PFAFFIAN GRAPHS, T-JOINS AND CROSSING NUMBERS

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Abstract. We characterize Pfaffian graphs in terms of their drawings in the plane. We generalize the techniques used in the proof of this characterization, and prove a theorem about the numbers of crossings in T-joins in different drawings of a fixed graph. As a corollary we give a new proof of a theorem of Kleitman on the parity of crossings in drawings of K_{2j+1} and $K_{2j+1,2k+1}$.

1. Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. For a graph G we denote its vertex set by $V(G)$ and its edge set by $E(G)$. If u and v are vertices in a graph G, then uv denotes the edge joining u and v. A *perfect matching* is a set of edges in a graph that covers each vertex exactly once. For sets X and Y we denote their symmetric difference by $X \triangle Y$.

In a directed graph we denote by uv an edge directed from u to v. A *labeled* graph (resp. digraph) is a graph (resp. digraph) with vertex set $\{1, 2, \ldots, n\}$ for some n. Let D be a labeled directed graph and let $M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ be a perfect matching of D. Define $\mathrm{sgn}_D(M)$, the *sign* of M, to be the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_n & v_n \end{pmatrix}.$$

Note that the sign of a perfect matching is well-defined as it does not depend on the order in which the edges of M are listed. We say that an orientation D of a labeled graph G is *Pfaffian* if the signs of all perfect matchings in D are positive. It is well-known and easy to verify that the existence of a Pfaffian orientation does not depend on the numbering of $V(G)$. Thus we say that a graph with an arbitrary vertex-set is *Pfaffian* if it is isomorphic to a labeled graph that admits a Pfaffian orientation. Pfaffian orientations have been introduced by Kasteleyn [5, 6, 7], who demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time, and

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has received considerable attention since then. We refer to [16] for a recent survey.

In [7] Kasteleyn proved the following theorem.

Theorem 1.1. *Every planar graph is Pfaffian.*

In this paper we characterize Pfaffian graphs in terms of their drawings in the plane. By *a drawing* Γ of a graph G we mean an immersion of G in the plane such that edges are represented by homeomorphic images of $[0,1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn in the plane let $cr(e,f)$ denote the number of times the edges e and f cross. For a set $J \subseteq E(G)$ let $cr(J,\Gamma)$, or $cr(J)$ if the drawing is understood from context, denote $\sum cr(e,f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in J$.

Theorem 1.2. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

The "if" part of this theorem was known to Kasteleyn [7] and was proved by Tesler [15]; however our proof of this part is different.

We prove Theorem 1.2 in Section 2. A preliminary version of the results in that section has previously appeared in [12]. In the following sections we generalize the techniques used in the proof of Theorem 1.2. In Section 3 we prove a technical theorem about the numbers of crossings in T-joins in different drawings of a fixed graph, generalizing one of the lemmas from Section 2. In Section 4 we apply the results of Section 3 to give a new proof of a result of Kleitman on the parity of the number of crossings in a graph. A well-known theorem of Hanani and Tutte follows as a corollary.

2. Drawing Pfaffian graphs

In this section we derive Theorem 1.2 from a more general result. To state it we need a definition. Let Γ be a drawing of a graph G in the plane. We say that $S \subseteq E(G)$ is a *marking* of Γ if $cr(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G.

Theorem 2.1. *For a graph G the following are equivalent:*
(a) G is Pfaffian;
(b) some drawing of G in the plane has a marking;
(c) every drawing of G in the plane has a marking;
(d) there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.

We say that Γ is *a standard drawing* of a labeled graph G if the vertices of Γ are arranged on a circle in order and every edge of Γ is drawn as a straight line.

Theorem 2.1 immediately follows from the next two lemmas.

Lemma 2.2. *There exists a one-to-one correspondence between Pfaffian orientations of a labeled graph G and markings of its standard drawing Γ.*

Proof. Let D be an orientation of G. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. The sign of M is the sign of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Let $i(P)$ denote the number of inversions in P. We have

$$sgn_D(M) = sgn(P) = (-1)^{i(P)} = \prod_{1\le i<j\le 2k} sgn(P(j) - P(i)) =$$

$$= \prod_{1\le i<j\le k} sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i))\times$$

$$\times \prod_{1\le i\le k} sgn(v_i - u_i). \tag{1}$$

In Γ edges u_iv_i and u_jv_j cross if and only if, in the circle containing the vertices of Γ, each of the two arcs with ends u_i and v_i contains one of the vertices u_j and v_j, in other words if and only if

$$sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) = -1.$$

Define $S_D = \{uv \in E(D) | u > v\}$. From (1) we deduce that

$$sgn(M) = (-1)^{cr(M)} \times (-1)^{|M\cap S_D|}.$$

Therefore M has a positive sign if and only if $cr(M)$ and $|M \cap S_D|$ have the same parity. It follows that D is a Pfaffian orientation of G if and only if S_D is a marking of the standard drawing of G. □

