Graph Minor Theory

Sergey Norin

March 13, 2017

Abstract

Lecture notes for the topics course on Graph Minor theory.

Winter 2017.

Contents

1 Background **2**
1.1 Minors 2
1.2 Connectivity 3
1.3 Planarity and coloring. 4

2 Excluding a small clique and Hadwiger's conjecture **4**
2.1 Hadwiger's conjecture for $t \leq 3$ 4
2.2 Hadwiger's conejceture for $t = 4$ 5

3 Excluding a forest **7**
3.1 Excluding a path. Treedepth 8
3.2 Excluding a star 8
3.3 Pathwidth 9

4 Tree decompositions **12**
4.1 Definition and basic properties 12
4.2 Brambles, cops and robbers 14
4.3 Tangles 16

5 Applications of tree decompositions **18**
5.1 Algorithms on graphs of bounded treewidth 18
5.2 Erdős-Pósa Property 19
5.3 Balanced separations 20

6 The grid theorem **21**
6.1 From a tangle to a mesh 21
6.2 Cleaning up the mesh. 23

7 Well quasi-ordering **25**
7.1 Basic properties 25
7.2 Kruskal's theorem 27
7.3 Well-quasi-ordering graphs of bounded treewidth. 28

8 The graph minor structure theorem **28**
8.1 Surfaces 28
8.2 Vortices 29
8.3 The clique sum structure 30
8.4 The two paths theorem 31
8.5 Transactions in vortices 35
8.6 Towards the proof of the graph minor structure theorem 36

9 Balanced separations **38**
9.1 Alon-Seymour-Thomas theorem 38
9.2 Counting K_t-minor-free graphs 40

1 Background

The graphs in these notes are assumed to be simple, unless explicitly stated otherwise.

1.1 Minors

By *contraction of an edge uv* in a graph G we mean identification of u and v, i.e. replacement of u and v by a new vertex w adjacent to all of the neighbors of u and v. We denote the graph obtained this way by $G\backslash uv$.

A graph H is *a minor* of a graph G if H can be obtained from G by repeatedly deleting vertices and edges and contracting edges. We say that G *contains* H *as a minor*, and write $G \geq H$, if a graph isomorphic to H is a minor of G. It is easy to see that the minor relation is transitive, that is if $G \geq H$ and $H \geq F$ then $G \geq F$.

A *subdivision of a graph* H is a graph obtained from a graph isomorphic to H by replacing some of its edges by internally vertex disjoint paths. The following is easy.

Lemma 1.1. *If a subdivision of* H *is a subgraph of* G *then* $H \leq G$.

The converse of Lemma 1.1 does not generally hold and we need the following more involved definition. A *model of a graph* H *in a graph* G is a function μ assigning to the vertices of H vertex disjoint connected subgraphs of G, such that if $uv \in E(H)$ then some edge of G joins a vertex of $\mu(u)$ to a vertex of $\mu(v)$.

Lemma 1.2. *There exists a model of a graph H in a graph G if and only if $H \leq G$.*

Proof. If there exists a model μ of H in G then by repeatedly contracting the edges of $\mu(v)$ we can identify all the vertices in $V(\mu(v))$ to a single vertex x_v for every $v \in V(H)$. Deleting all the remaining vertices not in the set $\{x_v : v \in V(H)\}$ and all the edges not of the form $x_u x_v$ for $uv \in E(H)$ we obtain a graph isomorphic to H, which is a minor of G.

In the opposite direction, suppose that $H \leq G$. We show that there exists a model of H in G by induction on $|V(G)|$. We may assume, by replacing G by a subgraph if necessary, that H is obtained from G by contraction operations only. Let uv be the first contracted edge, let $G' = G \backslash uv$, and let w be the vertex obtained by identifying u and v. By the induction hypothesis there exists a model μ' of H in G'. Suppose that $w \in V(\mu'(x))$ for some $x \in V(H)$. We modify $\mu'(x)$ by deleting w and adding u, v, the edge uv and the edges from u and v to the neighbors of w in $\mu'(x)$. It is easy to verify that this modification produces a model of H in G. $\square$

Lemma 1.3. *If H is a graph with maximum degree at most three, and a graph G contains H as a minor, then a subdivision of H is a subgraph of G.*

Proof. Assume without loss of generality that no proper subgraph of G contains H as a minor, and let μ be a model of H in G. Then for every $v \in V(H)$ the subgraph $\mu(v)$ of G is a tree such that every leaf of $\mu(v)$ is incident for some neighbor $u \in V(H)$ to the unique edge of G joining a vertex of $\mu(v)$ to a vertex of $\mu(u)$. We denote such a leaf by l_{vu}. As H has maximum degree at most three it follows that there exists a vertex $x_v \in V(\mu(v))$ and paths from x_v to l_{vu} for each neighbor u of v, disjoint except for x_v. Joining such paths together, we obtain a subdivision of H is a subgraph of G with the vertices $\{x_v \; v \in V(H)\}$ corresponding to the vertices of H. $\square$

1.2 Connectivity

A *separation* of a graph G is a pair (A, B) such that $A \cup B = V(G)$ and no edge of G has one end in $A - B$ and the other in $B - A$. The *order* of a separation (A, B) is $|A \cap B|$. The separation is *non-trivial* if $A - B \neq \emptyset$ and $B - A \neq \emptyset$.

A graph G is *k-connected* if $|V(G)| \geq k + 1$ and $G \setminus X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. If G is k-connected then G has no non-trivial separations of order less than k. The following is a very useful variant of Menger's theorem.

Theorem 1.4 (Menger)**.** *Let G be a graph, k a positive integer, and $Q, R \subseteq V(G)$. Then exactly one of the following holds:*

(i) *There exist pairwise vertex disjoint paths $P_1, P_2, \dots P_k$ each with one end in Q and the other end in R, or*

(ii) *there exists a separation (A, B) of G of order less than k such that $Q \subseteq A$, $R \subseteq B$.*

1.3 Planarity and coloring.

A graph G is *planar* if it can be drawn in the plane with vertices represented by distinct points, and edges by the curves joining the corresponding points, disjoint except for their ends.

Theorem 1.5 (Wagner). *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

A *(vertex) k-coloring* of a graph G is a function $c : V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for every $uv \in E(G)$. The *chromatic number* $\chi(G)$ of a graph G is the minimum k such that G admits a k-coloring.

Theorem 1.6 (The Four Color Theorem, Appel and Haken). *$\chi(G) \leq 4$ for every planar graph G.*

2 Excluding a small clique and Hadwiger's conjecture

2.1 Hadwiger's conjecture for $t \leq 3$

The following famous conjecture of Hadwiger motivates many of the results in these notes.

Conjecture 2.1 (Hadwiger). *If $\chi(G) > t$ then $G \geq K_{t+1}$.*

The conjecture is easy for $t = 1, 2$. We discuss the cases $t = 3, 4$ next.

Theorem 2.2. *Every 3-connected graph contains a K_4 minor.*

Proof. Let G be a 3-connected graph, $G \not\geq K_4$. Choose distinct $u, v \in V(G)$. As G is 3-connected there exist three paths P, Q and R from u to v, disjoint except for their ends. Without loss of generality, there exist vertices $p \in V(P) - \{u, v\}$ and $q \in V(Q) - \{u, v\}$. By connectivity there exists a path S from p to q in $G \setminus u \setminus v$. By choosing a shortest path joining internal vertices of two distinct paths among P, Q and R, we may assume that S is internally disjoint from P, Q and R. In this case, $P \cup Q \cup R \cup S$ is a K_4-subdivision in G. □

Corollary 2.3. *If $G \not\geq K_4$ then G contains a vertex of degree at most two.*

Corollary 2.4. *A graph G does not contain K_4 as a minor if and only if it can be obtained from an empty graph by the following operations*

- *adding a vertex of degree at most one,*
- *adding a vertex of degree two with two adjacent neighbors,*
- *subdividing an edge.*

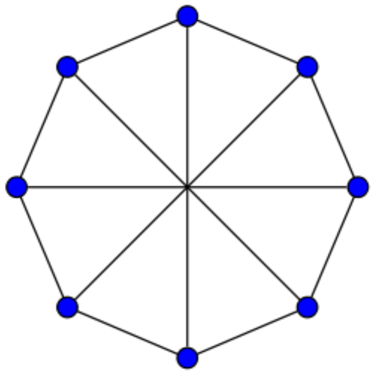

Figure 1: The Wagner graph V_8

Corollary 2.4 can be reinterpreted using the following convenient definition. Let G_1 and G_2 be two vertex disjoint graphs, and let $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_1)$ be two cliques with $|X_1| = |X_2| = k$. Let $f : X_1 \to X_2$ be a bijection, and let G be obtained from $G_1 \cup G_2$ by identifying x and $f(x)$ for every $x \in X_1$ and possibly deleting some edges with both ends in the clique of size k resulting from the identification. We say that G is a *k-sum* of G_1 and G_2.

Lemma 2.5. *If a graph G is a k-sum of G_1 and G_2 then $\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\}$.*

Theorem 2.6.

1. *$G \not\geq K_2$ if and only if G can be obtained from one vertex graphs by 0-sums,*
2. *$G \not\geq K_3$ if and only if G can be obtained from complete graphs on at most 2 vertices by 0- and 1-sums,*
3. *$G \not\geq K_4$ if and only if G can be obtained from complete graphs on at most 3 vertices by 0-, 1- and 2-sums.*

Theorem 2.6 and Lemma 2.5 give a uniform, if somewhat heavy handed proof of Hadwiger's conjecture for $t \leq 3$.

2.2 Hadwiger's conejceture for $t = 4$

The exact structure of graphs not containing K_5 as a minor is also known.

Theorem 2.7 (Wagner)**.** *A graph G does not contain K_5 as a minor if and only if G can be obtained from planar graphs and V_8 by 0-, 1-, 2- and 3-sums. (The graph V_8 is shown on Figure 1.)*

By Lemma 2.5, Theorem 2.8 and the Four Color Theorem imply Hadwiger's conjecture for $t = 4$. Let us also mention a result complementary to Theorem 2.8, establishing the structure of $K_{3,3}$-minor-free graphs.

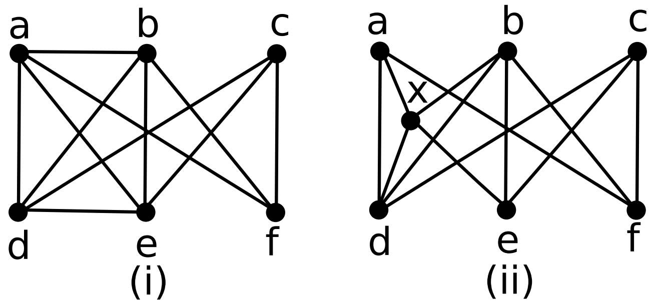

Figure 2: Extending a $K_{3,3}$ subdivision.

