)|}{2}$. Thus $\frac{|A_i|}{|E(G_i)|}$ has mean $\frac{1}{2}$ and variance $\frac{1}{4|E(G_i)|}$. Since the G_i are connected, $|E(G_i)| \to \infty$. Thus the mean is constant and the variance tends to 0 and so we have almost sure convergence of $\frac{|A_i|}{|E(G_i)|}$ to $\frac{1}{2}$.

Finally, consider the limit of the quantity $\frac{k(G_i|A_i)}{|V(G_i)|}$. We again use the technique of showing that the expected value converges while the variance tends to 0. For a vertex v and a graph H, let $c(v,H)$ be 1 over the size of the component containing v in H. Then $k(H) = \sum_{v \in V(H)} c(v,H)$. For every positive integer R and a graph H define $k_R(H)$ to be the number of components of size less than R in H, and let $c_R(v,H) = c(v,H)\chi\left(c(v,H) > \frac{1}{R}\right)$. We have $k_R(H) = \sum_{v \in V(H)} c_R(v,H)$, and in particular

$$k(H) - \frac{|V(H)|}{R} \leq k_R(H) \leq k(H). \tag{4.3}$$

For a rooted graph g, let $c_{1/2}(g)$ be the random variable equal to $c_R(v,h)$, where h is the subgraph of g obtained by taking a subset of edges of g uniformly at random. If a vertex lies in a component of size less than R, all the information about the component is entirely contained in $B(v,R)$. Let S be the set of all rooted graphs of radius R. Thus clearly

$$\begin{aligned} &\mathbb{E}_{A_i}\left(\frac{k_R(G_i|A_i)}{|V(G_i)|}\right) \\ &\quad = \frac{1}{|V(G_i)|}\sum_{v \in V(V(G_i))} c_{1/2}(v, B(v,R)) \\ &\quad \& = \sum_{g \in S} \mathbb{P}_v(B(v,R) = g) c_{1/2}(g) \end{aligned} \tag{4.4}$$

It follows that $\mathbb{E}_{A_i}\left(\frac{k_R(G_i|A_i)}{|V(G_i)|}\right)$ converges to a limit α_R for every R as $i \to \infty$. Moreover from (4.3) we have

$$\left|\mathbb{E}_{A_i}\left(\frac{k_R(G_i|A_i)}{|V(G_i)|}\right) - \mathbb{E}_{A_i}\left(\frac{k(G_i|A_i)}{|V(G_i)|}\right)\right| \leq \frac{1}{R}.$$

It follows that there exists a limit $\lim_{R\to\infty}\alpha_R = \alpha$, and that $\mathbb{E}_{A_i}\left(\frac{k(G_i|A_i)}{|V(G_i)|}\right)$ converges to this limit, as desired.

We now need only show that $Var\left(\frac{k_R(G_i|A_i)}{|V(G_i)|}\right)$ tends to 0. Fix $\epsilon > 0$. We use the same c and c_R as above. However, since no confusion is likely to arise, we will omit the $G_i|A_i$. We will also simply write V for $V(G_i)$. Note that $c_R(v) \leq 1$ and $c(v) - c_R(v) < 1/R$ for all vertices v.

$$\begin{aligned}
\frac{Var\left(\sum_{v\in V} c(v)\right)}{|V(G)|^2} &= \frac{Var\left(\sum_{v\in V} c_R(v) + (c(v) - c_R(v))\right)}{|V(G)|^2} \\
&= \frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} + \frac{Var\left(\sum_{v\in V} c(v) - c_R(v)\right)}{|V(G)|^2} \\
&\quad + \frac{Cov\left(\sum_{v\in V} c_R(v), \sum_{v\in V} c_R(v) - c(v)\right)}{|V(G)|^2} \\
&\leq \frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} + \frac{E\left[\left(\sum_{v\in V} c(v) - c_R(v)\right)^2\right]}{|V(G)|^2} \\
&\quad + \frac{E\left[\sum_{v\in V} c_R(v) \cdot \sum_{v\in V} c_R(v) - c(v)\right]}{|V(G)|^2} \\
&\leq \frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} + \frac{|V(G)|^2/R^2}{|V(G)|^2} + \frac{|V(G)|^2/R}{|V(G)|^2} \\
&\leq \frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} + \frac{2}{R}
\end{aligned}$$

Choose R large enough that $\frac{2}{R} < \epsilon$.

