A short course on matching theory, ECNU Shanghai, July 2011.

Sergey Norin

LECTURE 3
Tight cuts, bricks and braces.

3.1. Outline of Lecture

- Ear decomposition of bipartite graphs.
- Tight cut decomposition.
- Bricks and braces.

3.2. The dimension of perfect matching polytope of bipartite graphs.

We will use the description of the inequalities defining the perfect matching polytope to determine its dimension. The dimension is in particular of interest as it provides a lower bound on the number of perfect matching of a graph. From now on we will concentrate our attention on connected graphs in which every edge belongs to a perfect matching. We call these graphs *matching-covered*. As before, our discussion is separated into two parts with analysis of bipartite graphs preceding that of general graphs.

If a polytope $P \in \mathbb{R}^m$ is given by a system of inequalities, then only the inequalities which are satisfied on all of P with equality affect the dimension of P. The (linear) dimension of P is then determined by

$$\dim P = m - \operatorname{rank} A,$$

where A is the matrix of the system of equations corresponding to those inequalities. The next lemma therefore allows us to determine the dimension of $\mathcal{PM}(G)$ for bipartite G.

Lemma 1. *Let A be the incidence matrix of a matching covered graph G with n vertices then*

(a) $\operatorname{rank} A = n - 1$, *if G is bipartite,*
(b) $\operatorname{rank} A = n$, *if G is not bipartite.*

Proof. Clearly $\operatorname{rank} A \leq n$, as A has n rows. If G is bipartite then rows of A are linearly dependent: The sum of rows corresponding to one class of the bipartition is equal to that of the other. Hence $\operatorname{rank} A \leq n - 1$ for bipartite graphs.

The rank of A can not be lower than the value stated in the lemma. Indeed, consider a linear relation between rows of A. Let a row corresponding to the vertex v be taken with coefficient λ_v. Then $\lambda_v + \lambda_u = 0$ for every edge $uv \in E(G)$. Hence, if $\lambda = \lambda_v$ for some $v \in V(G)$, the coefficient of every other vertex is either λ or $-\lambda$, depending on the parity of a path from v to this vertex. Hence G must be bipartite and the relation must be equivalent to the relation described above. □

The next theorem is now the immediate consequence of Lemma 1, Theorem 3 from Lecture 2 and the discussion above.

Theorem 1. *For a bipartite, matching-covered graph G we have*

$$\dim \mathcal{PM}(G) = |E(G)| - |V(G)| + 1.$$

We will give an alternative proof of Theorem 1 using structural methods, instead of linear algebraic ones. An *ear decomposition* of a bipartite graph is a sequence $(P_0, P_1, P_2, \ldots, P_k)$, so that, if G_i is defined as $P_0 \cup P_1 \cup \ldots P_i$, then

- $G_0 = P_0$ is isomorphic to the one edge graph K_2,
- P_i is an odd path with both ends in G_{i-1} and is otherwise disjoint from it for $1 \leq i \leq k$, and
- $G_k = G$.

We will show that matching-covered graphs allow ear decompositions, but first we need an important definition and a lemma, presented as an exercise. We say that a subgraph H of G is *central* if $G - V(H)$ has a perfect matching.

Exercise 1. A bipartite graph with bipartition (A, B) is matching covered if and only if $G - \{u, v\}$ has a perfect matching for every $u \in A$ and $v \in B$.

Theorem 2. *A bipartite graph G is matching-covered if and only if it allows an ear-decomposition.*

Proof. We start with the "if" direction. Clearly G is connected. Let P_i and G_i be as in the definition of the matching decomposition, for $0 \leq i \leq k$. The proof is by induction on k. By the induction hypothesis we may assume that G_{k-1} is matching covered. Let $P_k = u_0 v_0 \dots u_l v_l$, then any perfect matching M of G_{k-1} extends to a perfect matching $M + v_0 u_1 + \dots v_{l-1} u_l$ of G. Further, a perfect matching M' of $G_{k-1} - \{u_0, v_l\}$, which exists by Exercise 1, extends to a perfect matching $M' + u_0 v_0 + \dots + u_l v_l$ of G. It follows that every edge of G belongs to a perfect matching.