Theorem 2.8 (Wagner). *A graph contains no $K_{3,3}$ minor if and only if it can be obtained from planar graphs and K_5 by* 0-, 1-, *and* 2-*sums.*

To avoid some of the technicalities in the proof of Theorem 2.8 we will derive the Hadwiger's conjecture for $t = 4$ from the following weaker result.

Theorem 2.9. *Every non-planar* 4*-connected graph contains K_5 as a minor.*

Proof. The proof follows the strategy of the proof of Theorem 2.2, but is more involved. Let G be a non-planar 4-connected graph. By Theorem 1.5 and Lemma 1.3 we may assume that G contains a subdivision H of $K_{3,3}$ as a subgraph, and let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the vertices of H of degree three corresponding to vertices of two parts of the bipartition of $K_{3,3}$. For $x \in A$, let H_x denote the component of $H \setminus B$ containing x, and define H_x for $x \in B$, symmetrically. We choose a path P in $G \setminus B$ joining some two vertices in A such that $P \cup H$ is minimal. Without loss of generality, let a and b be the ends of P. By minimality, P is a union of a path P_a in H_a, a path P_b in H_b, and a path joining an end of P_a and an end of P_b, which is internally disjoint from H. Without loss of generality we assume that $P \cap H_d = \emptyset$. We proceed to choose a path Q in $G \setminus B$ joining d to another vertex in B such that $Q \cup H$ is minimal. Let e be the second end of Q. Again using the minimality of our choice, we see that Q is a union of a path Q_d in H_d, a path Q_e in H_e, and a path joining an end of P_a and an end of P_b, which is internally disjoint from H, and a path joining an end of Q_d and an end of Q_e, which is internally disjoint from H.

We claim that $H \cup P \cup Q$ contains a K_5 minor. Suppose first that Q and P are disjoint. Contracting all the edges P_a, P_b,Q_d and Q_e, we obtain a subdivision of the graph on Figure 2 (i), which has a K_5 minor, obtained by contracting the edge cf. Suppose next that Q and P intersect. The path Q_d is disjoint from P by our assumption, and Q_e intersects at most one of the paths P_a and P_b. Assume, by symmetry that Q_e and P_b are disjoint. We delete the edges of a path in H joining a to e which are not in $P \cup Q$. Further, contract all the edges Q_d and P_b, and of Q_e and P_a, if the last two paths are disjoint. Finally, contract all the remaining edges of P and Q, except for the edges incident to a, b, d and e. We obtain a subdivision of a graph shown on Figure 2 (ii), which again contains a K_5 minor, obtained by contracting edges af and ce. □

We now derive the Hadwiger's conjecture for $t = 4$ from Theorem 2.9.

Corollary 2.10. *If $G \not\geq K_5$ then $\chi(G) \leq 4$.*

Proof. By induction on $|V(G)|$.

Consider a non-trivial separation (A, B) of G of minimum order, and let $X = A \cap B$. If G is 4-connected the corollary follows from Theorem 2.9 and 1.6. Thus we assume that $|X| \leq 3$. We consider only the case $|X| = 3$, the other cases are easier.

Let G_1 and G_2 be the graphs obtained from $G[A]$ and $G[B]$, respectively, by adding vertices z_1 and z_2, respectively, adjacent to all vertices of X. As (A, B) is minimum we have $G_i \leq G$, and therefore, $G_i \not\geq K_5$ for $i = 1, 2$.

There are two subcases to consider. Assume first that X is independent. Let G'_i be the graph obtained from G_i by contracting all the edges incident to x_i, i.e. identifying all the vertices of X in $G[A]$ and $G[B]$. We have $\chi(G'_i) \leq 4$ by the induction hypothesis. Thus there exist 4-colorings of $G[A]$ and $G[B]$ in which all vertices of X receive the same color. Combining these colorings produces a 4-coloring of G.

Finally, suppose that some two vertices of X are adjacent, and let v be the remaining vertex of X. Let G''_i be obtained by contracting the edge $x_i v$ in G_i for $i = 1, 2$, i.e. adding edges to make X a clique in $G[A]$ and $G[B]$. Again we have, $\chi(G''_i) \leq 4$, and these 4-colorings can be combined to produce the required coloring of G. □

We say that a graph G is *apex* if $G \setminus v$ is planar for some $v \in V(G)$. Robertson, Seymour and Thomas established Hadwiger's conjecture for $t = 5$ by proving that a minimum counterexample is apex. The following beautiful conjecture would provide a more streamlined proof of their result.

Conjecture 2.11 (Jorgensen)**.** *If a 6-connected graph G contains no K_6-minor then G is apex.*

While Conjecture 2.11 is still open, Kawarabayashi, Norin, Thomas and Wollan have proved the following related result, which unfortunately has no direct consequences for the Hadwiger's conjecture.

Theorem 2.12. *There exists N such that, if a 6-connected graph G with $|V(G)| \geq N$ contains no K_6-minor, then G is apex.*

3 Excluding a forest

In this section we examine approximate structure of graphs which do not a forest in one of the several natural classes as a minor.

3.1 Excluding a path. Treedepth

We start with paths. Let (T, r) be a rooted tree. The *depth* of (T, r) is the number of vertices in the longest path in T starting at r. We consider the following partial order on $V(T)$, called *the tree order of* (T, r). For $u, v \in T(T)$ we have $u \leq v$ if only if u is a vertex of the unique path in T with ends r and v. The closure $\mathrm{clo}(T, r)$ is a graph obtained from T by joining any pair of comparable vertices by an edge. We say that a rooted spanning tree (T, r) of a graph G is *normal* if the ends of every edge of T are comparable in the above order, i.e. if G is a subgraph of $\mathrm{clo}(T, r)$.

Lemma 3.1. *Every connected graph contains a normal spanning tree.*

Proof. Any *depth first search tree* is normal. To find such a depth first search spanning tree in a connected graph G we construct a subtree T of G as follows. We start with a root vertex r and add it to a stack. At each step of the construction we consider the vertex v at the top of the stack. If v has a neighbor u which is not yet in $V(T)$, we add u and the edge uv to T, and add u to the top of the stack. Otherwise, we remove v from the stack. □

The *treedepth* of a connected graph G is defined as the minimum depth of a rooted tree (T, r) such that $G$is a subgraph of $\mathrm{clo}(T, r)$. The treedepth $\mathrm{td}(G)$ of a general graph G is the maximum treedepth of a component of G.

Lemma 3.2. *If* $\mathrm{td}(G) = k$ *then* G *contains a path on* k *vertices and no path on* 2^k *vertices.*

Proof. The first assertion is trivial. We prove the second assertion by induction on k. We may assume that G is connected. Let P be the longest path in G, and let r be the root of a normal spanning tree of G of minimum depth. Every component of $G \setminus r$ had treedepth at most $k-1$, and so by the induction hypothesis every component of $P \setminus r$ has at most $2^{k-1} - 1$ vertices. As there are at most two such components, we have $|V(P)| < 2^k$. □

Thus treedepth is a graph parameter "tied" to the length of the longest path in a graph. (Note that G contains P_k as a minor, if and only if G contains P_k as a subgraph.)

3.2 Excluding a star

We turn to stars next. Let S_k denote the star with k leaves.

Lemma 3.3. *For a connected graph* G *and an integer* $k \geq 3$*, the following are equivalent:*

1. $S_k \leq G$,
2. *a tree with at least* k *leaves is a subgraph of* G,
3. G *contains a spanning tree with at least* k *leaves.*

Proof. Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

To see that (1) implies (2). Consider a model μ of S_k in G. Let v be the center of S_k. Then as in the proof of Lemma 1.3 we may assume that $\mu(v)$ is a tree. By adding to $\mu(v)$ a single vertex from $\mu(u)$ for each leaf u of S_k together with an edge joining this vertex to a vertex of $\mu(u)$ we obtain a subgraph of G which is a tree with at least k leaves.

It remains to show that (2) implies (3). Choose a subtree T of G with at least k leaves such that $|V(T)|$ is maximum. If T is not spanning, we find $u \in V(G)-V(T)$ with a neighbor $v \in V(T)$. Adding u and the edge vu to T we obtain a subtree T' of G with at least as many leaves as T, contradicting the choice of T. Thus T is spanning as desired. $\square$

Lemma 3.4. *Let $k \geq 3$ be an integer. If every spanning tree of a connected graph G has less than k leaves then G is a subdivision of a graph on at most $10k-23$ vertices. Conversely, if G is a subdivision of a connected graph on at most k vertices then every spanning tree of G has at most $k(k-1)$ leaves.*

Proof. It suffices to prove the first statement for the graph G containing no vertices of degree two. Suppose for a contradiction that $|V(G)| \geq 10k-23$. Choose a spanning tree T of G with maximum number of leaves. As T has less than k leaves, it has at most $k-3$ vertices of degree at least three, and so T has at least $8k-19$ vertices of degree two. It follows that there exists a path $v_1v_2v_3v_4v_5$ in T consisting of vertices of degree two. The degree of v_3 in G is at least three, and let u be th neighbor u of v_3 in $V(G)-\{v_2v_4\}$. Then either $T \setminus v_1v_2 + uv$ or $T \setminus v_4v_5 + uv$ is a spanning tree of G with more leaves in G, yielding the desired contradiction.

For the second statement, it suffices to note that if P is a path in a graph G with every internal vertex of P of degree two, and T is a subtree of G, then T has at most two leaves in $V(P)$. $\square$

3.3 Pathwidth

Finally, we discuss a rough characterization of graphs not containing a general forest as a minor. Describing this characterization requires more substantial preparation. A *path decomposition* of a graph G is a sequence $\mathcal{W} = (W_1, \ldots, W_s)$ of subsets of $V(G)$, such that the following three conditions hold:

(P1) $\cup_{i=1}^{s} W_i = V(G)$,

(P2) every edge of G has both ends in some set W_i,

(P3) if $1 \leq i \leq j \leq k \leq s$ then $W_i \cap W_k \subseteq W_j$.

The *width* of a path decomposition $(W_1, \ldots, W_s)$ is equal to $\max_{i=1}^{s} (|W_i|-1)$. The *pathwidth* $\mathrm{pw}(G)$ is equal to the minimum width of a path decomposition of G. We will show that the graph not containing a fixed tree as a minor have bounded pathwidth. Conversely, the following holds.

Lemma 3.5. *Let H_d be the complete ternary tree of depth d. Then* $\mathrm{pw}(H_d) \geq d - 1$*.*

Proof. We will prove the following more general statement, which immediately implies the lemma. Let G be a graph and let G_1, G_2, G_3 be three vertex disjoint connected subgraphs of the graph $G \setminus v$ for some $v \in V(G)$ such that v has a neighbor in each of them. Then $\mathrm{pw}(G) \geq 1 + \min_{i=1}^{3} \mathrm{pw}(G_i)$.