The variable $c_R(v)$ is independent of $c_R(w)$ for all but $O(|V(G)|)$ pairs of vertices (v, w). This is because there are $O(|V(G)|)$ pairs of vertices within distance R of each other. The variance of $c_R(v)$ and the covariance of $(c_R(v), c_R(w))$ are both bounded above by 1. Thus

$$\frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} = \frac{O(|V(G)|)}{|V(G)|^2} = o(1)$$

and so $\frac{Var\left(\sum_{v\in V} c_R(v)\right)}{|V(G)|^2} \to 0$. Since $|V(G_i)| \to \infty$ we have

$$\lim_{i\to\infty} \frac{Var\left(\sum_{v\in V} c(v)\right)}{|V(G)|^2} < \epsilon$$

for all $\epsilon > 0$ and so $Var\left(\frac{k_R(G_i|A_i)}{|V(G_i)|}\right) \to 0$.

Thus the expectation converges and the variance goes to 0 and so $\frac{k_R(G_i|A_i)}{|V(G_i)|} \to \alpha$ almost surely.

Thus, putting all of this together, we have that $\frac{r(G_i|A_i)}{|V(G_i)|} \to 1-\alpha$ and $\frac{s(G_i|A_i)}{|E(G_i)|} \to \frac{1}{2} - \frac{1}{2d} + \frac{\alpha}{2d}$ almost surely. □

In general, it may not be easy to find α. However, section 5 does this in the case of random regular graphs.

5. Generating functions for the number of components in random subgraphs of random regular graphs

Let $G = (V, E)$ be a d-regular graph with $|V| = n$. Suppose we take a subgraph $F \subseteq E$ by independently and uniformly choosing each edge with probability $1/2$. Then, using generating functions we find a concentration in the limiting coefficients of the Tutte polynomial.

Lemma 5.1. *Let $G \in G(n, d)$ be random and let F be a random subset of G. As $n \to \infty$, the probability that a component chosen uniformly at random contains a cycle tends to 0.*

Proof. As $n \to \infty$, for $i \geq 3$, the number of i-cycles in a random $G \in G(n, d)$ is asymptotically Poisson with mean $(d-1)^i/(2i)$, [Bol, p. 56]. In particular, the mean converges to a finite limit as does the variance.

We claim that the mean number of components in F is at least $\frac{n2^{-d}}{d+1}$. Each vertex has probability 2^{-d} of not having any neighbours in F. Now construct an independent set in G as follows. We work with two sets: A the set of accepted vertices and P the set of potential vertices. Start with A empty and $P = V$. Now choose any vertex $v \in P$ and add it to A. Remove all the neighbours of v from P. Continue this procedure until P is empty. A cannot contain any vertices which are neighbours in G. Also, since every step removes at most $r + 1$ vertices from P while adding one to A, $|A| \geq \frac{n}{d+1}$. Now the number of these vertices which are alone in their components in F follows a binomial with probability 2^{-r}. This probability does not depend on n. Hence, by the strong law of large numbers, the number of vertices in A which are isolated in F converges almost surely to $2^{-r}|A|$. Furthermore, the number of components is bounded above by n. Thus, $k(F) = \Theta(n)$ almost surely.

We first consider components in F of size at most m for each m. These can only contain cycles of size up to m. The expected number of such cycles in G is

$$\sum_{i=3}^{m} \frac{(d-1)^i}{2i} \leq (d-1)^{m+1}$$

In particular, this expectation does not depend on n. Furthermore, since the distribution is asymptotically the sum of independent Poisson random variables, it is a Poisson.

Now let $n \to \infty$. Choose a vertex v_n uniformly at random in each graph. The property of v_n being in a component of size m can only depend on edges incident to less than a bounded number of vertices N_m (vertices in the component and their neighbours). Thus the probability that v_n lies in a component of size at most m stays constant when $n > N_m$. Then the probability that v_n is in a component of size at most m converges to a positive limit p_m. Hence the number of such components is $\Theta(n)$. Hence the probability that any given component of size at most m in a graph of size n contains a cycle is $\Theta(1/n)$.

Since all components in any finite graph must have finite size, $\sum_{m\geq 1} p_m = 1$. Now let $\epsilon > 0$. Choose M large enough that $\sum_{m\geq M} p_m < \epsilon/2$. Now the probability that a given component of size at most M contains a cycle is $\Theta(1/n)$. Thus it tends to 0. Choose N large enough that for $n \geq N$, it is at most $\epsilon/2$. Thus the probability of a randomly selected component containing a cycle is less than ϵ.

