SET INTERSECTIONS, PERFECT GRAPHS, AND VOTING IN AGREEABLE SOCIETIES

DEBORAH E. BERG, SERGUEI NORINE, FRANCIS EDWARD SU*, ROBIN THOMAS, AND PAUL WOLLAN

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My idea of an agreeable person is a person who agrees with me.
—Benjamin Disraeli [7]

1. Introduction

When is agreement possible? An important aspect of group decision-making is the question of how a group makes a choice when individual preferences may differ. Clearly, when making a single group choice, people cannot all have their "ideal" preferences, i.e, the options that they most desire, if those ideal preferences are different. However, for the sake of agreement, people may be willing to accept as a group choice an option that is merely "close" to their ideal preferences.

A good example of such a situation is voting for candidates along a political spectrum. We normally think of this spectrum as one-dimensional, with conservative positions on the right and liberal positions on the left, as in Figure 1. We call each position on the spectrum a *platform* that a candidate may choose to adopt. While one may represent one's ideal platform by some point x on this line, one might be willing to vote for a candidate that positions himself at some point "close enough" to x, i.e., in an interval about x.

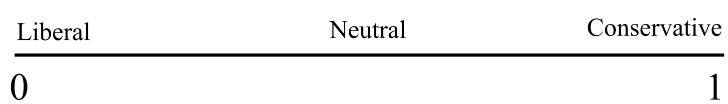

Figure 1. A one-dimensional political spectrum, modeled by a unit interval.

In this article, we ask the following: given such preferences on a political spectrum, when can we guarantee that some fraction (say, a majority) of the population will agree on some candidate? By "agree", we mean in the sense of *approval voting*, in which voters declare which candidates they find acceptable.

*Corresponding author.

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Author affiliations:
Berg, Su — Department of Mathematics, Harvey Mudd College, Claremont CA 91711, dberg@hmc.edu, su@math.hmc.edu.
Norine, Thomas — School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, snorine@math.gatech.edu, thomas@math.gatech.edu.
Wollan — Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, pwollan@math.uwaterloo.ca .

Approval voting has not yet been adopted for political elections in the United States. However, many scientific and mathematical societies, such as the Mathematical Association of America and the American Mathematical Society, use approval voting for their elections. Additionally, countries other than the United States have used approval voting or an equivalent system. For details, see Brams and Fishburn [2], who give many reasons why they believe approval voting is advantageous. In what follows, our study of *agreeability* will help us understand when we can guarantee a majority under approval voting.

Consider the 2003 California gubernatorial recall election, with 135 candidates in the mix [6]. We might imagine these candidates positioned at 135 points on the line in Figure 1. If each California voter approves of candidates within some range of positions (call this the voter's *approval set*), we might wonder if and when there might be a point on the line covered by a majority of the voter approval sets, i.e., a platform on which a majority of the voters agree.

In this setting, we may assume that each approval set is a closed interval on $\mathbb{R}$ and we call a collection of voters, together with their approval sets, a *linear society.* Call a society *super-agreeable* if for every pair of voters there is some platform they would both approve, i.e., each pair of approval sets has a non-empty intersection. For linear societies this local condition guarantees a strong global property, namely, that there is a platform that *every* voter approves! As we shall see in Theorem 3, this is a consequence of Helly's theorem about intersections of convex sets.

In this article, we consider a variety of similar theorems. For instance, we relax the condition above and call a society *agreeable* if it has at least three voters and among every three voters, there is some pair of voters who agree on some platform. Then we prove the following:

Theorem 1 (The Agreeable Linear Society Theorem)**.** *In an agreeable linear society, there is a platform which has the approval of a majority of voters, i.e., a winning platform.*

For example, Figure 2 shows approval sets for an agreeable linear society of six voters, and indeed there are platforms that a majority of voters approve. As another application of our theorem, consider a situation in which each voter's approval set is a closed subinterval of $[0, 1]$ of length at least $1/3$. Then Theorem 1 guarantees a winning platform, since among any three approval sets there must be a pair that intersect. We consider other degrees of "agreeability" and prove a more general result in Theorem 8 giving a lower bound for the size of the plurality in approval voting. We also briefly study societies whose approval sets are convex subsets of $\mathbb{R}^d$.