Let $\mathcal{W} = (W_1, \ldots, W_s)$ be a path decomposition of G of width $\mathrm{pw}(G)$. As $\mathcal{W}$ induces a path decomposition of each of G_1, G_2 and G_3 it suffices to show that every back containing a vertex of G_i contains a vertex in $V(G) - V(G_i)$ for some $1 \leq i \leq 3$. Let $I_i = \{k : W_k \cap V(G_i) \neq 0\}$ be the set of indices of bags of $\mathcal{W}$ containing vertices of G_i for $1 \leq i \leq 3$. Then I_1, I_2 and I_3 are intervals. Let $I = I_1 \cup I_2 \cup I_3$, $m = \min I$, $M = \max I$. Without loss of generality we assume that $m = \min(I_2 \cup I_3)$ and $M = \max(I_2 \cup I_3)$. We will show that W_k contains a vertex of $V(G) - V(G_1)$ for every $m \leq k \leq M$. As $(W_m \cap V(G_1), \ldots, W_M \cap V(G_1))$ is a path decomposition of G_1 this will imply that $\mathrm{pw}(G_1) \leq \mathrm{pw}(G) - 1$, as desired.

Suppose that there exists $k \notin I_2 \cup I_3$ for some $m \leq k \leq M$. Then without loss of generality $I_2 \subseteq [1, k-1]$, $I_3 \subseteq [k+1, s]$. Finally let $I_v = \{k : v \in W_k\}$. Then I_v is again an interval and $I_v \cap I_2, I_v \cap I_3 \neq \emptyset$. It follows that $k \in I_v$, finishing the proof of the above claim and the lemma. $\square$

The main result of this section is a qualitative converse of Lemma 3.5.

Theorem 3.6 (Bienstock, Robertson,Seymour, Thomas)**.** *Let G be a graph such that* $\mathrm{pw}(G) \geq n$*. Then G contains every tree on at most $n+1$ vertices as a minor.*

We present the proof of Theorem 3.6 due to Diestel. We start with a few definitions. Let H and G be graphs, $\phi : V(H) \to V(G)$ an injective function. We say that a model μ of H in G is *ϕ-rooted* if $\phi(v) \in V(\mu(v))$ for every $v \in V(H)$, and if $X = \mathrm{Im}(\phi)$ we also say that μ is *X-rooted*. We say that G contains an X-rooted H-minor, if G contains an X-rooted model of H.

For a set A of vertices of a graph G, let the *boundary ∂A of A* be the set of vertices in A which are adjacent to vertices in $V(G) - A$. We say that A has an *H-saturated boundary* if $G[A]$ contains a ∂A-rooted model of H.

We say that a sequence $\mathcal{A} = (A_0, A_1, \ldots, A_s)$ is an *A-chain*, if $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_s = A$. If $A \subseteq V(G)$ then the *width of the A-chain* $(A_0, A_1, \ldots, A_s)$ is defined as $\max_{i=1}^{s} |(A_i - A_{i-1}) \cup \partial A_{i-1}|$. We say that A is *n-fractured* if there exists an A-chain of width at most n. The following easy lemma shows the connection between the pathwidth and our new notion.

Lemma 3.7. *Let G be a graph an n a positive integer. Then* $\mathrm{pw}(G) \leq n - 1$ *is and only if $V(G)$ is n-fractured.*

Proof. Given a path decomposition $\mathcal{W} = (W_1, \ldots, W_s)$ define $A_i = W_1 \cup \ldots W_i$ for $i = 0, \ldots, s$. Then $(A_0, \ldots, A_s)$ is a $V(G)$-chain, and $(A_i - A_{i-1}) \cup \partial A_{i-1} \subseteq W_i$ for every $i = 1, \ldots, s$. Thus if the width of $\mathcal{W}$ is at most $n-1$ then the resulting $V(G)$-chain has width at most n.

Conversely, if $(A_0, \ldots, A_s)$ is a $V(G)$-chain of width at most n then

$$(A_1, (A_2 - A_1) \cup \partial A_1, \ldots, (V(G) - A_{s-1}) \cup \partial A_{s-1})$$

is a path decomposition of G of width at most $n-1$. $\square$

A *linkage* $\mathcal{P}$ in a graph G is a collection of pairwise vertex disjoint paths in G. We say that a linkage $\mathcal{P}$ is a (Q,R)-*linkage* for $Q, R \subseteq V(G)$ if every path in $\mathcal{P}$ has one end in Q and the other end in R. We say that $Q, R \subseteq V(G)$ are *linked* if $|Q| = |R|$, and there exists a (Q,R)-linkage $\mathcal{P}$ in G with $|\mathcal{P}| = |Q|$. The following technical lemma is used in the proof of Theorem 3.6.

Lemma 3.8. *Let $A \subseteq B \subseteq V(G)$ be such that B is n-fractured, and ∂A is linked to a subset of ∂B. Then A is n-fractured.*

Proof. Let $(B_0, B_1, \ldots, B_s)$ be a B-chain of width at most n. Let $A_i = B_i \cap A$ for $0 \leq i \leq s$. Clearly, $(A_0, \ldots, A_s)$ is an A-chain. We will show that its width is at most n, which will imply the lemma. We have $A_i - A_{i-1} \subseteq B_i - B_{i-1}$, and so it suffices to show that $|\partial A_{i-1}| \leq |\partial B_{i-1}|$. Let $\mathcal{P}$ be a $(\partial A, \partial B)$-linkage which covers ∂A. Then every vertex $z \in \partial A_{i-1} - \partial B_{i-1}$ lies on some path in $\mathcal{P}$, which in turn intersects ∂B_{i-1} in some vertex $z' \in \partial B_{i-1} - \partial A_{i-1}$. This correspondence gives an injection of $\partial A_{i-1} - \partial B_{i-1}$ into $\partial B_{i-1} - \partial A_{i-1}$, implying the desired inequality. $\square$

Proof of Theorem 3.6. Let T be a tree on $n+1$ vertices. We assume without loss of generality that the graph G with $\mathrm{pw}(G) \geq n$ is connected, and show that $T \leq G$.

Let $T_0 \leq T_1 \subseteq T_2 \subseteq \ldots T_{n+1} = T$ be the sequence of subtrees of T, such that T_i is obtained from T_{i+1} by deleting a leaf. (In particular, $|V(T_i)| = i$.) We choose maximum $1 \leq k \leq n$ such that there exists $A \subseteq V(G)$ with the following properties:

(i) A is n-fractured,

(ii) A has T_k-saturated boundary,

(iii) if $A \subset B \subseteq V(H)$, $A \neq B$ and $|\partial B| \leq k$ then B is not n-fractured.

Such a choice is possible, as $|V(G)|$ is not n-fractured by Lemma 3.7, and thus a maximal n-fractured subset A of $V(G)$ with $|\partial A| = 1$ satisfies the above conditions for $k = 1$.

Let μ be a ∂A-rooted model of T_n in $G[A]$, let v be the unique vertex in $V(T_{k+1}) - V(T_k)$ and let u be the unique neighbor of v in T. Let x be the vertex in $\mu(u) \cap \partial A$, and let y be the neighbor of x in $V(G) - A$. Setting $V(\mu(v)) = \{y\}$ we extend the model of T_k to a model of T_{k+1}. If $k = n$ then $G \geq T$.

Thus we assume that $k < n$ and aim for a contradiction. Let $A' = A \cup \{y\}$. Appending A' to an A-chain of width at most n, we obtain an A'-chain. Moreover, $|(A' - A) \cup \partial A| = k + 1 \leq n$, and so the resulting chain has width at most n. Choose maximal $A'' \supseteq A'$ such that $|\partial A''| \leq k+1$ and A'' is n-fractured. We have, $|\partial A''| = k+1$, by condition (iii) above, as $A \subseteq A''$.

We claim that $\partial A'$ and $\partial A''$ are linked. If not then there exists a separation (X, Y) of G of order at most k such that $A' \subseteq X \subseteq A''$ and ∂X is linked to a subset of $\partial A''$. By Lemma 3.8 the set X is n-fractured once again contradicting condition (iii). This finished the proof of the claim.

Note that any $(\partial A', \partial A'')$-linkage is internally disjoint from A' and is contained in $G[A'']$. We can use such a linkage to extend a $\partial A'$-rooted model of T_{k+1} in $G[A']$ to a $\partial A''$-rooted model of T_{k+1} in $G[A'']$. It follows that A'' satisfies the conditions (i),(ii) and (iii) above with k replaced by $k+1$, a contradiction. $\square$

4 Tree decompositions

Tree decompositions generalize path decompositions and are central to the graph minor theory.

4.1 Definition and basic properties

A *tree decomposition* of a graph G is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W}$ is a family $\{W_t \mid t \in V(T)\}$ of vertex sets $W_t \subseteq V(G)$, such that the following three conditions hold:

(T1) $\cup_{t \in V(T)} W_t = V(G)$,

(T2) every edge of G has both ends in some W_t,

(T3) If $t, t', t'' \in V(T)$ are such that t' lies on the path in T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of a tree decomposition $(T, \mathcal{W})$ is defined as $\max_{t \in V(T)} (|W_t| - 1)$, and the *treewidth* of G is defined as the minimum width of a tree decomposition of G. We denote the treewidth by $\mathrm{tw}(G)$.

Let $(T, \mathcal{W})$ be a tree decomposition of a graph G. For a subtree S of T let $W_S \cup_{t \in V(S)} W_t$. For an edge $e = t_1 t_2 \in E(T)$ let $W_e = W_{t_1} \cap W_{t_2}$. We say that W_e is *an adhesion set* of $(T, \mathcal{W})$. We define the *adhesion* of $(T, \mathcal{W})$ as the maximum size of an adhesion set. Conversely, for $v \in V(G)$ let T_v be the subgraph of T induced by $\{t \in V(T) \mid v \in W_t\}$. We start by deriving a number of direct useful properties of tree decompositions.

Lemma 4.1. *Let $(T, \mathcal{W})$ be a tree decomposition of a graph G. Then T_v is a subtree of T for every $v \in V(G)$.*

Proof. The subgraph T_v is connected by (T3) property of tree decompositions, which can be equivalently restated as

(T3') For every $v \in V(G)$, if $t, t'' \in V(T_v)$ and t' lies on the path in T between t and t'', then $t' \in V(T_v)$. $\square$

Lemma 4.2. *Let $(T, \mathcal{W})$ be a tree decomposition of a graph G. Let e be an edge of T, and let T_1 and T_2 be the two components of $T \setminus e$. Then (W_{T_1}, W_{T_2}) is a separation of G, and $W_{T_1} \cap W_{T_2} = W_e$.*

Proof. Showing that (W_{T_1}, W_{T_2}) is a separation of G is equivalent to showing that $W_{T_1} \cup W_{T_2} = V(G)$, and that no edge of G has one end in $W_{T_1} - W_{T_2}$ and the other in $W_{T_2} - W_{T_1}$. The first condition holds by (T1), and so we suppose for a contradiction that there exist adjacent $v \in W_{T_1} - W_{T_2}$ and $u \in W_{T_2} - W_{T_1}$. Then $T_v \subseteq T_1$, $T_u \subseteq T_2$, but $V(T_v) \cap V(T_u) \neq \emptyset$ by (T2), a contradiction.

