Small Graph Classes and Bounded Expansion

Zdeněk Dvořák[1]

Institute for Theoretical Computer Science (ITI), Charles University, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic

Serguei Norine[2]

Department of Mathematics, Fine Hall, Washington Road, Princeton NJ, 08544-1000

Abstract

A class of simple undirected graphs is *small* if it contains at most $n!\alpha^n$ labelled graphs with n vertices, for some constant α. We prove that for any constants $c, \varepsilon > 0$, the class of graphs with expansion bounded by the function $f(r) = c^{r^{1/3-\varepsilon}}$ is small. Also, we show that the class of graphs with expansion bounded by $6 \cdot 3^{\sqrt{r \log(r+e)}}$ is not small.

We work with simple undirected graphs, without loops or parallel edges. A class of graphs is *small* if it contains at most $n!\alpha^n$ different (but not necessarily non-isomorphic) labelled graphs on n vertices, for some constant α. For example, the class of all trees is small, as there are exactly $n^{n-2} < n!e^n$ trees on n vertices.

Norine, Seymour, Thomas and Wollan [8] showed that all proper minor-

Email addresses: `rakdver@kam.mff.cuni.cz` (Zdeněk Dvořák), `snorin@math.princeton.edu` (Serguei Norine)

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closed classes of graphs are small, answering the question of Welsh [9]. This question was motivated by the results of McDiarmid, Steger and Welsh [2] regarding random planar graphs. These results in fact hold for any class of graphs that is small and addable. A class $\mathcal{G}$ is *addable* if

- $G \in \mathcal{G}$ if and only if every component of G belongs to $\mathcal{G}$, and
- if $G_1, G_2 \in \mathcal{G}$, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then the graph obtained from the disjoint union of G_1 and G_2 by adding the edge $\{v_1, v_2\}$ belongs to $\mathcal{G}$.

Many naturally defined graph classes are addable (for example, minor-closed classes defined by excluding a set of 2-connected minors), and this condition is usually easy to verify. The more substantial assumption thus is that the class is small.

Let $\mathcal{G}$ be a class that is small and addable, and let $N(n)$ be the number of labelled graphs in $\mathcal{G}$ with n vertices. In [2] the following results (among others) were shown:

- The limit $c = \lim_{n\to\infty}(N(n)/n!)^{1/n}$ exists and is finite.
- If $K_{1,k+1} \in \mathcal{G}$, then there exist constants d and n_0 such that letting $a_k = d/(c^k(k+2)!)$, the probability that a random graph in $\mathcal{G}$ on $n \geq n_0$ vertices has fewer than $a_k n$ vertices of degree k is at most $e^{-a_k n}$. Also, a similar result is shown for the number of appearances of arbitrary connected subgraphs.
- The probability that a random graph in $\mathcal{G}$ on n vertices has an isolated vertex is at least $a_1/e + o(1)$ (on the other hand, the probability that

such a graph is connected is greater than zero as well).

Let us now recall the notion of classes of graphs with bounded expansion, as defined by Nešetřil and Ossona de Mendez [6, 3, 4, 5]. The *grad (Greatest Reduced Average Density) with rank* r of a graph G is equal to the largest average density of a graph G' that can be obtained from G by removing some of the vertices (and possibly edges) and then contracting vertex-disjoint subgraphs of radius at most r to single vertices (arising parallel edges are suppressed). The grad with rank r of G is denoted by $\nabla_r(G)$. In particular, $2\nabla_0(G)$ is the maximum average degree of a subgraph of G. Given a function $f : \mathbb{Z}_+ \to \mathbb{R}_+$, a graph has *expansion bounded by* f if $\nabla_r(G) \leq f(r)$ for every integer r. A class $\mathcal{G}$ of graphs has *expansion bounded by* f if the expansion of every $G \in \mathcal{G}$ is bounded by f. Finally, we say that a class of graphs $\mathcal{G}$ has *bounded expansion* if there exists a function f such that the expansion of $\mathcal{G}$ is bounded by f.

The concept of classes of graphs with bounded expansion proves surprisingly powerful. Many classes of graphs have bounded expansion (proper minor-closed classes, classes of graphs with bounded maximum degree, classes of graphs excluding subdivision of a fixed graph, ...), and many results for proper minor-closed classes (existence of colorings, small separators, light subgraphs, ...) generalize to classes of graphs with bounded expansion (possibly with further natural assumptions). The classes of graphs with bounded expansion are also interesting from the algorithmic point of view, as the proofs of the mentioned results usually give simple and efficient algorithms. Furthermore, fast algorithms and data structures for problems like deciding whether a graph contains a fixed subgraph, or for determining the distance between a pair vertices (assuming that the distance is bounded by a fixed constant), have been derived. The reader is referred to [7] for a survey of the results regarding the bounded expansion.