Lemma 2.3. *Let Γ_1 and Γ_2 be two drawings of a labeled graph G in the plane. Then Γ_1 has a marking if and only if Γ_2 has one. If some drawing of a labeled graph G in the plane has a marking then there exists another drawing of G in the plane that has an empty set as a marking.*

Proof. One can derive the lemma from a more general Theorem 3.1, and we will do so in in Section 3. In fact, Theorem 3.1 can be considered as a generalization of this lemma.

Here we would like to present a simple, albeit informal, argument. We may assume without loss of generality that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2. We transform the

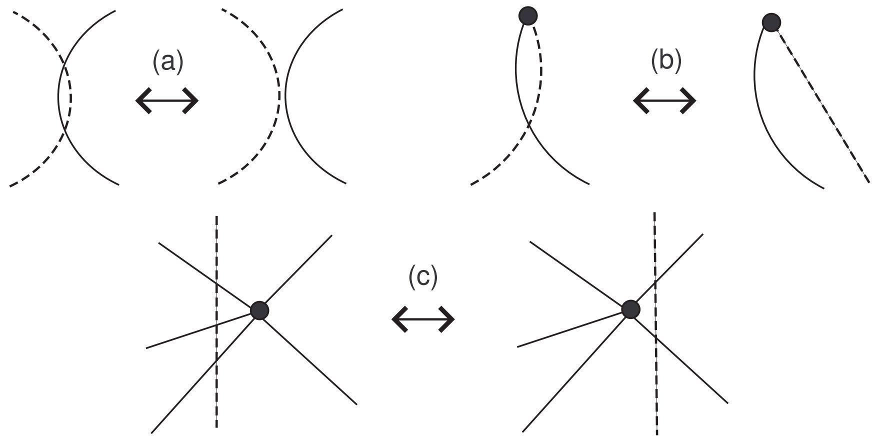

FIGURE 1. Changing the drawing.

drawing Γ_1 into the drawing Γ_2 by smoothly changing the images of edges, one edge at a time. We consider changes in the number of crossings between edges. One can classify events that cause these changes into three types (see Figure 1). We show that none of these events affects the existence of a marking.

In the event of type (a) and (b) the parity of the number of crossings between any two non-adjacent edges remains unchanged. In the event of type (c) the image of an edge e passes through an image of a vertex v, such that e is not incident to v. The number of crossings in any perfect matching containing e changes by one. Therefore one can replace a marking S of a drawing prior to this event by a marking $S \triangle \{e\}$ of a drawing after the event.

The argument above also shows how to obtain a drawing with an empty set as a marking from a drawing with an arbitrary marking. One has to transform every edge belonging to the marking so that this edge passes through a single vertex in the course of transformation. $\square$

3. A THEOREM ABOUT DRAWINGS OF T-JOINS

A pair (G, T) consisting of a graph G and a set $T \subseteq V(G)$ of even cardinality is called *a graft*. *A T-join* is a subset $J \subseteq E(G)$ such that every vertex $v \in V(G)$ is incident with an odd number of edges in J if and only if $v \in T$.

T-joins were first introduced in relation to the Chinese Postman problem, which can be reformulated as follows: find the minimum set of edges in a graph whose doubling results in an Eulerian graph. Note that such set of edges is a T-join, where T is the set of all vertices of odd degree. Perfect matching are other example of a T-join, where $T = V(G)$. Since their introduction T-joins have been extensively studied (see for example [14], sections 6.5 and 6.6 of [10], [3], section 2 of [2]).

We say that an unordered pair $\{e, f\}$ of adjacent edges in G is *an angle*. We denote the set of all edges and angles in a graph G by $Æ(G)$. If $J \subseteq E(G)$ we say that $e \in E(G)$ *lies* in J if $e \in J$, and we say that an angle $\{e, f\}$ *lies* in J if $e, f \in J$. For $J \subseteq E(G)$ and $S \subseteq Æ(G)$ we denote by $J \sqcap S$ the set of elements of S which lie in J.

The following theorem is the main result of this section. While the theorem itself is rather technical, it has a number of interesting applications. Throughout this section all integer identities are modulo 2.