Thus the probability that a randomly selected component contains a cycle tends to 0 as $n \to \infty$. □

Lemma 5.2. *Fix an integer radius R. Let $G_n \in G(n, r)$ be chosen uniformly at random. Let v_n be chosen uniformly at random from the vertices of G_n. Then as $n \to \infty$, the probability that there is a cycle contained in $B_{G_n}(v_n, R)$ tends to 0.*

Proof. First of all, we know that $|B_{G_n}(v_n, R)| \leq \sum_{d=1}^{R} d^d \leq d^{R+1}$ since this would be the size of a tree with this degree rooted at v_n. Hence $B_{G_n}(v_n, R)$ cannot contain cycles of length more than d^{R+1}. Hence we may ignore such long cycles.

The number of cycles of a given length in a random regular graph of size n is asymptotically Poisson with finite mean and variance. Let $C_{n,l}$ be the number of cycles of length at most l in $G(n, d)$. The sum of independent Poisson random variables is Poisson so $C_{n,l}$ is asymptotically Poisson with finite mean and variance. Then, for any $\epsilon > 0$, there is an M such that $\lim_{n\to\infty} P(C_{n,l} > M) < \epsilon$.

Fix $\epsilon > 0$, set $l = d^{R+1}$ and choose M large enough that $\lim_{n\to\infty} P(C_{n,l} > M) < \epsilon$. Let A_n be the event that $B_{G_n}(v_n, R)$ contains a cycle.

$$\begin{aligned}\lim_{n\to\infty} P(A_n) &= \lim_{n\to\infty} P(A_n \cap \{C_{n,l} > M\}) + \lim_{n\to\infty} P(A_n \cap \{C_{n,l} \leq M\}) \\ &< \lim_{n\to\infty} P(C_{n,l} > M) + \lim_{n\to\infty} P(A_n \cap \{C_{n,l} \leq M\}) \\ &< \epsilon + \lim_{n\to\infty} P(A_n \cap \{C_{n,l} \leq M\})\end{aligned}$$

If $C_{n,l} \leq M$, then there are at most lM vertices in cycles shorter than l. If $B_{G_n}(v_n, R)$ contains an entire cycle, then it must contain an element of a cycle of length at most l. The probability that a given vertex is an element of such a cycle if $C_{n,l} \leq M$ is at most $\frac{Ml}{n}$. Hence $P(A_n \cap \{C_{n,l} \leq M\}) \leq \frac{Ml^2}{n}$. This tends to 0 as $n \to \infty$. Thus, for all $\epsilon > 0$,

$$\lim_{n\to\infty} P(A_n) < \epsilon$$

and so $P(A_n) \to 0$ which is want. □

This in fact simply states that a sequence of random regular graphs of increasing size will almost surely converge to a tree in the sense of Benjamini-Schramm. Thus we may apply Theorem 4.1. Clearly, the average degree will be the constant degree d. The tricky part is to find α. For this we use 4.4. The limiting probability of having a tree neighbourhood is 1 for any radius so it suffices to find the expected inverse of component size in a tree. For convenience, set $k = d - 1$.

Definition 5.3. Let v be a vertex in a k-ary tree whose root may have $k+1$ children. Randomly select a subset of the edges of the tree by randomly and independently including each edge with probability 1/2.

Let $g(x) = \sum_{j=1}^{\infty} a_j x^j$ be the function such that for small j, a_j is the probability that the component of a vertex v has size j.

Let f be the analogous generating function in the case where v is the root and it only has k children.

We obtain a recurrence relation for f by adding a layer at the root. The new root can have from 0 to k children. Each of these will have a certain number of descendants with probabilities given by coefficients of f. Furthermore, we have added one vertex to our tree. Thus we have

$$f(x) = x\frac{(1+f(x))^k}{2^k}$$

In fact the root of the tree may have as many as $k+1$ children. Thus

$$g(x) = x\frac{(1+f(x))^{k+1}}{2^{k+1}} = \frac{1}{2}(1+f(x))f(x)$$

Lemma 5.4. *The expected value of 1 over the size of a component in such a random subset of a k-ary tree whose root may have up to $k+1$ children is $\frac{1}{2}\left(f(1) - \frac{k-1}{2}f(1)^2\right)$.*

When $k = 2$, $f(1) = 1$ and when $k \geq 3$, $f(1)$ is the solution to $2^k p = (1+p)^k$ in the interval $(0,1)$.