To verify the last condition, consider $v \in W_{T_1} \cap W_{T_2}$. Let $t_i \in V(T_i) \cap V(T_v)$ for $i = 1, 2$. Then the path in T joining t_1 and t_2 is also a path in T_v and so both ends of e are vertices of T_v. It follows that $v \in W_e$, as desired. □

Lemma 4.3. *Let $(T, \mathcal{W})$ be a tree decomposition of a graph G. Let $S \subseteq V(G)$. Then*

- *either $S \subseteq W_t$ for some $t \in V(T)$, or*
- *for some $e \in E(T)$ some two vertices of S lie in different components of $G \setminus W_e$.*

In particular, if S is a clique in G then $S \subseteq W_t$ for some $t \in V(T)$.

Proof. Suppose that the second condition of the lemma does not hold. By Lemma 4.2, for every $e \in E(T)$ there exists a component T' of $T \setminus e$ such that $S \subseteq W_{T'}$. We orient e towards T'. As $|E(T)| < |V(T)|$, there exists $v \in V(T)$ such that all edges of T incident to v are oriented towards v. We claim that $S \subseteq W_t$. Suppose not then there exists $s \in S$ such that $v \notin V(T_s)$. Let e be an edge of T incident to v such that T_s and v lie in different components of $T \setminus e$. Then e is oriented away from v by construction, a contradiction. □

Corollary 4.4. *If a graph G is a k-sum of two graphs G_1 and G_2 then*

$$\operatorname{tw}(G) \leq \max(\operatorname{tw}(G_1), \operatorname{tw}(G_2)).$$

In particular, $\operatorname{tw}(G) \leq w$ if and only if G can be obtained from graphs on at most $w + 1$ vertices by k-sums for $k \leq w$.

Proof. Let $(T_i, \mathcal{W}^i)$ be tree decompositions of G_i of width at most w for $i = 1, 2$, such that $V(T_1) \cap V(T_2) = \emptyset$. To prove the first statement of the lemma it suffices to show that G has a tree decomposition of width at most w. Let S_i be the clique in G_i so that G is obtained by identifying S_1 and S_2. By Lemma 4.3 there exists $t_i \in V(T_i)$ such that $S_i \subseteq W^i_{t_i}$. Let T be a tree obtained from $T_1 \cup T_2$ by adding an edge t_1 and t_2, and let $\mathcal{W}$ be a collection of subsets of $V(G)$ obtained from $\mathcal{W}^1 \cup \mathcal{W}^2$ by identifying vertices of S_1 and S_2. It is easy to check that the resulting pair $(T, \mathcal{W})$ is indeed a tree decomposition of G of width at most w.

To prove the second statement of the lemma it now suffices to verify that if $\operatorname{tw}(G) \leq w$ then G can be obtained from graphs on at most $w + 1$ vertices by k-sums for $k \leq w$. The proof is by induction on $|V(G)|$. For the induction step, let $(T, \mathcal{W})$ be a tree decomposition of G of width at most w, chosen with $|V(T)|$ minimum. By adding edges to G if necessary, we may assume that every to vertices of G belonging to the same bag of $\mathcal{W}$ are adjacent. Consider a leaf l of T. Then there exists $v \in V(G)$ such that $V(T_v) = \{l\}$, as otherwise $(T \setminus l, \mathcal{W} - \{W_l\})$ is a tree decomposition of G contradicting the choice of $(T, \mathcal{W})$. Let $G_1 = G[W_l]$ and $G_2 = G[W_{T \setminus l}]$ then $|V(G_1)| \leq w + 1$, $|V(G_2)| < |V(G)|$ and G is a k-sum of G_1 and G_2 for $k \leq w$. Applying the induction hypothesis to G_1 and G_2, we conclude that G satisfies the lemma. □

It follows from Corollary 4.4 and Theorem 2.6 that for $w \leq 2$ we have $\mathrm{tw}(G) \leq w$ if and only if G does not contain K_{w+2} as a minor.

Lemma 4.5. *If $H \leq G$ then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.*

Proof. The treewidth is clearly monotone under taking subgraphs, and so it suffices to show that if H is obtained from a graph G by contracting an edge uv to a new vertex w then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$. Let $(T, \mathcal{W})$ be a tree decomposition of G of width $\mathrm{tw}(G)$. Let $\mathcal{W}'$ be obtained from $\mathcal{W}$ by removing u and v from all the bags, and adding w to the bags that either u or v belonged to. (I.e. $T_w = T_u \cup T_v$ in the resulting decomposition.) It is easy to check that $(T, \mathcal{W}')$ is a tree decomposition of H and its width is equal to the width of $(T, \mathcal{W})$. □

4.2 Brambles, cops and robbers

Next we consider a concept dual to the tree decomposition. A collection of subsets $\mathcal{B}$ of the vertex set of a graph G is called a *bramble* if for all $B, B' \in \mathcal{B}$ the subgraph $G[B \cup B']$ of G induced by $B \cup B'$ is connected. (In particular, $G[B]$ is connected for every $B \in \mathcal{B}$.) We say that a set $S \subseteq V(G)$ is a *cover* of $\mathcal{B}$ if $S \cap B \neq \emptyset$ for every $B \in \mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a cover of $\mathcal{B}$. The *bramble number* $\mathrm{bn}(G)$ of G is the maximum order of the bramble in G.

Grids provide a key example of graphs with unbounded bramble number. An $n \times n$-*grid* $G_{n \times n}$ is a graph with vertex set $V(G_{n \times n}) = \{(i,j) | i,j \in [n]\}$, and edges of the form $(i,j)(i,j+1)$ and $(i,j)(i+1,j)$.[1] Let $B_{i,j} = \{(i,k) | k \in [n]\} \cup \{(k,j) | j \in [n]\}$ be the union of the ith row and jth column of $G_{n \times n}$. Then $\mathcal{B} = \{B_{i,j}\}_{i,j \in [n]}$ is a bramble in $G_{n \times n}$. The order of $\mathcal{B}$ is n, as a set S is a cover of $\mathcal{B}$ if and only if it intersects every row or every column of $G_{n \times n}$. One can construct a slightly larger bramble as follows. Let $\mathcal{B}'$ be a bramble in $G_{(n-1) \times (n-1)}$ constructed as above. Let $P = \{(n,k) | k \in [n]\}$ and $Q = \{(k,n) | k \in [n-1]\}$. Then $\mathcal{B}' \cup \{P, Q\}$ is a bramble in $G_{n,n}$ of order $n+1$. We will see that $\mathrm{bn}(G_{n \times n}) = n+1$.

Brambles generalize complete subgraphs and models of complete graphs as obstructions to tree decompositions in a sense captured in the following lemma.

Lemma 4.6. *Let $(T, \mathcal{W})$ be a tree decomposition of a graph G and let $\mathcal{B}$ be a bramble in G. Then W_t is a cover of $\mathcal{B}$ for some $t \in V(T)$. In particular,* $\mathrm{bn}(G) \leq \mathrm{tw}(G) + 1$.

[1] In other words, $G_{n \times n}$ is *a Cartesian product* of two paths on n vertices.

Proof. Our argument mirrors the proof of Lemma 4.3. Suppose that W_e is not a cover of $\mathcal{B}$ for every $e \in E(T)$. Then for every $e \in E(T)$ there exists a unique component T' of $T \setminus e$ such that $B \subseteq W(T') - W_e$. We orient e towards T'. Let $t \in V(T)$ be such that all edges of T incident to t are oriented towards t. Suppose for a contradiction that W_t is not a cover of $\mathcal{B}$ then there exists $B \in \mathcal{B}$ such that $B \subseteq W_{T'} - W_t$ for some component T' of $T \setminus t$. Let e be an edge of T joining t to T'. Then e is oriented away from t by construction, a contradiction. □

The following duality characterization of treewidth by Seymour and Thomas, strengthens Lemma 4.6 and provides an important tool in the area.

Theorem 4.7. *For every graph G*

$$\mathrm{bn}(G) = \mathrm{tw}(G) + 1.$$

The notion of bramble has a fun interpretation in terms of a cops-and-robbers game on a graph. The game is played by a robber and k cops and all the participants are visible to each other. At any point in the game, the robber stands at a vertex of a graph not occupied by any cop. He can travel along a path in a graph arbitrarily fast, he is not allowed however to run through a cop. Cops travel in helicopters, that is a cop can be temporarily removed from the game and land on a new vertex. The robber will see the helicopter approaching and can avoid capture by moving, if possible. The robber's objective is to avoid capture indefinitely, and cops' is to land a helicopter on a robber. We say that cops capture robber using *a monotone strategy* if no vertex vacated by a cop is revisited during the course of the chase.

Lemma 4.8. *If $tw(G) \leq k-1$ then k cops can capture a robber on a graph G using a monotone strategy.*

Proof. Let $(T, \mathcal{W})$ be a tree decomposition of G such that $|W_t| \leq k$ for every $t \in V(T)$. Fix a root r of T. For a vertex $x \in V(T)$, let T^x denote the subtree of T rooted at x, i.e. the subtree induced by the set of all vertices $y \in V(T)$ such that $y \geq x$ in the tree order of (T, r). Similarly, for an edge $e \in E(T)$, with ends x and y such that $y \geq x$ in the order of (T, r), denote by T^e the subtree rooted at e, that is the subtree induced by $V(T^y) \cup \{x\}$.

With the notation in place, we are ready to describe the strategy. Cops start on the vertices of W_r. For each i, at the end of ith step of the game the cops will occupy all the vertices of W_{x_i} for some $x_i \in V(T)$, while the robber will be confined to the set of vertices $S_e = \{v \in V(G) | T_v \subseteq T^e\}$ for some $e \in E(T^{x_i})$ incident to x_i. Moreover, we will maintain $x_i \leq x_{i+1}$ for every i, where the comparison is once again in the order of (T, r).