Theorem 3.1. *Let (G, T) be a graft and let Γ_1 and Γ_2 be two drawings of G in the plane. Then there exists $S = S(T, \Gamma_1, \Gamma_2) \subseteq Æ(G)$ such that for every T-join $J \subseteq E(G)$ the following identity holds modulo 2*

$$cr(J, \Gamma_1) = cr(J, \Gamma_2) + |J \sqcap S|. \tag{2}$$

Proof. For any n and any two sequences $(a_1, a_2, .., a_n)$ and $(b_1, b_2, .., b_n)$ of pairwise distinct points in the plane, there clearly exists a homeomorphism of the plane that takes a_i to b_i for all $1 \leq i \leq n$. See for example [11, Chapter 13, Theorem 7] for a more general result. Therefore without loss of generality we assume that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2.

We say that the drawings Γ_1 and Γ_2 are *adjacent* if they differ only in the position of a single edge $e = u_1u_2$. We start by proving Theorem 3.1 for adjacent drawings.

Let e_1 and e_2 denote the images of e in Γ_1 and Γ_2 correspondingly. By changing these images within the regions of $\Gamma_1 \setminus e_1$ we can assume that e_1 and e_2 have finitely many intersections and each intersection is a crossing. Define $C = e_1 \cup e_2$. The closed curve C separates its complement into two sets P_1 and P_2 with the property that every simple curve with ends $a \in P_i$ and $b \in P_j$ crosses C an even number of times if and only if $i = j$.

For $x \in (V(G) \cup E(G)) \setminus \{e\}$ we will not distinguish between x and its representation in Γ_1 and Γ_2. Define F_i to be the set of all edges $f \in E(G) \setminus \{e\}$ such that f is adjacent to u_j for some $j \in \{1, 2\}$ and $f \cap U \subseteq P_i \cup \{u_j\}$ for some open set $U \ni u_j$ in the plane. Define

$$S = \{\{e, f\} | f \in F_1\}$$

if $|T \cap P_1|$ is even, and

$$S = \{\{e, f\} | f \in F_1\} \cup \{e\}$$

if $|T \cap P_1|$ is odd.

If $e \notin J$ then $cr(J,\Gamma_1) = cr(J,\Gamma_2)$ and (2) trivially holds, so we assume $e \in J$. We have

$$cr(J,\Gamma_1) + cr(J,\Gamma_2) = 2 \sum_{\{f,g\} \subseteq J \setminus \{e\}} cr(f,g) + \sum_{f \in J \setminus \{e\}} (cr(f,e_1) + cr(f,e_2)) = \sum_{f \in J \setminus \{e\}} cr(f,C)$$

Therefore it suffices to prove that

$$|J \sqcap S| = \sum_{f \in J \setminus \{e\}} cr(f,C),$$

or equivalently that

$$|J \cap F_1| + |T \cap P_1| = \sum_{f \in J \setminus \{e\}} cr(f,C). \tag{3}$$

From the definition of T-join we can deduce that for any $X \subseteq V(G)$

$$|T \cap X| = |\{uv \in J | u \in X, v \notin X\}|.$$

In particular

$$|T \cap P_1| = |\{uv \in J | u \in P_1, v \notin P_1\}| = |\{uv \in J | u \in P_1, v \in P_2\}| + \\ + |\{uv \in J | u \in P_1, v \in \{u_1,u_2\}\}|. \tag{4}$$

Let $J_1 = \{uv \in J \cap F_2 | u \in P_1\}$ and $J_2 = \{uv \in J \cap F_1 | u \in P_2\}$. Note that

$$(J \cap F_1) \triangle \{uv \in J | u \in P_1, v \in \{u_1,u_2\}\} = J_1 \cup J_2,$$

as

$$J \cap F_1 = \{uv \in J \cap F_1 | u \in P_1\} \cup J_2,$$
$$\{uv \in J | u \in P_1, v \in \{u_1,u_2\}\} = \{uv \in J \cap F_1 | u \in P_1\} \cup J_1,$$

and J_1, J_2 and $\{uv \in J \cap F_1 | u \in P_1\}$ are disjoint. Therefore

$$|J \cap F_1| + |\{uv \in J | u \in P_1, v \in \{u_1,u_2\}\}| = |J_1 \cup J_2|. \tag{5}$$

Let $J_3 = \{uv \in J | u \in P_1, v \in P_2\}$. The sets J_1, J_2 and J_3 are pairwise disjoint. From (4) and (5) we have

$$|J \cap F_1| + |T \cap P_1| = |J_1 \cup J_2 \cup J_3|. \tag{6}$$

But $J_1 \cup J_2 \cup J_3$ is exactly the set of those edges $f \in J \setminus \{e\}$ which cross C an odd number of times. Therefore (3) follows from (6) and the proof of Theorem 3.1 for adjacent drawings is complete.