A general theme of this article is that classical (and new) convexity theorems have interesting social interpretations, and these social questions motivate the study of set intersections and perfect graphs, since they have natural interpretations in this voting context.

2. Definitions

In this section, we fix terminology and explain some of the basic concepts upon which our results rely. Let us suppose that the set of possible preferences is modeled by a set X, called the *spectrum.* Each element of the spectrum is a *platform.* Assume that there is a finite set V of *voters*, and each voter v has an *approval set* A_v of platforms.

We define a *society* S to be a triple $(X, V, \mathcal{A})$ consisting of a spectrum X, a set of voters V, and a collection $\mathcal{A}$ of approval sets for all the voters. Of particular interest to us will be the case for a *linear society*, where X is $\mathbb{R}$ and approval sets in $\mathcal{A}$ are closed intervals, but in general X could be any set and approval sets could be any class of subsets of X. In Figure 2 we illustrate a linear society, where for ease of display we have separated the approval sets vertically so that they can be distinguished.

We have seen that politics provides natural examples of linear societies. For a different example, X could represent a temperature scale, V a set of inhabitants of a house, and each A_v is a range of temperatures that inhabitant v finds comfortable.

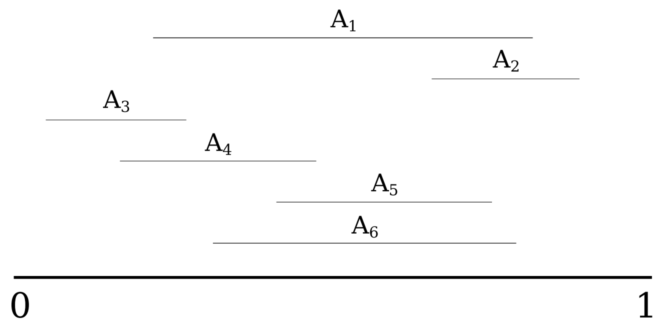

FIGURE 2. Approval sets of a linear society of six voters.

Let $1 \leq k \leq m$ be integers. Call a society (k, m)-*agreeable* if it has at least m voters, and for any subset of m voters, there is at least one platform that at least k of them can agree upon. Said another way, for any subset of m voters, there is a point common to at least k of the voters' approval sets. Thus $(2, 3)$-agreeable is the same as agreeable defined earlier, and for societies with at least two voters super-agreeable is the same as $(2, 2)$-agreeable. One may check that the society of Figure 2 is $(2, 3)$-agreeable. It is not $(3, 4)$-agreeable, however, because among voters $2, 3, 4, 5$ there are no triples whose approval sets share a common point.

For a society S, the *agreement number* of a platform, $a(p)$, is the number of voters in S who approve of platform p. The *agreement number* $a(S)$ of a society S is the maximum agreement number over all platforms in the spectrum, so

$$a(S) = \max_{p \in X} a(p).$$

The *agreement proportion* of S is simply the agreement number of S divided by the number of voters of S. This concept is useful when we are interested in percentages of the population rather than the number of voters. The *agreement set* of S consists of platforms that receive $a(S)$ votes. The society of Figure 2 has agreement number 4, which may be seen in Figure 3 where the shaded rectangles cover the agreement set.

3. HELLY'S THEOREM AND SUPER-AGREEABLE SOCIETIES

Let us say that a society is $\mathbb{R}^d$-*convex* if the spectrum is $\mathbb{R}^d$ and each approval set is a closed convex subset of $\mathbb{R}^d$. Note that a linear society is an $\mathbb{R}^1$-convex society. An $\mathbb{R}^d$-convex society can arise when considering a multi-dimensional spectrum, such as when evaluating political platforms over several axes (e.g., conservative vs. liberal, pacifist vs. militant, interventionist vs. isolationist). Or, the spectrum might be arrayed over more personal dimensions: the dating website *eHarmony* claims to use up to 29 of them [8]. In such situations, the convexity of approval sets might, for instance, follow from an independence-of-axes assumption and convexity of approval sets along each axis.