The strategy is not difficult to implement. If at the end of the ith step the robber is confined to the set S_e as above, denote the second end of e by x_{i+1}. Move the cops from W_{x_i} to $W_{x_{i+1}}$ while the cops positioned on W_e stay in place. Clearly, the robber can not escape from S_e. It remains to note that if $e_1, e_2 \in E(T^{x_{i+1}})$ are two edges incident to x_{i+1} then the vertices of S^{e_1} and S^{e_2} belong to a different component of $G[S_e]$. Thus the robber will be confined to the set of vertices $S_{e'}$ for some $e' \in E(T^{x_{i+1}})$ incident to x_{i+1}, as claimed.

The above strategy is monotone and terminates in the cop victory after the number of steps not exceeding the depth of (T, r). $\square$

The next definition describes a potential strategy for a robber. For $X \subseteq V(G)$ an X-*flap* is a component of $G \setminus X$. A *haven* β of order k in G is a function assigning to every $X \subseteq V(G)$ with $|X| < k$ an X-flap $\beta(X)$, such that $\beta(X) \cup \beta(Y)$ is connected for every pair $X, Y \subseteq V(G)$ with $|X|, |Y| < k$. If a graph G contains a haven of order k then a robber can escape $k-1$ cops by always positioning himself in the set $\beta(X)$, where X is the set of vertices occupied by cops. Havens are brambles are closely related as seen in the following lemma.

Lemma 4.9. *A graph G contains a haven of order k if and only if G contains a bramble of order k.*

Proof. If β is a haven of order k in G then $\{\beta(X) | X \subseteq V(G), |X| < k\}$ is a bramble. Conversely, if $\mathcal{B}$ is a bramble of order k in G then for every $X \subseteq V(G), |X| < k$ there exists a unique component of $G \setminus X$ such that $\beta(X)$ contains some $B \in \mathcal{B}$, and the resulting function βX is a haven of order k. $\square$

The results of Theorem 4.6 and Lemmas 4.8 and 4.9 are summarized in the following corollary, which gives a number of equivalent definitions of treewidth.

Corollary 4.10. *For a graph G and an integer $k \geq 1$ the following are equivalent:*

- $\mathrm{tw}(G) \leq k - 1$,
- $\mathrm{bn}(G) \leq k$,
- *G foes not contain a haven of order $k + 1$,*
- *k cops can capture a visible robber,*
- *k cops can capture a visible robber using a monotone strategy.*

4.3 Tangles

Finally, we introduce another notion, closely related to treewidth, which can be considered as an abstraction of the concepts of a clique minor or a bramble as A *tangle* $\mathcal{T}$ of order $k \geq 1$ in G is a collection of separations of G, satisfying the following:

(i) for every separation (A, B) of G of order $< k$ either $(A, B) \in \mathcal{T}$, or $(B, A) \in \mathcal{T}$,

(ii) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $G[A_1] \cup G[A_2] \cup G[A_3] \neq G$.

We will need the following technical lemmas, concerning separations.

Lemma 4.11. *Let (A, B) and (C, D) be a pair of separations of the graph G of order less than k. Then $(A \cap D, B \cup C)$ and $(A \cup D, B \cap C)$ are also separations of G, and at least one of them has order less than k.*

Proof. It is easy to check that $(A \cap D, B \cup C)$ and $(A \cup D, B \cap C)$ are indeed separations of G. The second statement of the lemma follows from the next inequality:

$$|(A \cap D) \cap (B \cup C)| + |(A \cup D) \cap (B \cap C)| \leq |A \cap B| + |C \cap D|. \tag{1}$$

To verify (1) note that every vertex of G contributes at least as much to the right side of the inequality as to the left side. $\square$

Lemma 4.12. *For an integer $k \geq 1$ and a graph G, let $\mathcal{T}$ be a collection of separations of order $< k$ in G satisfying (i) in the definition of the tangle, but not (ii). Suppose further, that*

(i') *if (A, B),(A', B') are separations of G of order $< k$ such that $A' \subseteq A, B \subseteq B'$ and $(A, B) \in \mathcal{T}$, then $(A', B') \in \mathcal{T}$.*

Then there exist $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ such that $G[A_1] \cup G[A_2] \cup G[A_3] = G$, $A_1 \cup A_2 = B_3$, $A_1 \cup A_3 = B_2$, and $A_2 \cup A_3 = B_1$.

Proof. Choose $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ violating (ii) such that $|A_1| + |A_2| + |A_3| - |B_1| - |B_2| - |B_3|$ is minimum. Note that $B_3 \subseteq A_1 \cup A_2$, $B_2 \subseteq A_1 \cup A_3$, and $B_1 \subseteq A_2 \cup A_3$.

Suppose for a contradiction that $A_1 \not\subseteq B_2$. By Lemma 4.11 applied to the separations (A_1, B_1) and (B_2, A_2), the order of one the separations $(A_1 \cap B_2, A_2 \cup B_1)$ and $(A_2 \cap B_1, A_1 \cup B_2)$ is less than k. Suppose first, that $(A_1 \cap B_2, A_2 \cup B_1)$ is such a separation. By (i') $(A_1 \cap B_2, A_2 \cup B_1) \in T$. Moreover, $G[A_1] \setminus E(G[A_2]) \subseteq G[A_1 \cap B_2]$. Therefore, $G[A_1 \cap B_2] \cup G[A_2] \cup G[A_3] = G$. However, $|A_1 \cap B_2| < |A_1|$. Thus $(A_1 \cap B_2, A_2 \cup B_1), (A_2, B_2), (A_3, B_3)$ contradicts our choice of the triple of separations violating (ii).

The case when $(A_2 \cap B_1, A_1 \cup B_2)$ is a separation of order less than k, instead, is similar. In this case $(A_1, B_1), (A_2 \cap B_1, A_1 \cup B_2), (A_3, B_3)$ contradicts our choice as $|A_1 \cup B_2| > |B_2|$. $\square$

Lemma 4.13. *Let $\mathcal{T}$ be a tangle in the graph G. Then for every $(A, B) \in \mathcal{T}$ there exists $(A', B') \in \mathcal{T}$ such that $A \subseteq A'$, $B' \subseteq B$, and $G[B' - A']$ is connected.*

Proof. Suppose that for some $(A, B) \in \mathcal{T}$ the lemma does not hold. Let $C_1, C_2, \ldots, C_k$ be the vertex sets of components of $G[B - A]$. Choose $i \leq k$ maximum such that $(A \cup C_1 \cup \ldots \cup C_{i-1}, B - (C_1 \cup \ldots \cup C_{i-1})) \in \mathcal{T}$. Clearly, $i < k$, and $(B - (C_1 \cup \ldots \cup C_i), A \cup C_1 \cup \ldots \cup C_i) \in \mathcal{T}$. Further, by our assumption, $((A \cap B) \cup C_i, A \cup (B - C_i)) \in \mathcal{T}$. However, the subgraphs of G induced by $A \cup C_1 \cup \ldots \cup C_{i-1}$, $B - (C_1 \cup \ldots \cup C_i$ and $(A \cap B) \cup C_i$ cover G, contradicting the definition of a tangle.

$\square$

The tangle number $\mathrm{tn}(G)$ is the maximum order of a tangle in G.

The relation between the tangle number and the treewidth of a graph is captured in the following theorem of Robertson and Seymour.

Theorem 4.14. *Let G be a graph with $\mathrm{tn}(G) \geq 2$. Then the treewidth $\mathrm{tw}(G)$ of G satisfies*

$$\mathrm{tn}(G) \leq \mathrm{tw}(G) + 1 \leq \frac{3}{2}\,\mathrm{tn}(G).$$

Proof. By Theorem 4.7 it suffices to show that

$$\mathrm{tn}(G) \leq \mathrm{bn}(G) \leq \frac{3}{2}\,\mathrm{tn}(G).$$

We start by showing that $\mathrm{bn}(G) \leq \frac{3}{2}\,\mathrm{tn}(G)$. That is, we show if G contains a bramble $\mathcal{B}$ of order $3k/2$ then G contains a tangle of order k. We define a tangle $\mathcal{T}$ as follows. For every separation (X,Y) of G of order $< k$, there exists $B \in \mathcal{B}$ such that $B \subseteq Y - X$ or $B \subseteq X - Y$. In the first case, let $(X,Y) \in \mathcal{T}$, and otherwise $(Y,X) \in \mathcal{T}$. Clearly, the collection of separations defined this way satisfies condition (i) in the definition of a tangle and the condition (i') of Lemma 4.12. It remains to show that $\mathcal{T}$ satisfies (ii). If not, then by Lemma 4.12 there exist $(A_1,B_1),(A_2,B_2),(A_3,B_3) \in \mathcal{T}$ such that $G[A_1] \cup G[A_2] \cup G[A_3] = G$, and $A_1 \cup A_2 = B_3$, $A_1 \cup A_3 = B_2$, and $A_2 \cup A_3 = B_1$. Let $X_i = A_i \cap B_i$ for $i = 1,2,3$, and let $X = X_1 \cup X_2 \cup X_3$. Every vertex of X belongs to at least two of the sets A_1, A_2 and A_3, and therefore to at least two of the sets X_1, X_2 and X_3. Thus $|X| \leq 3(k-1)/2 < 3k/2$. It follows that there exists $B \in \mathcal{B}$ such that $B \cap X = \emptyset$. Thus $B \subseteq B_1 \cap B_2 \cap B_3 = \emptyset$, a contradiction.

It remains to show that $\mathrm{bn}(G) \geq \mathrm{tn}(G)$. Let $\mathcal{T}$ be a tangle of order k in G. Let

$$\mathcal{B} = \{B - A | (A,B) \in \mathcal{T}, G[B-A] \text{ is connected}\}.$$

It is easy to check that $\mathcal{B}$ is a bramble. It remains to show that the order of $\mathcal{B}$ is at least k. If not let X be a cover of $\mathcal{B}$ with $|X| < k$. Then $(X, V(G)) \in \mathcal{T}$ and by Lemma 4.13 there exists $(A,B) \in \mathcal{T}$ such that $X \subseteq A$ and $G[B-A]$ is connected. Therefore, $B - A \in \mathcal{B}$ and $(B-A) \cap X = \emptyset$, contradicting the choice of X. □

5 Applications of tree decompositions

5.1 Algorithms on graphs of bounded treewidth

If $P \neq NP$ then there are no polynomial time algorithms to compute the following parameters of a graph G:

- the independence number $\alpha(G)$,or equivalently the clique number $\omega(G)$,
- $\chi(G)$,
- $\mathrm{tw}(G)$.

However, if $\mathrm{tw}(G) \leq k$, then there is polynomial time algorithm that finds a tree decomposition of G of width $O(k^4)$.