For two arbitrary drawings Γ_1 and Γ_2 of G there always exist an integer n and a sequence of drawings $\Gamma_1 = \Gamma'_1, \Gamma'_2, \ldots, \Gamma'_n = \Gamma_2$ of G such that Γ'_i is adjacent to Γ'_{i+1} for all $i \in \{1,2,\ldots,n-1\}$. We have proved that there exist sets $S_i \subseteq Æ(G)$ for all $i \in \{1,2,\ldots,n-1\}$ such that

$$cr(J,\Gamma'_i) = cr(J,\Gamma'_{i+1}) + |J \sqcap S_i| \tag{7}$$

for all T-joins J. Let $S = S_1 \triangle S_2 \triangle \ldots \triangle S_{n-1}$. Summing up (7) over all $i \in \{1, 2, \ldots, n-1\}$ we get (2), thereby completing the proof of Theorem 3.1 for arbitrary drawings. □

We now derive Lemma 2.3 from Theorem 3.1.

Proof of Lemma 2.3. Perfect matchings are T-joins in the graft $(G, V(G))$. Therefore by Theorem 3.1 there exists $S \subseteq Æ(G)$ such that for every perfect matching M of G we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \sqcap S|.$$

Let $S' = S \cap E(G)$. As no perfect matching contains an angle we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \cap S'|$$

for every perfect matching M of G. Let S_1 be a marking of Γ_1. Then

$$cr(M, \Gamma_2) = cr(M, \Gamma_1) - |M \cap S'| = |M \cap S_1| - |M \cap S'| = |M \cap (S' \triangle S_1)|$$

for every perfect matching M of G. Therefore $S' \triangle S_1$ is a marking of Γ_2.

It remains to show that if some drawing of a labeled graph G in the plane has a marking then there exists another drawing of G in the plane that has an empty set as a marking. Consider a drawing of G in the plane with a marking S. Suppose there exists $e \in S$. We change the way e is drawn, so that the closed curve C which is composed from the old and the new drawing of e separates one vertex of G from the rest. From the proof of Theorem 3.1 it follows that $S \setminus \{e\}$ is a marking in the new drawing. By repeating the procedure we produce a drawing of G such that the empty set is a marking. □

4. Parity of the number of crossings

In this section we demonstrate an application of Theorem 3.1 to the theory of crossing numbers. We say that a set $\mathcal{J}$ of T-joins in a graft (G, T) is *nice* if every $x \in Æ(G)$ lies in an even number of elements of $\mathcal{J}$.

Lemma 4.1. *Let $\mathcal{J}$ be a nice set of T-joins in a graft (G, T). Then the parity of*

$$\sum_{J \in \mathcal{J}} cr(J, \Gamma) \tag{8}$$

is independent of the choice of a drawing Γ of G in the plane.

Proof. By Theorem 3.1 it suffices to prove that

$$\sum_{J \in \mathcal{J}} |J \sqcap S|$$

is even for any $S \subseteq Æ(G)$. This is true by the definition of a nice set of T-joins. $\square$

We derive the next theorem from Lemma 4.1.

Theorem 4.2 (Kleitman [8])**.** *Let $G = K_{2j+1}$ or $G = K_{2j+1,2k+1}$ for some positive integers j and k. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of a drawing of G in the plane.*

Proof. By Lemma 4.1 it suffices to find $T \subseteq V(G)$ and a nice set $\mathcal{J}$ of T-joins such that

$$|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$$

is odd for every two non-adjacent edges e, f of G. (By the definition of a nice set, $|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$ is even for every angle $\{e, f\}$.)

For $G = K_{2j+1,2k+1}$ we choose $T = \emptyset$ and we choose $\mathcal{J}$ to be the set of all cycles of length 4 in G. Every edge belongs to $4jk$ elements of $\mathcal{J}$, every angle belongs to either $2j$ or $2k$ of such elements, and every pair of non-adjacent edges belongs to a unique element of $\mathcal{J}$.

For $G = K_{2j+1}$ the construction is slightly more complicated. Choose $v \in V(G)$ and let $T = V(G) \setminus \{v\}$. Let $\mathcal{J}_1$ be the set of all perfect matchings of $G \setminus \{v\}$. For distinct vertices $u_1, u_2 \in T$ let

$$J_{u_1 u_2} = \{vw | w \in T \setminus \{u_1, u_2\}\} \cup \{u_1 u_2\}$$

and let $\mathcal{J}_2 = \{J_{u_1 u_2} | \{u_1, u_2\} \subseteq T, u_1 \neq u_2\}$. Let $J_3 = \{vw | w \in T\}$. Finally, if j is odd let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ and if j is even let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \{J_3\}$.