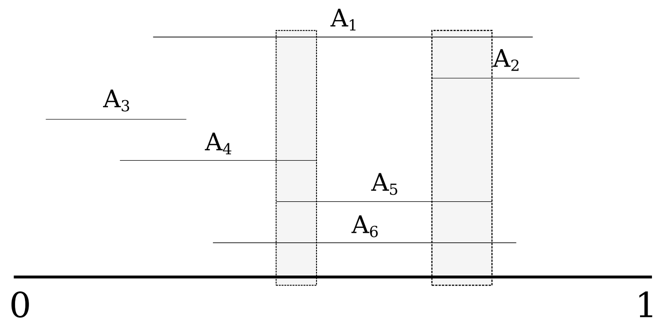

FIGURE 3. The society of Figure 2 with agreement number 4 and agreement set marked.

For $\mathbb{R}^d$-convex societies, work concerning set intersections can be applied to the agreement number problem. The most well known theorem in this area is Helly's theorem. This theorem was proven by Helly in 1913, but the result was not published until 1921, by Radon [14].

Theorem 2 (Helly). *Given t convex sets in $\mathbb{R}^d$ where $d < t$, if every $d+1$ of them intersect at a common point, then they all intersect at a common point.*

Helly's theorem has a nice interpretation for $\mathbb{R}^d$-convex societies, especially when $d = 1$, where the Helly condition for approval sets is equivalent to the condition for a linear super-agreeable society:

Theorem 3. *For every $d \geq 1$, a $(d+1, d+1)$-agreeable $\mathbb{R}^d$-convex society must contain at least one platform that is acceptable to all voters.*

In particular, when $d = 1$ and the approval sets of every pair of voters intersect, we have:

Corollary 4 (The Super-Agreeable Society Theorem). *A linear super-agreeable society must contain at least one platform that is acceptable to all voters.*

We provide a simple proof of the Super-Agreeable Society Theorem (equivalently, Helly's theorem in dimension 1) as it will be needed later. A proof of Helly's theorem for general d may be found in [12].

Proof. Since each voter agrees on at least one platform with every other voter, we see that the sets A_i must be non-empty. Thus, each A_i is a non-empty closed interval in $[0, 1]$. Let $x = \max_i\{\min\{p \in A_i\}\}$ and $y = \min_j\{\max\{p \in A_j\}\}$.

We claim that $x \leq y$. Why? Let i be the voter whose approval set minimum is maximal, and let j be the voter whose approval set maximum is minimal. Since the approval sets of i and j intersect, the only way this could hold is if $x \leq y$.

Therefore, every approval set contains the non-empty interval $[x, y]$; hence there is a platform common to all approval sets. □

Besides Helly's theorem, another famous theorem about set intersections is the KKM lemma [10], which is concerned with set intersections on simplices. There is a variant of this theorem for trees (e.g., see [13]) that generalizes both Helly's theorem and the KKM lemma, and since a line is a tree, Theorem 4 also follows as a consequence.

Here is an example demonstrating that the convexity assumption is essential. Let $n \geq 2$ be an integer and let the spectrum of a society S consist of all 2-element subsets of

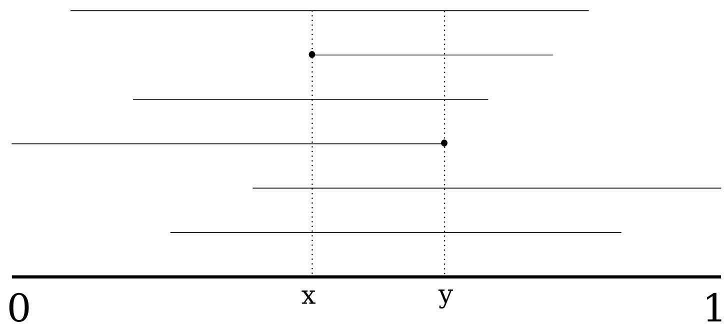

FIGURE 4. A super-agreeable society of 6 voters, with agreement number 6.

$\{1, 2, \ldots, n\}$. Let S have n voters numbered $1, 2, \ldots, n$, and let the approval set of voter i consist precisely of those 2-element subsets of $\{1, 2, \ldots, n\}$ that include i. Then S is a $(2, 2)$-agreeable (super-agreeable) society with agreement number 2, which stands in sharp contrast to the conclusion of Theorem 3 (since there is no 2-element subset that is contained in every approval set).