Theorem 5.1 (Arnborg, Proskurowski, '89). *Given a graph G and a set $Z \subseteq V(G)$, $|Z| \leq k+1$ with k fixed, we want to compute some information $P(G,Z)$. Suppose that*

(1) $P(G,Z)$ can be computed in constant time if $|V(G)| \leq k+1$,

(2) if $Z' \subseteq Z$, then $P(G, Z')$ can be computed from $P(G, Z)$ in constant time,

(3) if (A, B) is a separation of G such that $A \cap B \subseteq Z$, then $P(G, Z)$ can be computed from $P(G[A], Z \cap A)$ and $P(G[B], Z \cap B)$ in constant time.

Then, if a tree decomposition of G of width $\leq k$ is given, $P(G, \emptyset)$ can be computed in time linear in $|V(G)|$.

Proof. Let $(T, \mathcal{W})$ be a tree decomposition of G of width $\leq k$. Assume that T is rooted at r. For $v \in V(T)$, let G_v be the subgraph of G induced by the union of bags W_t taken over all $t \geq v$ in the tree order of (T, r). We will recursively compute $P(G_v, W_v)$. Once $P(G_r, W_r)$ is computed, we obtain $P(G, \emptyset)$ from it in constant time by (2), as $G = G_r$.

If v is a leaf, $P(G_v, W_v)$ can be computed by (1).

If v has children $u_1, u_2, \ldots, u_d$, we compute $P(G_v, W_v)$ from $\{P(G_{u_i}, W_{u_i})\}$ in time linear in d, as follows. Let G^i be the subgraph of G induced by $V(G_{u_i}) \cup W_v$. Then $P(G^i, W_v)$ can be computed from $P(G_{u_i}, W_{u_i})$ in the following way: By (2), we can compute $P(G_{u_i}, W_{u_i} \cap W_v)$, by (1), we can compute $P(G[W_v], W_v)$, and by (3), we can compute $P(G^i, W_v)$. Applying (3) repeatedly we compute $P(G_v, W_v)$ from $\{P(G^i, W_v)\}$. □

Corollary 5.2. *For fixed k, given a tree decomposition of G of width $\leq k$, we can compute $\alpha(G)$ in linear time.*

Proof. For $Y \subseteq Z \subseteq V(G)$, let $\alpha_{YZ}(G)$ be the maximum size of an independent set S of G such that $S \cap Z = Y$. Let $P(G, Z) = (\alpha_{YZ}(G) | Y \subseteq Z)$, then conditions (1) and (2) of Theorem 5.1 hold trivially. For (3),

$$\alpha_{YZ}(G) = \alpha_{Y \cap A, Z \cap A}(G[A]) + \alpha_{Y \cap B, Z \cap B}(G[B]) - |Y \cap A \cap B|.$$

Thus $P(G, \emptyset) = (\{alpha_{YZ}(G) | Y \subseteq Z)$ by Theorem 5.1. □

5.2 Erdős-Pósa Property

Let $\mathcal{H}$ be a class of graphs (closed under isomorphism). We say that $\mathcal{H}$ has *Erdős-Pósa property* if for every integer $k \geq 1$ there exists $f_{\mathcal{H}}(k) = f(k)$ such that for every graph G either G contains k vertex disjoint subgraphs in $\mathcal{H}$ or $G \setminus X$ contains no subgraphs in $\mathcal{H}$ for some $X \subseteq V(G), |X| \leq f(k)$.

Consider the example of $\mathcal{H} = \{K_2\}$. Then k vertex disjoint subgraphs in $\mathcal{H}$ is a matching of size k, and $G \setminus X$ has no subgraphs in $\mathcal{H}$ for $|X| \leq f(k)$ is equivalent to G having a vertex cover of size $f(k)$. Thus $\mathcal{H}$ has Erdős-Pósa property with a $f_{\mathcal{H}}(k) = 2(k-1)$. Indeed, the vertex set of any maximal matching of G is a vertex cover. In fact, for any graph H the class $\mathcal{H}$ of all graphs isomorphic to H has the Erdős-Pósa property with $f(k) = |V(H)|(k-1)$. The following theorem will be proved in the next section.

Theorem 5.3 (Robertson, Seymour, the Grid Theorem)**.** *For every planar graph H, there exists w such that if $G \not\geq H$, then $\mathrm{tw}(G) \leq w$.*

Note that Theorem 5.3 does not hold for non-planar graphs since there exists planar graphs with arbitrarily large treewidth.

In the next theorem we show that the class of graphs containing a fixed planar graph as a minor has Erdös-Pósa property. We denote by kH the union of k vertex disjoint copies of the graph H.

Theorem 5.4. *For every fixed planar connected graph H and integer k, there exists $f(k)$ such that for every graph G*

- *either G contains vertex disjoint subgraphs $G_1, G_2, \ldots, G_k$ such that $G_i \geq H$ for $i = 1, 2, \ldots, k$ and we denote it as $G \geq kH$;*
- *or $G \setminus X \ngeq H$ for some $X \subseteq V(G), |X| \leq f(k)$.*

Proof. For fixed H, the proof is by induction on k. The base case is trivial.

For the induction step suppose that $f(k)$ exists, and let w_k be such that if a graph G does not contain the union of $(k+1)$ vertex disjoint copies of H as a minor, then $\mathrm{tw}(G) < w_k$. Such a w_k exists by Theorem 5.3. We will show that $f(k+1) \leq 3w_k + 2f(k)$ satisfies the conditions of the theorem.

Let G be a graph such that $G \ngeq (k+1)H$. Suppose first that there exists a separation (A, B) of G of order at most w_k such that $G[A-B]$ and $G[B-A]$ both contain an H minor. If $G \ngeq (k+1)H$, then $G[A-B], G[B-A] \ngeq kH$. Thus there exists $X_1 \subseteq A-B, X_2 \subseteq B-A$ such that $G[A-B] \setminus X_1, G[B-A] \setminus X_2 \ngeq H$¿ Let $X = X_1 \cup X_2 \cup (A \cap B)$. Then $G \setminus X \ngeq H$, and $|X| \leq 2f(k) + w_k$.

Things are even better if $G[A-B], G[B-A] \ngeq H$ for some separation (A, B) as above, as $G \setminus (A \cap B) \ngeq H$. Thus we may assume that for every (A, B) separation of G of order $\leq w_k$, exactly one of $G[A-B], G[B-A]$ has an H minor.

Let $\mathcal{T}$ consist of separations (A, B) of order at most k such that $G[B-A] \geq H$. If $\mathrm{tw}(G) \geq w_k$, then $G \geq (k+1)H$. By the choice of w_k we have $\mathrm{tw}(G) < w_k$ and thus $\mathcal{T}$ is not a tangle. However, it satisfies the first condition in the definition of a tangle, by our assumption. Thus there exist $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ such that $G[A_1] \cup G[A_2] \cup G[A_3] = G$. In particular, $A_1 \cup A_2 \cup A_3 = V(G)$. Let $X = (A_1 \cap B_1) \cup (A_2 \cap B_2) \cup (A_3 \cap B_3)$. Then $|X| \leq 3w_k$. Sand $G \setminus X$ has no H minor. Indeed if C is a connected component of $G \setminus X$ such that $C \geq H$ minor, then $V(C) \subseteq (B_1 - A_1) \cap (B_2 - A_2) \cap (B_3 - A_3) = \emptyset$, a contradiction. □

Note that if H is non-planar, then there is no analogue of Theorem 5.4.

5.3 Balanced separations

A separation (A, B) of G with $|V(G)| = n$ is *balanced* if

$$|A - B|, |B - A| \leq \frac{2n}{3},$$

or equivalently,

$$|A|, |B| \geq \frac{n}{3}.$$

Theorem 5.5. *If* $\mathrm{tn}(G) = k$*, the* G *contains a balanced separation of order* k*.*

Proof. Suppose there is no such separation. Let $|V(G)| = n$. Then for every separation (A, B) of G of order at most k, either $|A| < \frac{n}{3}$ or $|B| < \frac{n}{3}$. Define $\mathcal{T}$ by letting $(A, B) \in \mathcal{T}$ if $|A| < \frac{n}{3}$ and $(B, A) \in \mathcal{T}$, otherwise. Since $\mathrm{tn}(G) = k$, $\mathcal{T}$ is not a tangle, so there exists $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ such that $A_1 \cup A_2 \cup A_3 = V(G)$, a contradiction. $\square$

Define *sn*(G) to be the smallest k such that every subgraph H of G has a balanced separation of order at most k. The parameter $\mathrm{sn}(G)$ is closely related to treewidth.

Theorem 5.6 (Dvořák, Norin)**.** *For any graph* G*,* $\mathrm{sn}(G) \leq \mathrm{tw}(G) \leq 105\,\mathrm{sn}(G)$*.*

6 The grid theorem

Our main goal in this section is to prove the following theorem.

Theorem 6.1. *For every* n*, there exists* N *such that if* $tw(G) \geq N$*, then* G *contains an* $n \times n$ *grid* $H_{n\times n}$ *as a minor.*

Note that Theorem 6.1 implies Theorem 5.3 as for every planar graph G, there exists n such that $G \leq H_{n\times n}$.

The proof informally proceeds by successively proving that a graph of large treewidth contains an large (but much smaller) increasingly structured object in the following sequence.

$$\text{Tree-width} \to \text{Tangle} \to \text{Mesh} \to \text{Fence} \to \text{Grill} \to \text{Grid}$$

Theorem 4.14 accomplishes the first step. The next step is to obtain a mesh.

6.1 From a tangle to a mesh

An (n, m)*-mesh* in a graph G is a linkage $\mathcal{P}$ of order n such that for all $P_i, P_j \in \mathcal{P}$, there is a linkage $\mathcal{Q}_{ij}$ in G of order m such that every path in $\mathcal{Q}_{ij}$ has one end in P_i and the other end in P_j, and is otherwise disjoint from $\mathcal{P}$.

Let G be a graph. A set $Z \subseteq V(G)$ is *properly linked* if for any $X, Y \subseteq Z$ with $|X| = |Y|$, there exists an (X, Y)-linkage $\mathcal{Q}$ of order $|X|$ that is internally disjoint from Z.

Lemma 6.2. *Let* w *be an integer,* G *a graph with* $tn(G) \geq w + 1$*. There exists a separation of* (A, B) *of* G *such that*

- $|A \cap B| = w$*,*
- $G[A]$ *contains an* $(A \cap B)$*-rooted model of a path on* w *vertices,*
- $A \cap B$ *is properly linked in* $G[B]$*.*

Proof. Let $\mathcal{T}$ be a tangle in G of order $\geq w + 1$. Let (A, B) be a separation of G of order $1 \leq k \leq w$ such that the following holds:

a) $(A, B) \in \mathcal{T}$,

b) $G[A]$ contains an $(A \cap B)$-rooted model of a path on k vertices,

c) There is no separation $(A', B') \in \mathcal{T}$ of order $< k$ such that $A \subseteq A', B \supseteq B'$.

d) Subject to the previous conditions, $|A| - |B|$ is maximum.