For the "only if" direction, let H be a maximal central subgraph of G allowing an ear-decomposition. Clearly, we can choose such H. We claim that $H = G$. If not, let M be a perfect matching of $G - V(H)$ and let M' be a perfect matching of G containing some edge $e \in \nabla(V(H))$. Then the M-augmenting path P contained in $M \cup M'$ and containing e is an odd path with both ends in $V(H)$ and otherwise disjoint from it. We can add this path to the ear decomposition of H to obtain an ear decomposition of $H \cup P$, contrary to the choice of H. □

Proof of Theorem 1. From Lemma 1 we have $\dim \mathcal{PM}(G) \leq |E(G)| - |V(G)| + 1$, and so it suffices to show that $\dim \mathcal{PM}(G) \geq |E(G)| - |V(G)| + 1$. The proof is by induction on $|E(G)|$. Consider an ear decomposition $(P_0, P_1, \dots, P_k)$ of G. It is easy to see that $k = m - n$. By the induction hypothesis $\dim \mathcal{PM}(G_{k-1}) = k$, where G_{k-1} is defined as above. As in the proof of Theorem 2, every perfect matching of G_{k-1} extends to a perfect matching of G. The resulting matchings form a subset of $\mathcal{PM}(G)$ of dimension k and none of them contains an edge of P_k incident to its end. Hence, $\dim \mathcal{PM}(G) > k$ as desired. □

As an immediate corollary of the above results we have $m(G) \geq |E(G)| - |V(G)| + 2$ for every matching covered bipartite graph G. This is tight for all even values of $|V(G)|$: consider two vertices joined by l disjoint paths of length 3. Every perfect matching M of G contains the two end edges of one of these paths and the middle edges of all the others. We have $m(G) = l$ and $|V(G)| = 2l + 2$, $E(G) = 3l$.

If v is a vertex of degree two in a graph G adjacent to two distinct neighbors and H is a graph obtained from G by contracting both edges incident to v then we say that H is obtained from G by *bicontracting the vertex v*. This notion will be useful later.

Exercise 2. a) Show that if H is obtained from G by bicontracting a vertex, then $m(G) = m(H)$.

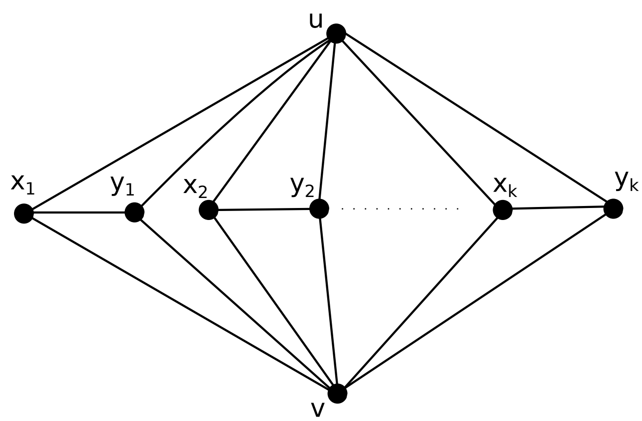

Figure 1. A graph with few perfect matchings.

b) Show that $m(G) = |E(G)| - |V(G)| + 2$, if and only if a graph on two vertices can be obtained from G by repeated bicontraction. (This is due to de Carvalho, Lucchesi and Murty.)

3.3. Tight cuts

Inequalities of type (iii) in Theorem 5 of Lecture 2 can reduce the dimension of $\mathcal{PM}(G)$, if they turn identically to equalities on the whole polytope. This observation motivates a definition. A cut C in a graph G is *tight* if $|C \cap M| = 1$ for every perfect matching M of G. A cut C is *trivial* if $C = \nabla(v)$ for some $v \in V(G)$, and is *non-trivial* otherwise.

By the discussion in the previous section we have the following result.

Theorem 3. *Let G be a non-bipartite, matching-covered graph with no non-trivial tight cuts. Then* $\dim \mathcal{PM}(G) = |E(G)| - |V(G)|$.

Unfortunately, the above formula for the dimension does not hold in general. It is not even true that $m(G) \geq |E(G)| - |V(G)| + 1$. For an example, consider the graph on Figure 1. It has $2k + 2$ vertices, $5k$ edges and only $2k$ perfect matchings. The formula has to account for tight cuts.

Let us first present examples of tight cuts:

1. A set of vertices X in a matching-covered graph G is called *a barrier* if $c_o(G - X) = |X|$. Let X be a barrier in a graph G, let Y be an odd component of $G - X$. Then $\nabla(Y)$ is called a *barrier cut.*

2. Let u, v be distinct vertices in a matching-covered graph G, such that $G - \{u, v\}$ is disconnected. Let Y be an even component of $G - \{u, v\}$. Then the cut $\nabla(Y \cup \{u\})$ is called a 2-*separation cut.*

Exercise 3. Show that barrier and 2-separation cuts are tight.

The following deep theorem of Edmonds, Lovász and Pulleyblank will allow us to characterize graphs without tight cuts. We give it without proof.