4. Graph Representations and Interval and Perfect Graphs

If we are to understand other kinds of agreeability beyond super-agreeability, it will be helpful to examine methods of pictorially representing the agreeableness of a society. A *graph* G consists of a finite set $V(G)$ of *vertices* and a set $E(G)$ of 2-element subsets of $V(G)$, called edges. If $e = \{u, v\}$ is an edge, then we say that u, v are the *ends* of e, and that u and v are *adjacent* in G. We use uv as shorthand notation for the edge e.

Given a society S, we construct the *agreement graph* G of S by letting the vertices $V(G)$ be the voters of S and the edges $E(G)$ be all pairs of voters u, v whose approval sets intersect with each other. Thus u and v are connected by an edge if there is a platform that both u, v would approve. Note that the agreement graph of a super-agreeable society is a complete graph (but the converse is false— see later).

The *clique number* of G, written $\omega(G)$, is the greatest integer q such that G has a set of q pairwise adjacent vertices, called a *clique* of size q. An immediate consequence of the Super-Agreeable Society Theorem (Corollary 4) is that the clique number can tell us how many people can agree on a platform:

Fact 1. *For the agreement graph of a linear society, the clique number of the society is the agreement number.*

This fact does not necessarily hold if the society is not linear. For instance, there is an $\mathbb{R}^2$-convex society with three voters such that every two voters agree on a platform, but all three of them do not.

Now, to get a handle on the clique number, we shall make a connection between the clique number and colorings of the agreement graph. The *chromatic number* of G, written $\chi(G)$, is the minimum number of colors necessary to color the vertices of G such that no two adjacent vertices have the same color. Thus two voters may have the same color as long as they do not agree on a platform. Note that in all cases, $\chi(G) \geq \omega(G)$.

A graph G is called an *interval graph* if every vertex x represents a closed interval $I_x \subseteq \mathbb{R}$ and $xy \in E(G)$ if and only if $I_x \cap I_y \neq \emptyset$. Hence

Fact 2. The agreement graph of a linear society is an interval graph.

An *induced subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and the edges of H are the edges of G that have both ends in $V(H)$. If every induced subgraph H of a graph G satisfies $\chi(H) = \omega(H)$, then G is called a *perfect graph*.

Theorem 5. *Interval graphs are perfect.*

Proof. Let G be an interval graph, and for $v \in V(G)$, let I_v be the interval representing the vertex v. Since every induced subgraph of an interval graph is an interval graph, it suffices to show that $\chi(G) = \omega$, where $\omega = \omega(G)$. We proceed by induction on $|V(G)|$. The assertion holds for the null graph, and so we may assume that $|V(G)| \geq 1$, and that the statement holds for all smaller graphs. Let us select a vertex $v \in V(G)$ such that the right end of I_v is as small as possible. It follows that N, the set of neighbors of v in $V(G)$, are pairwise adjacent because their intervals must all contain the right end of I_v, and hence $|N| \leq \omega - 1$. See Figure 5. By induction, the graph $G\backslash v$ obtained from G by deleting v can be colored using ω colors, and since v has at most $\omega - 1$ neighbors, this coloring can be extended to a coloring of G, as desired. □

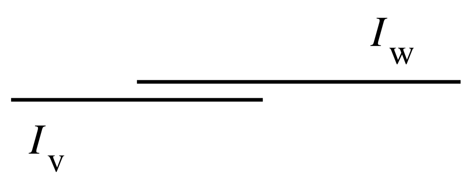

FIGURE 5. If I_v, I_w intersect and the right end of I_v is smaller than the right end of I_w, then I_w must contain the right end of I_v.

Perfect graphs appear in many different contexts in mathematics, theoretical computer science, and operations research. The concept was introduced by Berge [1] in 1961, who was motivated by a question in communication theory, specifically, the determination of the Shannon capacity of a graph [16]. Chvátal later proved that if A is a $0,1$-matrix, then the linear program

$$\max c \cdot x \text{ subject to } x \geq 0 \text{ and } Ax \leq 1. \tag{1}$$

has an integral optimum solution for every objective function c if and only if the matrix A arises from a perfect graph in a specified way [5, 15, 17]. As pointed out in [15], algorithms to solve semi-definite programs grew out of the theory of perfect graphs.