Proof. If g has power series $\sum_{j=1}^{\infty} a_j x^j$, then the expected number of components is $n\sum_{j=1}^{\infty} \frac{1}{j}a_j$. Multiplying the terms by x^j yields the power series for

$$K(x) = \int_0^x \frac{g(t)}{t}dt \tag{5.1}$$

To compute this, implicitly differentiate the recurrence relation for f and simplify by the same relation.

$$\begin{aligned}
f'(x) &= \frac{(1+f(x))^k}{2^k} + kxf'(x)\frac{(1+f(x))^{k-1}}{2^k} \\
f'(x) &= \frac{f(x)}{x} + kf'(x)\frac{f(x)}{1+f(x)} \\
f'(x)\left(1 - k\frac{f(x)}{1+f(x)}\right) &= \frac{f(x)}{x} \\
f'(x)\left(1+f(x)-kf(x)\right) &= \frac{(1+f(x))f(x)}{x} \\
\frac{d}{dx}\left(f(x) - \frac{k-1}{2}(f(x))^2\right) &= 2\frac{g(x)}{x}
\end{aligned}$$

Thus, by the fundamental theorem of calculus, and since $f(0) = 0$,

$$K(x) = \frac{1}{2}\left(f(x) - \frac{k-1}{2}(f(x))^2\right) \tag{5.2}$$

Thus the expected number of components is

$$\frac{1}{2}\left(f(1) - \frac{k-1}{2}f(1)^2\right)$$

To give us an idea, we now compute $p = f(1)$. This is the total probability that a random k-ary tree will have finite size. Thus it lies in the interval $[0,1]$. It solves the equation $2^k p = (1+p)^k$. For $k = 2$, the only solution is $p = 1$. For $r \geq 3$, there are two solutions in the interval. Now we know that all the coefficients of f are non-negative. Thus f is increasing. Also $f(0) = 0$ and f is continuous. Thus, we must have $f(1)$ being the smaller of the two roots by the intermediate value theorem. This incidentally means that, with positive probability, the random tree is infinite when $k \geq 3$. □

Recall that the degree of the graph is $k+1$. The generating functions $f(x)$ and $g(x)$ were defined in 5.3.

Definition 5.5. We thus have $\alpha = \alpha(k+1)$ given by the formula $\alpha = K(1)$, where the function $K(x)$ was defined in 5.1 and computed in 5.2.

The values of α are given in a table below:

k	p	α
2	1	$\frac{1}{4}$
3	$\sqrt{5}-2$	$\frac{5\sqrt{5}-11}{2}$
4	≈ 0.087378	≈ 0.03796

We may then apply the proof of Theorem 4.1 to obtain convergence to a specific delta function.

Corollary 5.6. *Let $G \in G(n,d)$. Then the coefficient measures μ_R defined in 4.2 converge weakly as $n \to \infty$ to the δ-measure*

$$\delta(d_1, 2d_2/d) = \delta\left(1-\alpha, \frac{2(\alpha-1)}{d} + \frac{1}{2}\right),$$

where $\alpha = \alpha(d), d = k+1$ was defined in 5.5.

6. Further questions

The questions considered in this paper can be formulated for different classes of graphs. It seems interesting to consider planar graphs; for example, every knot can be represented using its planar projections, giving rise to 4-regular planar graphs. However, regular planar graphs are quite different from random regular graphs; in particular, as $|E(G)| < 3|V(G)|$ for every non-null planar graph G d-regular planar graphs only exist for $d \leq 5$. Also, random 3-regular planar graphs are typically *not* 3-connected. The probability that a random 3-regular planar graph is 3-connected is exponentially small in the number of vertices; we refer to [Kan, Thm 6.4.1] for precise asymptotics of that probability. Similar results hold for $d > 3$. Accordingly, Theorem 2.3 does not apply for random planar regular graphs. Also, the number of spanning trees in random regular planar graphs grows slower [JR, Ly] than in general regular graphs [McK]. It would be interesting to study the limiting shapes of the Newton polygons, and the limiting distribution of the coefficient measures for random planar regular graphs (and more generally for random planar graphs of bounded degree).

It seems interesting and challenging to extend our results to the Tutte polynomial. (The Tutte polynomial of graph G can be defined from the rank polynomial by

$$T_G(x,y) = (x-1)^{-n+1} R_G(x-1, y-1).$$

It also seems very interesting to explore in more detail the restrictions of the 2-variable polynomials considered in this paper to some specific curves; and to study the distribution of the corresponding zeros, e.g. of the chromatic polynomials, or of Alexander polynomial of a random knot, considered in [Riv16]. We remark that the *expected value* of T_G for subgraphs obtained by randomly deleting edges from G were considered in [Wel, Thm 6.3].

It seems interesting to study the limiting distribution of zeros of R_G (or, equivalently, T_G), considered as subsets of $\mathbb{R}^2$ and $\mathbb{C}^2$.

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