Theorem 4. *A matching covered graph G has a non-trivial tight cut if and only if it has a non-trivial barrier cut or a non-trivial* 2*-separation cut.*

Let $C = \nabla(X)$ be a cut in a graph G. As in the proof of Theorem 5 in Lecture 2, we consider graphs G_1, obtained from G by identifying all vertices in X to a single vertex and G_2, obtained from G by identifying all vertices in $V(G) - X$ to a single vertex. (Note that for convenience we assume that $E(G_1), E(G_2) \subseteq E(G)$.) Then G_1, G_2 are called C-*contractions* of G. If C is tight then every perfect matching in G induces (by restriction) perfect matchings in G_1 and G_2. In fact, many properties of $\mathcal{M}(G)$ can be read off $\mathcal{M}(G_1)$ and $\mathcal{M}(G_2)$. The following theorem is an example.

Lemma 2. *Let C be a tight cut in a matching-covered graph G and let G_1 and G_2 be C-contractions. Then*

$$\dim \mathcal{PM}(G) = \dim \mathcal{PM}(G_1) + \dim \mathcal{PM}(G_2) - |C| + 1.$$

Proof. (This lemma and its proof are due to Lovász and Plummer.) For $i = 1, 2$, let $\mathcal{M}_i$ be the maximal family of perfect matchings in G_i, so that $\{\chi_{\mathbf{M}} | M \in \mathcal{M}_i\}$ is linearly independent. For $e \in C$ let $\mathcal{M}_i(e)$ be the set of perfect matchings in $\mathcal{M}_i$ containing e. For every $e \in C$ select one perfect matching $M_i(e) \in \mathcal{M}_i(e)$. This choice is possible as G_i is matching-covered. Consider the set of all perfect matchings of the form $M_1 \cup M_2(e)$, where $M_1 \in \mathcal{M}_1(e)$, and $M_1(e) \cup M_2$, where $M_2 \in \mathcal{M}_2(e)$. We claim that the characteristic vectors of perfect matchings in this set are independent and generate $\mathcal{PM}(G)$. This claim implies the lemma by routine counting.

To verify the claim, suppose first for a contradiction that this set of perfect matchings is linearly dependent, i.e. there exist $\lambda(M) \in \mathbb{R}$ for $M \in \mathcal{M}_1 \cup \mathcal{M}_2$ so that

$$\sum_{e \in C} \sum_{M_1 \in \mathcal{M}_1(e)} \lambda(M_1) \chi_{M_1 \cup M_2(e)} + \sum_{e \in C} \sum_{\substack{M_2 \in \mathcal{M}_2(e) \\ M_2 \neq M_2(e)}} \lambda(M_2) \chi_{M_2 \cup M_1(e)} = 0.$$

A restriction to $\mathbb{R}^{E(G_1)}$ gives us

$$(1) \quad \sum_{e\in C}\sum_{M_1\in\mathcal{M}_1(e)}\lambda(M_1)\chi_{M_1}+\sum_{e\in C}\left(\sum_{\substack{M_2\in\mathcal{M}_2(e)\\ M_2\neq M_2(e)}}\lambda(M_2)\right)\chi_{M_1(e)}=0.$$

It follows from the independence of the characteristic vectors of matchings in $\mathcal{M}_1$ that $\lambda(M)=0$ for $M\in M_1$, unless, possibly, $M=M_1(e)$ for some $e\in C$. By symmetry, $\lambda(M)=0$ for $M\in M_2$ unless, possibly, $M=M_2(e)$ for some $e\in C$. But then in (1) the second summand is zero, and we deduce that $\lambda(M)=0$ for all $M\in M_1$. Thus, the set of vectors we are considering is linearly independent. One can use a similar argument to show that it generates $\mathcal{PM}(G)$. □

3.4. Bricks and braces

By repeatedly performing cut contractions using non-trivial tight cuts we can reduce many problems related to perfect matchings to graphs with no such cuts. Let us now describe these graphs. A graph is called *k-extendable* if every matching of size $\leq k$ in it extends to a perfect matching. A connected, bipartite 2-extendable graph is called a *brace*. A graph G is *bicritical* if $G-\{u,v\}$ has a perfect matching for all pairs of distinct $u,v\in V(G)$. A *brick* is a 3-connected bicritical graph.

Exercise 4. For every matching-covered bipartite graph G the following conditions are equivalent

(a) G is a brace;
(b) for every $X\subseteq A$ either $N(X)=B$ or $|N(X)|\geq|X|+2$;
(c) for every four distinct vertices $a,a'\in A$, $b,b'\in B$ the graph $G-\{a,a',b,b'\}$ has a perfect matching.

Theorem 5 (Edmonds,Lovász and Pulleybank)**.** *A graph G has no non-trivial tight cuts if and only if G is a brick or a brace.*

Proof. By Theorem 4 it suffices to show that bricks and braces have no non-trivial barrier or 2-separation cuts, and that every graph with no non-trivial barrier an 2-separation cuts is a brick or a brace. We will check some of those implications.