Recently, Chudnovsky, Robertson, Seymour and Thomas [3] proved the following characterization of perfect graphs, conjectured by Berge in [1], and since then known as the Strong Perfect Graph Conjecture.

Theorem 6. *A graph is perfect if and only if it has no induced subgraph isomorphic to a cycle of odd length at least five, or a complement of such a cycle.*

Using this result, Chudnovsky, Cornuejols, Liu, Seymour and Vuskovic [4] found a polynomial-time algorithm to test whether a graph is perfect.

5. (k, m)-Agreeable Linear Societies

We now use the connection between perfect graphs, the clique number, and the chromatic number to obtain a lower bound for the agreement number of a (k, m)-agreeable linear society (Theorem 8). We first need a lemma that says that in the corresponding agreement graph, the (k, m)-agreeable condition prevents any coloring of the graph from having color classes that are too large. Thus, there must be many colors and, since the graph is perfect, a lower bound for a clique number.

Lemma 7. *Given integers $m \geq k \geq 2$, let positive integers r, ρ be defined by the division with remainder: $m - 1 = (k-1)r + \rho$, where $0 \leq \rho \leq k-2$. Let G be a graph on $n \geq m$ vertices with chromatic number χ such that every subset of $V(G)$ of size m includes a clique of size k. Then $n \leq \chi r + \rho$, or $\chi \geq (n-\rho)/r$.*

Proof. Let the graph be colored using the colors $1, 2, \ldots, \chi$, and for $i = 1, 2, \ldots, \chi$ let C_i be the set of vertices of G colored i. We may assume, by permuting the colors, that $|C_1| \geq |C_2| \geq \cdots \geq |C_\chi|$. Since $C_1 \cup C_2 \cup \cdots \cup C_{k-1}$ is colored using $k-1$ colors, it includes no clique of size k, and hence, $|C_1 \cup C_2 \cup \cdots \cup C_{k-1}| \leq m-1$. It follows that $|C_{k-1}| \leq r$, for otherwise $|C_1 \cup C_2 \cup \cdots \cup C_{k-1}| \geq (k-1)(r+1) \geq (k-1)r + \rho + 1 = m$, a contradiction. Thus each $|C_i| \leq r$ for $i \geq k$ and

$$n = \sum_{i=1}^{k-1} |C_i| + \sum_{i=k}^{\chi} |C_i| \leq m - 1 + (\chi - k + 1)r = (k-1)r + \rho + (\chi - k + 1)r = \chi r + \rho,$$

as desired. □

Theorem 8 (The (k, m)-Agreeable Linear Society Theorem)**.** *Let $2 \leq k \leq m$. If G is the agreement graph of a linear (k, m)-agreeable society, then $\omega(G) \geq \lceil (n-\rho)/r \rceil$, where r, ρ are defined by the division with remainder: $m - 1 = (k-1)r + \rho$, $\rho \leq k-2$. Consequently, every linear (k, m)-agreeable society has agreement proportion at least $(k-1)/(m-1)$.*

Note that this extends the Agreeable Linear Society Theorem (in which $k = 2, m = 3$ and the guaranteed agreement proportion is $1/2$) and the Super-Agreeable Society Theorem when $d = 1$ (in which $k = m$ and the guaranteed agreement proportion is 1).

Proof. By Fact 2 and Theorem 5 the graph G is perfect. Thus the chromatic number of G is equal to $\omega(G)$, and hence $\omega(G) \geq (n-\rho)/r$ by Lemma 7, as desired. The second assertion follows from Fact 1 and the inequality $(n-\rho)/r \geq n(k-1)/(m-1)$. □

Let us observe that the bound $\lceil (n-\rho)/r \rceil$ in Theorem 8 is best possible. Indeed, let $I_1, I_2, \ldots, I_r$ be disjoint intervals, for $i = r+1, r+2, \ldots, n-\rho$ let $I_i = I_{i-r}$, and let $I_{n-\rho+1}, I_{n-\rho+2}, \ldots, I_n$ be pairwise disjoint and disjoint from all the previous intervals, e.g., see Figure 6. Then the society with approval sets $I_1, I_2, \ldots, I_n$ is (k, m)-agreeable and its agreement graph has clique number $\lceil (n-\rho)/r \rceil$.