The aim of this paper is to prove that classes of graphs with expansion bounded by a slightly subexponential function ($f(r) = c^{r^{1/3-\varepsilon}}$ for any $c, \varepsilon > 0$) are small. This generalizes the result of [8], as proper minor-closed classes have expansion bounded by a constant. Also, we believe that our proof technique is simpler and more natural, although the calculations are somewhat involved. Furthermore, we show that the class of graphs with expansion bounded by $6 \cdot 3^{\sqrt{r \log(e+r)}}$ is not small.

Let us now describe the basic idea of the proof: As Nešetřil and Ossona de Mendez [4] showed, graphs in a class with expansion bounded by a subexponential function have separators of sublinear size (for more precise statement, see the following section). Therefore, each graph on n vertices in the class is a union of two smaller graphs (of order between $n/3$ and $2n/3$) from the same class, with a few vertices (forming the separator of $o(n)$ size) identified. This gives a recurrence for the number of graphs with n vertices, which we use to show that the class is small; see Section 2 for further details.

1. Separators

For a graph G, a set $S \subseteq V(G)$ is a *separator* if there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $A \cap B = S$, G contains no edges between $A - S$ and $B - S$, and $\max(|A \setminus S|, |B \setminus S|) \leq \frac{2}{3}|V(G)|$. We need the following result of Nešetřil and Ossona de Mendez [4].

Theorem 1. *There exists a constant C such that for every integer z, if a graph G on n vertices satisfies $2z(\nabla_z(G)+2) \leq \sqrt{n \log n}$, then G has a separator of size at most $C\frac{n \log n}{z}$.*

We use the following corollary:

Corollary 2. *Let $k \geq \frac{1}{2}$ be a constant and let $\mathcal{G}$ be the class of graphs with expansion bounded by a function f such that $f(r) \leq \frac{k}{r} e^{\frac{1}{2}\sqrt[3]{9\frac{r}{\log^2 r}}} - 2$ for $r \geq 2$. There exists a constant c such that every graph $G \in \mathcal{G}$ with $n \geq 3$ vertices has a separator of size $\frac{cn}{(\log n \log\log n)^2}$.*

Proof. We apply Theorem 1 with $z = \frac{1}{2k}\log^3 n \log^2 \log n$. The assumptions are satisfied, as $2z(\nabla_z(G)+2) \leq 2z(f(z)+2) \leq 2z(f(\log^3 n \log^2 \log n)+2) \leq e^{\frac{1}{2}\log n} \leq \sqrt{n \log n}$. □

2. Lower bound

Let us start with a technical lemma:

Lemma 3. *For every $n \geq 3$,*

$$\frac{1}{\log\log(2n/3)} - \frac{1}{\log\log n} \geq \frac{1}{3\log n \log^2 \log n}.$$

Proof. Let $x = \log n$ and $c = \log\frac{3}{2}$, and note that $x \geq 1$. Then $\frac{1}{\log\log(2n/3)} - \frac{1}{\log\log n} = \frac{1}{\log(x-c)} - \frac{1}{\log x} \geq \frac{\log x - \log(x-c)}{\log^2 x} \geq \frac{\log(1+\frac{c}{x})}{\log^2 x}$. The claim follows, as $\log(1+\frac{c}{x}) \geq \frac{1}{3x}$ for $x \geq 1$. □

We can now proceed with the main result of this section:

Theorem 4. *Let $c > 0$ be a constant and let $\mathcal{G}$ be a class of graphs closed under taking induced subgraphs, such that every graph $G \in \mathcal{G}$ with $n \geq 3$*

vertices has a separator of size at most $s(n) = \frac{cn}{(\log n \log\log n)^2}$. *Then* $\mathcal{G}$ *is small.*

Proof. We consider $\mathcal{G}$ as a class of unlabelled graphs; let $N(n)$ be the number of graphs in $\mathcal{G}$ with n vertices. Let $h(n) = \frac{12cn}{\log\log n}$ for $n \geq 3$, and let $n_0 \geq 3$ be an integer such that

- $h(n) < n$ and $s(n) \geq 1$ for all $n \geq n_0$,
- $h(n)$ is non-decreasing and concave on the interval $(n_0, +\infty)$, and
- $2n/3 + s(n) \leq n - 1$ for $n \geq n_0$.