We claim such a separation has the desired properties.

Claim 1: Every vertex of $(A \cap B)$ has a neighbor in $B - A$.

Suppose $v \in A \cap B$ does not. Let $B' = B - \{v\}$, $A' = A$, then (A', B') is a separation of order $k - 1$ in $\mathcal{T}$, violating the choice of (A, B).

Claim 2: There is no separation $(A', B') \in T$, $(A', B') \neq (A, B)$ of order k such that $A \subseteq A', B \supseteq B'$.

If such a separation exists, it would violate the choice of (A, B). One only needs to verify that $G[A']$ contains an $(A' \cap B')$-rooted model of a path, but since there is a linkage of order k from $A \cap B$ to $A' \cap B'$, by the property c) of the separation (A, B) one can extend the $(A \cap B)$-rooted model of a path in $G[A]$ to such a model.

Claim 3: $k = w$.

If $k < w$, let μ be a model of a path rooted on $A \cap B$. Let $u \in A \cap B$ be a vertex in a bag of μ to an endpoint of the path. By Claim 1 there exists $v \in B - A$ which is a neighbor of u. Let $A' = A \cup \{v\}, B' = B$. Then (A', B') violates the choice of (A, B) by Claim 2.

It remains to check that $A \cap B$ is properly linked in $G[B]$. Suppose not. Then exists $X, Y \subseteq A \cap B$, $|X| = |Y| = l$ disjoint, such that there is no (X, Y)-linkage in $G[B]$ that is internally disjoint from $A \cap B$. Let $Z = (A \cap B) - (X \cup Y)$. By Theorem 1.4, there exists a separation (C, D) of $G[B] \setminus Z$ with $X \subseteq C, Y \subseteq D$, $|C \cap D| < l$. Furthermore, assume its order is chosen to be minimal.

Consider the separation $(A \cup D, C \cup Z)$. Its order is

$$\begin{aligned} |(A \cup D) \cap (C \cup Z)| &= |Z| + |C \cap D| + |X| \\ &< |X| + |Y| + |Z| \\ &= |A \cap B| = k. \end{aligned}$$

If $(A \cup D, C \cup Z) \in \mathcal{T}$, it violates the choice of (A, B). So we must have $(C \cup Z, A \cup D) \in \mathcal{T}$. By symmetry, $(D \cup Z, A \cup C) \in \mathcal{T}$. But $G[C \cup Z] \cup G[D \cup Z] \cup G[A] = G$, a contradiction. $\square$

Corollary 6.3. *If $tw(G) \geq \frac{3}{2}mn$ then some minor of G contains an (n, m)-mesh.*

Proof. Let $w = mn$. By Theorem 4.14, as $tw(G) \geq \frac{3}{2}w$, we have $tn(G) \geq w + 1$. Let (A, B) be a separation of G as in Lemma 6.2. By contracting the edges of the model of a path, we may assume $A \cap B$ induces a path P. Since $nm \leq w$, we can find disjoint subpaths $P_1, \ldots, P_n$ of P each of length at least m. As $A \cap B$ is properly linked in $G[B]$, for any P_i, P_j, we can find m disjoint paths from P_i, P_j that are internally disjoint from P. Therefore $\mathcal{P} = \{P_1, \ldots, P_n\}$ forms an (n, m)-mesh. $\square$

6.2 Cleaning up the mesh.

A (k,l)-*fence* in a graph G is a pair of linkages $(\mathcal{P},\mathcal{Q})$ in G such that

- $|\mathcal{P}| = k, |\mathcal{Q}| \geq l$,
- $V(P) \cap V(Q) \neq \emptyset$ for every $P \in \mathcal{P}$, $Q \in \mathcal{Q}$,
- for some $(A,B) \subseteq V(G)$ such that $\mathcal{Q}$ is an (A,B)-linkage, there exists no (A,B)-linkage of order $|\mathcal{Q}|$ in $\mathcal{P} \cup \mathcal{Q} \setminus e$ for every $e \in E(\mathcal{Q}) - E(\mathcal{P})$.

Lemma 6.4. *For all k,l there exist n,m such that if a graph G contains an (n,m)-mesh then G contains a (k,l)-fence or $G \geq K_n$.*

Proof. Given an (n,m)-mesh $\mathcal{P}$ our goal is to repeatedly replace the linkages $\mathcal{Q}_{ij}$ by vertex disjoint paths R_{ij}. If we can accomplish this goal then G contains K_n as a minor. (Contracting the paths to single vertices and paths R_{ij} to edges produces such a minor.) We will show that if at any step we fail then G contains a (k,l)-fence.

The following definition is needed to make the above outline precise. Let $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$, and let $S \subseteq [n]^{(2)}$ be a collection of pairs of indices of paths in $\mathcal{P}$. We say that $\mathcal{P}$ together with a collection of $(V(P_i), V(P_j))$-linkages $\mathcal{Q}_{ij}$ internally disjoint from $V(\mathcal{P})$ is an S-*cleaned* (n,m)-mesh if

- $|\mathcal{Q}_{ij}| = m$ for $ij \notin S$, and $|\mathcal{Q}_{ij}| = 1$, otherwise, and
- $\mathcal{Q}_{ij}$ is vertex disjoint from $\mathcal{Q}_{i'j'}$ for every $\{i,j\} \in S$ and $\{i',j'\} \in [n]^{(2)}$, $\{i,j\} \neq \{i',j'\}$.

Note that -cleaned (n,m)-mesh if simply an (n,m)-mesh, and $[n]^{(2)}$-cleaned mesh yields a K_n minor, as discussed above. Thus the next claim will imply the lemma.

Claim: For all k,l,n and m there exists m' satisfying the following. If a graph G contains an S-cleaned (n,m')-mesh $(\mathcal{P}, \{\mathcal{Q}_{ij}\})$ and $S' = S \cup \{p,q\}$ for some $\{p,q\} \in [n]^{(2)} - S$ then either G contains an S'-cleaned (n,m)-mesh, or G contains a (k,l)-fence.

Let $m' = m + l$ and $m'' = n^2 l \binom{m}{l}$. By reducing the size of the linkages we assume that $|\mathcal{Q}_{ij}| = m'$ for $ij \notin S'$, while $|\mathcal{Q}_{pq}| = m''$. Let us further choose $\mathcal{Q}_{ij}$ for $ij \notin S'$, so that subject to the above properties $E(\mathcal{Q}_{ij}) - E(\mathcal{Q}_{pq})$ is minimal. If there exists $Q \in \mathcal{Q}_{pq}$ such that Q is vertex disjoint from at least m paths in $\mathcal{Q}_{ij}$ for every $ij \notin S'$ then G contains an S'-cleaned (n,m)-mesh, obtained by replacing $\mathcal{Q}_{pq}$ by Q, and $\mathcal{Q}_{ij}$ by the corresponding m paths for $ij \notin S'$. Thus we assume that no such path Q exists. By the choice of m'' it follows that there exists $\mathcal{Q}' \subseteq \mathcal{Q}_{ij}$ for$ij \notin S'$ and $\mathcal{P}' \subseteq \mathcal{Q}_{pq}$ such that

- $|\mathcal{P}'| = k$, $\mathcal{Q}' \geq m' - m \geq l$,
- $V(P) \cap V(Q) \neq \emptyset$ for every $P \in \mathcal{P}'$, $Q \in \mathcal{Q}'$,
- $V(P) \cap V(Q) = \emptyset$ for every $P \in \mathcal{P}'$, $Q \in \mathcal{Q}_{ij} - \mathcal{Q}'$.

We claim that $(\mathcal{P}, \mathcal{Q})$ ia a (k, l)-fence. It suffices to verify that there exists no $(V(\mathcal{Q}') \cap V(P_i), V(\mathcal{Q}') \cap V(P_j))$-linkage of order $|\mathcal{Q}'|$ in $\mathcal{P}' \cup \mathcal{Q}' \setminus e$ for every $e \in E(\mathcal{Q}') - E(\mathcal{P}')$. If such a linkage $\mathcal{Q}''$ exists then $\mathcal{Q}'_{ij} = \mathcal{Q}'' \cup (\mathcal{Q}_{ij} - \mathcal{Q}')$ is a $(V(P_i), V(P_j))$- linkage of order m' such that $E(\mathcal{Q}'_{ij}) \subseteq E(\mathcal{Q}_{pq}) \cup E(\mathcal{Q}_{ij}) - \{e\}$, contradicting the choice of e. $\square$

We improve a fence to a grill in the next step. An (r, l)-*grill* is a pair of linkages $(\mathcal{P}, \mathcal{Q})$ such that

- $|\mathcal{P}| = r$, $\mathcal{Q} = l$, and
- there exists an ordering $(P_1, P_2, \ldots, P_r)$ of paths in $\mathcal{P}$ so that every $Q \in \mathcal{Q}$ can be partitioned into subpaths $Q^1, \ldots Q^r$, appearing along Q in order, so that $V(P_i) \cap V(Q^j) \neq \emptyset$ if and only if $i = j$.

Lemma 6.5. *For all r, l there exist k such that if a graph G contains an (k, l)-fence then G contains an (r, l)-grill.*

Proof. We show that $k = (2l-1)(r+1)$ satisfies the lemma. Let $(\mathcal{P}, \mathcal{Q})$ be a (k, l)-fence in G. We assume without loss of generality that $|Q| = l$, and that $E(G) = E(\mathcal{P}) \cup E(\mathcal{Q})$. Let (A, B) satisfy the third property in the definition of a fence. Let $Q \in \mathcal{Q}$ be a path with ends $a \in A$ and $b \in B$. For each $1 \leq i \leq r+1$, let e_i be an edge of Q with ends x_i and y_i so that a, x_i, y_i and b appear along Q in order, $e_i \notin E(\mathcal{P})$ and exactly $(2l-1)i$ paths in $\mathcal{P}$ intersect the path $Q[a, x_i]$. By the definition of a (k, l)-fence there exists no (A, B)-linkage in $G \setminus e_i$ of order l, and thus by Theorem 1.4 there exists a separation (X_i, Y_i) of $G \setminus e_i$ of order $l-1$ such that $A \subseteq X_i, B \subseteq Y_i$. Choose such a separation so that $|X_i| - |Y_i|$ is maximum for every $1 \leq i \leq r$. Then for all $i < j$ we have $X_i \subseteq X_j$ and $Y_i \supseteq Y_j$. Indeed, otherwise, a separation $(X_i \cup X_j, Y_i \cap Y_j)$ of $G \setminus e_j$ violates the choice of (X_j, Y_j). (The order of $(X_i \cup X_j, Y_i \cap Y_j)$ is $l-1$ by inequality (1) in the proof of Lemma 4.11, as the order of separation $(X_i \cap X_j, Y_i \cup Y_j)$ of $G \setminus e_i$ is at least $l-1$.)