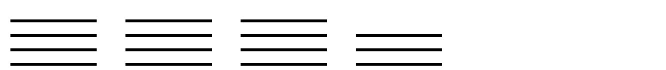

Figure 6. A linear $(4, 15)$-society with $r = 4$, $\rho = 2$, $n = 22$ voters, and clique number $\lceil (n-\rho)/r \rceil = 5$. (Approval sets have been separated vertically so that they may be distinguished.)

6. $\mathbb{R}^d$-convex Societies

In this section we prove a higher dimensional analogue of Theorem 8 by giving a lower bound on the agreement proportion of a (k, m)-agreeable $\mathbb{R}^d$-convex society. We need a different method than our method for $d = 1$, because for $d \geq 2$, neither Fact 1 nor Fact 2 holds. We use the following generalization of Helly's theorem, due to Kalai [11].

Theorem 9 (The Fractional Helly's Theorem)**.** *Let $d \geq 1$ and $n \geq d+1$ be integers, let $\alpha \in [0, 1]$ be a real number, and let $\beta = 1 - (1-\alpha)^{1/(d+1)}$. Let $F_1, F_2, \ldots, F_n$ be convex sets in $\mathbb{R}^d$ and assume that for at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-element index sets $I \subseteq \{1, 2, \ldots, n\}$ we have $\bigcap_{i\in I} F_i \neq \emptyset$. Then there exists a point in $\mathbb{R}^d$ contained in at least βn of the sets $F_1, F_2, \ldots, F_n$.*

The following is the promised analogue of Theorem 8.

Theorem 10. *Let $d \geq 1$, $k \geq 2$ and $m \geq k$ be integers. Then every (k, m)-agreeable $\mathbb{R}^d$-convex society has agreement proportion at least $1 - \left(1 - \binom{k}{d+1}/\binom{m}{d+1}\right)^{1/(d+1)}$.*

Proof. Let S be a (k, m)-agreeable $\mathbb{R}^d$-convex society, and let $A_1, A_2, \ldots, A_n$ be its voter approval sets. Let us call a set $I \subseteq \{1, 2, \ldots, n\}$ *good* if $|I| = d+1$ and $\bigcap_{i\in I} A_i \neq \emptyset$. By Theorem 9 it suffices to show that there are at least $\binom{k}{d+1}\binom{n}{d+1}/\binom{m}{d+1}$ good sets. We will do this by counting in two different ways the number N of all pairs (I, J), where $I \subseteq J \subseteq \{1, 2, \ldots, n\}$, I is good and $|J| = m$. Let g be the number of good sets. Since every good set is of size $d+1$ and extends to an m-element subset of $\{1, 2, \ldots, n\}$ in $\binom{n-d-1}{m-d-1}$ ways, we have $N = g\binom{n-d-1}{m-d-1}$. On the other hand, every m-element set $J \subseteq \{1, 2, \ldots, n\}$ includes at least one k-element set K with $\bigcap_{i\in K} A_i \neq \emptyset$ (because S is (k, m)-agreeable), and K in turn includes $\binom{k}{d+1}$ good sets. Thus $N \geq \binom{k}{d+1}\binom{n}{m}$, and hence $g \geq \binom{k}{d+1}\binom{n}{d+1}/\binom{m}{d+1}$, as desired. □

For $d = 1$, Theorem 10 gives a worse bound than Theorem 8, and hence for $d \geq 2$, the bound is most likely not best possible. However, a possible improvement must use a different method, because the bound in Theorem 9 *is* best possible.