If G is a brick then it clearly has no 2-separation cuts. It also has no barriers. Indeed, if $X\subseteq V(G)$ is such that $c_o(X)=|X|$ and $|X|\geq 2$ then deleting two vertices from X produces a graph without a prefect matching, contradicting the fact that G is bicritical.

Conversely, assume that G is non-bipartite, matching covered and is not bicritical. Then there exist $u,v\in V(G)$ and $X\subseteq V(G)-\{u,v\}$

such that $c_o(G - X - \{u, v\}) > |X|$. As $|V(G)|$ is even, we have $c_o(G - X - \{u, v\}) \geq |X| + 2$. Therefore $Z := X \sup\{u, v\}$ is a barrier. All components of $G - Z$ must be odd as G is matching covered and no perfect matching of G can contain an edge between Z and an even component of $G - Z$. Therefore, one of these components has size more than one, as otherwise, G is bipartite. The cut separating this component is a non-trivial barrier cut in G.

Finally, suppose that G is non-bipartite, matching covered and $G - \{u, v\}$ is disconnected for some $u, v \in V(G)$. If some component of $G - \{u, v\}$ is odd then $\{u, v\}$ is a barrier and we can find a tight cut as in the previous paragraph. Otherwise, we find a 2-separation cut.

We leave verification of the theorem for bipartite graphs as an exercise. □

Graphs C_4, $K_{3,3}$ and the cube are example of braces. Graphs K_4, triangular prism $\bar{C}_6$, the Wagner graph and the Petersen graph are examples of bricks.

A *tight cut decomposition procedure* for a graph G with a non-trivial tight cut C replaces it by the two C-contractions and recursively performs cut contractions along tight cuts in the resulting graphs. It produces a collection of bricks and braces. Lovász has proved that (surprisingly!) the final collection is, upto multiple edges independent on the choices of tight cuts in the decomposition. Let $\mathrm{b}(G)$ denote the number of bricks in the collection. Lemma 2 immediately implies the following.

Theorem 6. *Let G be a matching covered graph then*

$$\dim \mathcal{PM}(G) = |E(G)| - |V(G)| + 1 - \mathrm{b}(G).$$

Another theorem of Edmonds, Lovász and Pulleyblank implies that the number of bricks in a brick decomposition of a cubic bridgeless graph on n vertices is at most $n/4$. Therefore $m(G) \geq n/4$ for every such graph. We will prove asymptotically stronger lower bounds in later lectures.

We have learned how to generate all bipartite matching-covered graphs from single edges by ear-decomposition. One can similarly generate all bricks and braces. We say that H is *a retract* of a graph G if it is obtained from G by repeated bicontracting. (We do not identify parallel edges when we bicontract.) The first of the following two theorems is due to McCuaig and the second one to Norin and Thomas.

Theorem 7. *Every brace G other than C_4 and $K_{3,3}$ has an edge e such that the retract of $G - e$ is a brace.*

Theorem 8. *Every brick G other than K_4, the prism $\bar{C}_6$ and the Petersen graph has an edge e such that the retract of $G - e$ is a brick not isomorphic to the Petersen graph.*

One can use Theorem 8 to establish the following difficult result of Lovász.

Theorem 9. *Let $\mathbb{F}$ be a field and let G be a matching covered graph then the dimension of the span of characteristic vectors of perfect matchings over $\mathbb{F}$ is*

(a) $|E(G)| - |V(G)| + 2 - \mathrm{b}(G)$, *if* $\mathrm{char}\,\mathbb{F} \neq 2$,
(b) $|E(G)| - |V(G)| + 2 - \mathrm{b}(G) - \mathrm{p}(G)$, *if* $\mathrm{char}\,\mathbb{F} = 2$, *where* $\mathrm{p}(G)$ *is the number of bricks isomorphic to the Petersen graph in the tight cut decomposition of G.*

Exercise 5. Derive Theorem 9 from Theorem 8, using the argument from the proof of Lemma 2.

A brick G is called *minimal* if $G - e$ is not a brick for every $e \in E(G)$. A conjecture of Lovász, established by de Carvalho, Lucchesi and Murty that every minimal brick has a vertex of degree 3 also is immediately implied by Theorem 8. The following result due to Norin and Thomas is a consequence of the same theorem.

Theorem 10. *For $n \geq 10$ every minimal brick has at most $5n/2 - 7$ edges. Every minimal brick has at least three vertices of degree 3.*

Problem 1. Does there exist $\alpha > 0$ such that every minimal brick on n vertices has at least αn vertices of degree 3?