Let $C \geq e$ be a constant such that $N(n) \leq C^{n-h(n)}$ for $n_0 \leq n \leq 3n_0$. We show by induction that $N(n) \leq C^{n-h(n)}$ for every $n \geq n_0$. This implies that $\mathcal{G}$, considered as a class of labelled graphs, is small.

For $n \leq 3n_0$ the claim holds by the choice of C. Assume now that $n > 3n_0$, and that $N(k) \leq C^{k-h(k)}$ for $n_0 \leq k < n$. Let $s = \lfloor s(n) \rfloor$. A graph $G \in \mathcal{G}$ on n vertices has a separator S of size at most s (with the corresponding vertex sets A and B such that $A \cap B = S$), and since we can add vertices to the separator, we may assume that $|S| = s$. Note that the graphs $G[A]$ and $G[B]$ belong to $\mathcal{G}$. We conclude that

$$N(n) \leq \sum_{a=\lceil n/3 \rceil}^{\lfloor 2n/3 \rfloor + s} \binom{a}{s} \binom{n-a+s}{s} s! N(a) N(n-a+s),$$

since every graph in $\mathcal{G}$ on n vertices can be constructed in the following way: Choose an integer a such that $\lceil n/3 \rceil \leq a \leq \lfloor 2n/3 \rfloor + s$ and graphs $G_1, G_2 \in \mathcal{G}$ such that $|V(G_1)| = a$ and $|V(G_2)| = n-a+s$ (for a fixed a, this can be done in $N(a)N(n-a+s)$ ways). Choose subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ so that $|S_1| = |S_2| = s$ (this can be done in $\binom{a}{s}\binom{n-a+s}{s}$ ways). Choose a perfect matching between the vertices of S_1 and S_2 (in $s!$ ways), and identify the matched vertices in S_1 and S_2.

Note that $\binom{a}{s}s! \leq n^s$ and $\binom{n-a+s}{s} \leq n^s$. Also, $n_0 \leq n/3 \leq a < n$ and $n_0 \leq n - a + s < n$, thus by the induction hypothesis

$$N(n) \leq \sum_{a=\lceil n/3 \rceil}^{\lfloor 2n/3 \rfloor + s} n^{2s} C^{n+s-h(a)-h(n-a+s)}.$$

As h is concave, we get

$$h(a) + h(n-a+s) \geq h(n/3) + h(2n/3+s) \geq h(n/3) + h(2n/3).$$

It follows that

$$\begin{aligned} N(n) &\leq n^{2s+1} C^{n+s-h(n/3)-h(2n/3)} \\ &= C^{n+(2s+1)\log_C n + s - h(n/3) - h(2n/3)} \\ &\leq C^{n+(2s(n)+2)\log n - h(n/3) - h(2n/3)}. \end{aligned}$$

Moreover,

$$\begin{aligned} h(n/3) + h(2n/3) - h(n) &= 12cn \left(\frac{1/3}{\log\log(n/3)} + \frac{2/3}{\log\log(2n/3)} - \frac{1}{\log\log n} \right) \\ &\geq 12cn \left(\frac{1}{\log\log(2n/3)} - \frac{1}{\log\log n} \right), \end{aligned}$$

and by Lemma 3,

$$h(n/3) + h(2n/3) - h(n) \geq \frac{12cn}{3\log n \log^2 \log n} \geq (2s(n)+2)\log n.$$

It follows that $N(n) \leq C^{n-h(n)}$, as required. $\square$

Together with Corollary 2, this implies the following.

Corollary 5. *For any $k > 0$, the class of graphs with expansion bounded by a function $f(r) = \frac{k}{r} e^{\frac{1}{2}\sqrt[3]{9\frac{r}{\log^2(r+e)}}} - 2$ is small.*

Note that for any $c, \varepsilon > 0$, there exists k such that the function $h(r) = c^{r^{1/3-\varepsilon}}$ satisfies $h(r) \leq f(r)$, thus the class of graphs with expansion bounded by h is small.