A *box* in $\mathbb{R}^d$ is the Cartesian product of d closed intervals, and we say that a society is a *d-box society* if each of its approval sets is a box in $\mathbb{R}^d$. It follows from Theorem 3 that d-box societies satisfy the conclusion of Fact 1 (namely, that the clique number equals the agreement number), and hence their agreement graphs capture all the essential information about the society. Unfortunately, agreement graphs of d-box societies are, in general, not perfect. For instance, there is a 2-box society whose agreement graph is the cycle on five vertices. See Figure 7. For $k \leq m \leq 2k-2$, the following theorem and corollary will resolve the agreement proportion problem for all (k, m)-agreeable societies satisfying the conclusion of Fact 1, and hence for all (k, m)-agreeable d-box societies where $d \geq 1$.

Theorem 11. *Let $m, k \geq 2$ be integers with $k \leq m \leq 2k-2$, and let G be a graph on $n \geq m$ vertices such that every subset of $V(G)$ of size m includes a clique of size k. Then $\omega(G) \geq n - m + k$.*

Before we embark on a proof let us make a few comments. First of all, the bound $n-m+k$ is best possible, as shown by the graph consisting of a clique of size $n-m+k$ and $m-k$ isolated vertices. Second, the conclusion $\omega(G) \geq n-m+k$ implies that every subset of $V(G)$ of size m includes a clique of size k, and so the two statements are equivalent under the

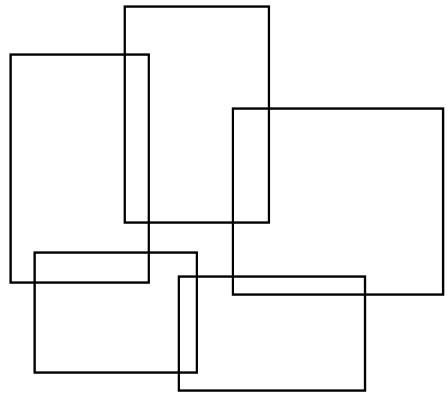

FIGURE 7. A 2-box society whose agreement graph is a 5-cycle.

hypothesis that $k \le m \le 2k-2$. Finally, this hypothesis is necessary, because if $m \ge 2k-1$, then for $n \ge 2(m-k+1)$, the disjoint union of cliques of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ satisfies the hypothesis of Theorem 11, but not its conclusion.

A *vertex cover* of a graph G is a set $Z \subseteq V(G)$ such that every edge of G has at least one end in Z. We say a set $S \subseteq V(G)$ is *stable* if no edge of G has both ends in S. We deduce Theorem 11 from the following lemma.

Lemma 12. *Let G be a graph with minimum vertex cover of size z such that $G \backslash v$ has a vertex cover of size at most $z-1$ for all $v \in V(G)$. Then $|V(G)| \le 2z$.*

Proof. Let Z be a vertex cover of G of size z. For every $v \in V(G) - Z$ let Z_v be a vertex cover in $G \backslash v$ of size $z-1$, and let $X_v = Z - Z_v$. Then X_v is a stable set. For $X \subseteq Z$ let $N(X)$ denote the set of neighbors of X outside Z. We have $v \in N(X_v)$ and $N(X_v) - \{v\} \subseteq Z_v - Z$, and so

$$|X_v| = |Z - Z_v| = |Z| - |Z \cap Z_v| = |Z_v| + 1 - |Z \cap Z_v| = |Z_v - Z| + 1 \ge |N(X_v)|.$$

On the other hand, if $X \subseteq Z$ is stable, then $|N(X)| \ge |X|$, for otherwise $(Z - X) \cup N(X)$ is a vertex cover in G of size at most $z-1$, a contradiction. We have

$$|Z| \ge |\bigcup X_v| \ge |\bigcup N(X_v)| \ge |V(G)| - |Z|, \tag{2}$$

where both unions are over all $v \in V(G) - Z$, and hence $|V(G)| \le 2z$, as required. To see that the second inequality holds let $u, v \in V(G) - Z$. Then

$$\begin{aligned} |X_u \cup X_v| &= |X_u| + |X_v| - |X_u \cap X_v| \ge |N(X_u)| + |N(X_v)| - |N(X_u \cap X_v)| \\ &\ge |N(X_u)| + |N(X_v)| - |N(X_u) \cap N(X_v)| = |N(X_u) \cup N(X_v)|, \end{aligned}$$

and, in general, the second inequality of (2) follows by induction on $|V(G) - Z|$. □