Again $\mathcal{J}$ is as required. The number of perfect matchings in every complete graph on an even number of vertices is odd. Therefore every edge not incident to v belongs to an odd number of elements of $\mathcal{J}_1$, to a unique element of $\mathcal{J}_2$ and does not belong to J_3. Every edge incident to v belongs to no element of $\mathcal{J}_1$, to $(j-1)(2j-1)$ elements of $\mathcal{J}_2$ and belongs to J_3. The only angles belonging to elements of $\mathcal{J}$ consist of two edges incident to v and each such angle belongs to J_3 and to $(j-1)(2j-3)$ elements of $\mathcal{J}_2$. It follows that $\mathcal{J}$ is nice. It remains to consider pairs of non-adjacent edges $e, f \in E(G)$. If neither e nor f is incident to v them $\{e, f\}$ belongs to an odd number of elements of $\mathcal{J}_1$ and to no other element of $\mathcal{J}$. If on the other hand e is incident to v then $\{e, f\}$ belongs to a single element of $\mathcal{J}_2$ and belongs to no other element of $\mathcal{J}$. $\square$

Kuratowki's theorem states that every non-planar graph has a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. One can therefore easily deduce the following well-known theorem from Theorem 4.2 and Kuratowski's theorem. This observation most likely is not new.

Theorem 4.3 (Hanani [4],Tutte [17])**.** *Let Γ be a drawing of a non-planar graph G in the plane. Then there exist distinct non-adjacent edges $e, f \in E(G)$ such that $cr(e, f)$ is odd.*

Further applications of Theorem 3.1 are considered in [13] .

Finally, let us note that parities of crossings have been studied in other contexts [1, 9, 17]. It might be interesting to analyze similarities and differences between methods employed in these papers and here.

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References

1. G. Cairns and Y. Nikolayevsky, *Bounds for generalized thrackles*, Discrete Comput. Geom. **23** (2000), no. 2, 191–206.
2. Gérard Cornuéjols, *Combinatorial optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 74, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001, Packing and covering.
3. Anna Galluccio and Martin Loebl, *On the theory of Pfaffian orientations. II. T-joins, k-cuts, and duality of enumeration*, Electron. J. Combin. **6** (1999), Research Paper 7, 10 pp. (electronic).
4. H. Hanani, *Überweentlich unplättbare kurven in dreidimensionalen raume*, Fundamenta Mathematicae **23** (1934), 135–142.
5. P. W. Kasteleyn, *The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice*, Physica **21** (1961), 1209–1225.
6. ______, *Dimer statistics and phase transitions*, J. Mathematical Phys. **4** (1963), 287–293.
7. ______, *Graph theory and crystal physics*, Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 43–110.
8. D. J. Kleitman, *A note on the parity of the number of crossings of a graph*, J. Combinatorial Theory Ser. B **21** (1976), no. 1, 88–89.
9. L. Lovász, J. Pach, and M. Szegedy, *On Conway's thrackle conjecture*, Discrete Comput. Geom. **18** (1997), no. 4, 369–376.
10. L. Lovász and M. D. Plummer, *Matching theory*, North-Holland Mathematics Studies, vol. 121, North-Holland Publishing Co., Amsterdam, 1986, Annals of Discrete Mathematics, 29.
11. Edwin E. Moise, *Geometric topology in dimensions* 2 *and* 3, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, Vol. 47.
12. Serguei Norine, *Drawing Pfaffian graphs.*, Graph Drawing (János Pach, ed.), Lecture Notes in Computer Science, vol. 3383, Springer, 2004, pp. 371–376.
13. Serguei Norine, *Matching structure and Pfaffian orientations of graphs*, Ph.D. thesis, Georgia Institute of Technology, 2005.
14. P. D. Seymour, *Matroids and multicommodity flows*, European J. Combin. **2** (1981), no. 3, 257–290.
15. Glenn Tesler, *Matchings in graphs on non-orientable surfaces*, J. Combin. Theory Ser. B **78** (2000), no. 2, 198–231.

16. Robin Thomas, *A survey of Pfaffian orientations of graphs*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 963–984.
17. W. T. Tutte, *Toward a theory of crossing numbers*, J. Combinatorial Theory **8** (1970), 45–53.

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