3. Upper bound

For any fixed $d > 2$, the results of Bender and Canfield [1] imply that the number of simple d-regular graphs on n vertices (with dn even) is $\Omega\left(\frac{(nd/2)!}{(d!)^n}\right)$. It follows that the class of 3-regular graphs (whose expansion is bounded by $f(r) = 3 \cdot 2^{r-1}$) is not small. We can improve this observation slightly in the following way: for a non-decreasing positive function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$, let $\mathcal{G}_g$ be the class of graphs such that $G \in \mathcal{G}_g$ if and only if there exists a 4-regular graph H such that G is obtained from H by subdividing each edge of H by $g(|V(H)|)$ vertices (we could use 3-regular graphs in the same construction, but the obtained bound would be similar and by using 4-regular graphs, we avoid the need to require that the number of vertices is even). Let $N(g, n)$ be the number of graphs in $\mathcal{G}_g$ with n vertices, and $N_4(n)$ the number of 4-regular graphs with n vertices, $N_4(n) = \Omega\left(\frac{(2n)!}{24^n}\right) = \Omega\left(\frac{(n!)^2}{7^n}\right)$. If $n = k(1 + 2g(k))$, then $N(g, n) \geq \binom{n}{k} N_4(k)(n-k)!$—we choose the vertices of a 4-regular graph H, order the remaining $n - k$ vertices arbitrarily, and distribute them to the edges of H according to some canonical ordering of $E(H)$. It follows that $N(g, n) \geq n! N_4(k)/k! = \Omega\left(\frac{n!k!}{7^k}\right)$. If $k \log k = \omega(n)$, this implies that $\mathcal{G}_g$ is not small. We can achieve this by choosing a function $g(x) = o(\log x)$.

Theorem 6. *The class of graphs with expansion bounded by the function* $f(r) = 6 \cdot 3^{\sqrt{r \log(r+e)}}$ *is not small.*

Proof. Let g be the function defined in the following way: $g(k) = t$, where t is the smallest integer such that $\log k \le t \log t$. Note that $g(k) = o(\log k)$, thus the class $\mathcal{G}_g$ is not small. We show that the expansion of $\mathcal{G}_g$ is bounded by f. Consider a graph $G \in \mathcal{G}_g$, and let k be the number of vertices of G of degree 4; i.e., $n = |V(G)| = k(1 + 2g(k))$. Let H be the 4-regular graph obtained from G by suppressing the vertices of degree two, $|V(H)| = k$. Let $r \ge 0$ be an integer and let us show that $\nabla_r(G) \le f(r)$. Note that $\nabla_0(G) \le 2 < f(0)$ and $\nabla_1(G) \le 6 < f(1)$, thus assume that $r > 1$. If $g(k) \le \sqrt{\frac{r}{\log r}}$, then $k \le e^{g(k) \log g(k)} \le e^{\frac{1}{2}\sqrt{r \log r}} \le f(r)$. Furthermore, note that suppressing the vertices of degree at most two does not decrease the maximum average density, thus $\nabla_r(G) \le \frac{|V(H)|-1}{2} \le k \le f(r)$. Therefore, assume that $g(k) \ge \sqrt{\frac{r}{\log r}}$. However, subgraph of G of radius r corresponds to a subgraph of H of radius at most $\left\lceil \frac{r}{g(k)+1} \right\rceil$, thus $\nabla_r(G) \le \nabla_{\lceil \frac{r}{g(k)+1} \rceil}(H) \le 2 \cdot 3^{\frac{r}{g(k)}+1} \le 6 \cdot 3^{\sqrt{r \log r}} \le f(r)$. □

4. Concluding remarks

For a function f, let $l(f) = \limsup_{r \to \infty} \frac{\log \log f(r)}{\log r}$. By Corollary 5, if $l(f) < 1/3$, then the class of graphs with expansion bounded by f is small. On the other hand, in Theorem 6 we proved that there exists a function f with $l(f) = 1/2$ such that the class of graphs with expansion bounded by f is not small.

Question 1. *What is the infimum of values of* $l(f)$ *taken over all functions*

$f : \mathbb{Z}_+ \to \mathbb{R}_+$, *such that the class of graphs with expansion bounded by* f *is not small?*

Instead of considering the expansion, we can formulate a similar question in the terms of the size of the separators. For a hereditary class of graphs $\mathcal{G}$, let $s(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\log |V(G)| - \log s(G)}{\log \log |V(G)|}$, where $s(G)$ is the size of the smallest separator in G. Theorem 4 shows that if $s(\mathcal{G}) > 2$, then $\mathcal{G}$ is small, and Theorem 6 shows an example of a class with $s(\mathcal{G}) = 1$ that is not small.

Question 2. *What is the supremum of values of* $s(\mathcal{G})$ *taken over hereditary classes of graphs* $\mathcal{G}$ *which are not small?*

We suspect that the answer to Question 2 is 1. This would imply that the answer to Question 1 is 1/2.

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