Proof of Theorem 11. We proceed by induction on n. If $n = m$, then the conclusion certainly holds, and so we may assume that $n \ge m+1$ and that the theorem holds for graphs on fewer than n vertices. We may assume that $m > k$, for otherwise the hypothesis implies that G is the complete graph. We may also assume that G has two nonadjacent vertices, say x and y, for otherwise the conclusion holds. Then in G, every clique contains at most one of x, y, so in the graph $G \backslash \{x, y\}$ every set of vertices of size $m-2$ includes a clique of size $k-1$. Since $k-1 \le m-2 \le 2(k-2)-2$ we deduce by induction on n that

$\omega(G) \geq \omega(G \backslash \{x, y\}) \geq n-2-(m-2)+k-1 = n-m+k-1$. We may assume in the last statement that equality holds throughout, because otherwise G satisfies the conclusion of the theorem. Let $\bar{G}$ denote the complement of G; that is, the graph with vertex set $V(G)$ and edge set consisting of precisely those pairs of distinct vertices of G that are not adjacent in G. Let us notice that a set Q is a clique in G if and only if $V(G)-Q$ is a vertex cover in $\bar{G}$. Thus the size of a minimum vertex cover in $\bar{G}$ is $m-k+1$. Since $2(m-k+1) \leq m \leq n$, by Lemma 12, the graph $\bar{G}$ has an induced subgraph H on exactly m vertices with no vertex cover of size $m-k$ or smaller. By hypothesis, the graph $\bar{H}$ has a clique Q of size k, but $V(H)-Q$ is a vertex cover in H of size $m-k$, a contradiction. □

Corollary 13. *Let $d \geq 1$ and $m, k \geq 2$ be integers with $k \leq m \leq 2k-2$, and let S be a (k, m)-agreeable d-box society with n voters. Then the agreement number of S is at least $n-m+k$, and this bound is best possible.*

Proof. The agreement graph G of S satisfies the hypothesis of Theorem 11, and hence it has a clique of size at least $n-m+k$ by that theorem. Since d-box societies satisfy the conclusion of Fact 1, the first assertion follows. The bound is best possible, because the graph consisting of a clique of size $n-m+k$ and $m-k$ isolated vertices is an interval graph. □

7. Speculation and Open Questions

As we have seen, set intersection theorems can provide a useful framework to model and understand the relationships between preference sets in many social contexts.

Additionally, recent results in discrete geometry have social interpretations. The piercing number [9] of approval sets can be interpreted as the minimum number of platforms that are necessary such that everyone has some platform of which he or she approves. Set intersection theorems on other spaces (such as trees and cycles) are derived in [13] and social applications are explored, including an approval voting interpretation when the society has a circular political spectrum.

We suggest several directions which the reader may wish to explore.

The most natural problem seems to be to determine the agreement proportion for $\mathbb{R}^d$-convex and d-box (k, m)-agreeable societies. The smallest case where we do not know the answer is $d = 2$, $k = 2$, and $m = 3$. Rajneesh Hegde (private communication) found an example of a $(2, 3)$-agreeable 2-box society with agreement proportion 3/8.

Additionally, we must examine our initial assumptions. For instance, we assumed that voters place candidates along a linear spectrum in exactly the same order, even though voters may order candidates along a spectrum differently. Also, while convexity seems to be a rational assumption in the linear case, in multiple dimensions, additional considerations may need to be made.

The original concept of an agreement graph could be applied to $\mathbb{R}^d$-convex societies to keep track of more information. For instance, two voters might not agree on every axis, meaning that their approval sets don't intersect, but it might be the case that many of the projections of their approval sets do. In this case, one may wish to consider an agreement graph with weighted edges.

Finally, we might wonder about the agreement parameters k and m for various issues which affect us personally. For instance, a society considering outlawing murder would probably be much more agreeable than that same society considering tax reform. Not only do the issues matter, however, but also the societies. Groups of similar people seem likely to be more agreeable than groups consisting of a more diverse population. Currently, we can empirically measure these parameters only by surveying large numbers of people about their preferences. It is interesting to speculate about methods for estimating k and m from limited data.

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