By the choice of e_i at most $(2l-1)i + l - 1$ paths in $\mathcal{P}$ intersect X_i, however at least $(2l-1)i - l + 1$ paths in $\mathcal{P}$ are contained in X_{i+1}. Thus there exists a path $P_i \in \mathcal{P}$ such that $V(P_i) \subseteq X_{i+1} - X_i$ for each $1 \leq i \leq r$, where we define $X_0 = \emptyset$ for convenience. Let $\mathcal{P}' = (P_1, \ldots, P_r)$ then $(\mathcal{P}', \mathcal{Q})$ is an (r, l)-grill. $\square$

Finally, a grill yields a grid.

Lemma 6.6. *For all n there exist r, l such that if a graph G contains an (r, l)-grill then G contains an $n \times n$-grid H_n as a minor.*

Proof. Let $l = n^{2n}$, and let $r = 2^{l^2}$. Let $(\mathcal{P}, \mathcal{Q})$ be an (r, l)-grill in G, let $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_l\}$. For each $1 \leq i \leq r$ define the graph H_i as follows. Let $V(H_i) = [l]$ and $\{s, t\} \in E(H_i)$ if and only if there exist a path $R \subseteq P_i$ such that R has one end in $V(Q_s)$ another end in $V(Q_t)$ and is otherwise disjoint from $\mathcal{Q}$.

We claim that H_i is connected. Indeed it suffices to show that for any partition (S, T) of $[l]$ there exists $st \in E(H_i)$ such that $s \in S$, $t \in T$. Choose a minimal subpath $R \subseteq P_i$ such that R intersects $V(Q_s)$ for some $s \in S$ and $V(Q_t)$ for some $t \in T$. Then R is internally disjoint from $\mathcal{Q}$, and so $st \in E(H_i)$ as desired.

By the pigeonhole principle and the choice of r,there exist $i_1, i_2, \ldots, i_{n^2} \in [r]$ such that $H_{i_1} = H_{i_2} = \ldots = H_{i_{n^2}} =: H$. By the choice of l either H contains a path of length n or a vertex x of degree n^2. Contracting subpaths of paths in $\mathcal{Q}$ joining the vertices of P_{i_j} for each j, we obtain a subdivision of a $n \times n^2$-grid as a subgraph of G in the first case. By further contracting all but paths in $\mathcal{Q}$ but Q_x to single vertices, we obtain a subdivision of K_{n^2,n^2} as a subgraph of G in the second case. $\square$

Corollary 6.3 and Lemmas 6.4, 6.5 and 6.6 imply Theorem 6.1 and thus our proof is finished. Clearly, we were wasteful in our argument and the bounds on $N(n)$ such that every graph G with $tw(G) \geq N(n)$ contains $H_{n \times n}$ as a minor are far from optimal. Our proof follows the argument of Diestel et al., as presented in Diestel, and of Leaf and Seymour, where the last one establishes that $N(n) = 2^{O(n \log n)}$ suffices. Recently, the first polynomial bounds on $N(n)$ have been found by Chekuri and Chuzhoy and improved by Chuzhoy, with the current record proving that taking $N(n) = O(n^{20})$ suffices.

7 Well quasi-ordering

7.1 Basic properties

A *partial order* is a pair $(S, \leq)$, where S is a set and $\leq$ is a binary relation on S satisfying the following properties

Reflexivity $a \leq a$ for every $a \in S$,

Antisymmetry if $a \leq b$ and $b \leq a$ then $a = b$, and

Transitivity if $a \leq b$ and $b \leq c$ then $a \leq c$.

A *quasi-order* is a reflexive and transitive binary relation, which is not necessarily antisymmetric. A (quasi-)order $(S, \leq)$ is a *well-(quasi)-order* if for every infinite sequence $s_0, s_1, \ldots, s_n, \ldots$ there exist $i < j$ such that $s_i \leq s_j$. We say that such a pair is *a good pair*, and a sequence that contains no good pair is a *bad sequence*.

The following is the most celebrated theorem of the Graph Minor theory.

Theorem 7.1. *The minor relation $\leq$ is a well-quasi-order on the set of finite graphs.*

While we will be unable to present the proof in these notes, the goal of this section is to make the first steps in the right direction and establish this theorem for graphs of bounded treewidth.

The infinite Ramsey theorem will help us better understand well-quasi-orders.

Theorem 7.2. *Let $c : \mathbb{N}^{(2)} \to [k]$ be a coloring of edges of the complete graph on $\mathbb{N}$ in k colors. Then there exists $i \in [k]$ and infinite $Z \subseteq \mathbb{N}$ such that $c(\{m, n\}) = i$ for all $\{m, n\} \subseteq Z$.*

Proof. We recursively will construct an infinite sequence of integers $0 < a_1 < a_2 < \ldots < a_n < \ldots$, colors $c_1, \ldots, c_n, \ldots$ and infinite sets $\mathbb{N} \supseteq X_1 \supseteq X_2 \supseteq \ldots$ so that

- $a_n < x$ for every $x \in X_n$
- $a_{n+1} \in X_n$ for every $n \in \mathbb{N}$, and
- $c(\{a_n, x\}) = c_n$ for every $x \in X_n$.

The construction is as follows. Given that $a_1, \ldots, a_n$ and $X_1, \ldots, X_n$ were constructed, we let $a_{n+1} = \min X_n$. As X_n is infinite, there exists a color $c_{n+1} \in [k]$ and an infinite $X_{n+1} \subseteq X_n - \{a_{n+1}\}$ such that $c(\{a_{n+1}, x\}) = c_{n+1}$ for every $x \in X_{n+1}$.

Given $(a_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ there exists $i \in [k]$ and ab infinite sequence of indices $1 \leq j_1 \leq j_2 \leq \ldots \leq j_n \leq \ldots$ such that $c_{j_k} = i$ for every $k \in \mathbb{N}$. Let $Z = \{a_{j_k}\}_{k\in\mathbb{N}}$, then Z is as desired. Indeed, $c(\{a_{j_k}, a_{j_l}\}) = c_{j_k} = i$ for every $k < l$, as $a_{j_l} \in X_{j_k}$. □

Recall, that a set $A \leq S$ is *an antichain* in a quasi-order $(S, \leq)$ if no two elements of A are comparable. We will write $s < t$ for $s, t \in S$ if $s \leq t$ and $s \not\geq t$.

Corollary 7.3. *Let $(S, \leq)$ be a quasi-order. Then every infinite sequence of elements of S infinite antichain or an infinite strictly decreasing sequence $s_1 > s_2 > \ldots > s_n > \ldots$ in S. If $(S, \leq)$ is a well-quasi-order then every infinite sequence in S contains an infinite non-decreasing subsequence.*

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of elements of S. We define a coloring $c : \mathbb{N}^{(2)} \to \{1, 2, 3\}$ as follows. For $m < n$ let

$$c(m, n) = \begin{cases} 1, & \text{if } a_m \leq a_n, \\ 2, & \text{if } a_m > a_n, \\ 3, & \text{if } a_m \text{ and } a_n \text{ are incomparable.} \end{cases}$$

By Theorem 7.2 there exists an infinite subset of $Z \subseteq \mathbb{N}$ such that all pairs in Z are colored the same color. This set corresponds to an infinite non-decreasing subsequence, an infinite strictly decreasing subsequence and an infinite antichain, respectively. □

Given a set S let $S^{(<\omega)}$ denote the set of all finite sequences of elements of S. Given a quasi-order $(S, \leq)$ we introduce a quasi-order $(S^{(<\omega)}, \leq)$ defined as follows. Given sequences $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_l)$ in $S^{(<\omega)}$ we write $A \leq B$ if there exists an order preserving injection $\phi : [k] \to [l]$[2] such that $a_i \leq b_{\phi(i)}$ for every $1 \leq i \leq k$. This quasi-order is easier to understand in the case $k = l$, as in such a situation the unique order preserving injection is the identity and $A \leq B$ if and only if $a_i \leq b_i$ for every $1 \leq i \leq k$.

The next theorem introduces an important *minimal bad sequence method.*

Theorem 7.4. *If $(S, \leq)$ is a well-quasi-order then so is $(S^{(<\omega)}, \leq)$.*

[2] that is $\phi(1) < \phi(2) < \ldots < \phi(k)$

Proof. Suppose not then there exists a bad sequence $(A_0, A_1, \ldots, A_n, \ldots)$ in $(S^{(<\omega)}, \leq)$. Choose such a sequence so that $|A_0|$ is as small as possible, subject to that $|A_1|$ is as small as possible, etc. Let $A_i = (a_i, B_i)$ for every i, that is a_i is the first element of A_i and B_i is the sequence of the remaining elements. By Corollary 7.3 there exists an infinite non-decreasing subsequence $a_{i_0} \leq a_{i_1} \leq \ldots \leq a_{i_n} \leq \ldots$ of $(a_n)_{n\in\mathbb{Z}_+}$. Consider now a sequence

$$A_0, A_1, \ldots, A_{i_0-1}, B_{i_0}, B_{i_1}, \ldots, B_{i_n}, \ldots.$$

By the choice of $(A_n)_{n\in\mathbb{Z}_+}$, the above sequence is not bad. Therefore either

- $A_i \leq A_j$ for some $i < j < i_0$, or
- $A_i \leq B_{i_j}$ for some $i < i_0 \leq i_j$, in which case $A_i \leq A_{i_j}$ or
- $B_{i_j} \leq B_{i_k}$ for some $i_j < i_k$, in which case $A_{i_j} \leq A_{i_k}$.

In each case we obtain a contradiction to our assumption that $(A_n)_{n\in\mathbb{Z}_+}$ is a bad sequence. $\square$

7.2 Kruskal's theorem

We are now ready to prove that finite trees are well-quasi-ordered by the minor relation. In fact a much stronger statement is true. Define the following quasi-order $\preceq$ on rooted finite trees. Let (T, r) and (T', r') be rooted trees, and let $\leq_T$ and $\leq_{T'}$ be the corresponding tree orders, as defined in Section 3. We have $(T, r) \preceq (T', r')$ if there exists an injection $\phi : V(T) \to V(T')$ such that if $s \leq_T t$ then $\phi(s) \leq_{T'} \phi(t)$. It is not hard to see that if $(T, r) \leq (T', r')$ then T' contains a subdivision of T as a subgraph, thus $\preceq$ refines the minor relation.

Theorem 7.5 (Kruskal)**.** *The relation $\preceq$ is a well-quasi-order on a set of finite rooted trees.*

Proof. As in the proof of Theorem 7.4, we suppose for a contradiction that a bad sequence

$$(T_0, r_0), (T_1, r_1), \ldots, (T_n, r_n), \ldots$$

is chosen so that $|V(T_0)|$ is minimum, subject to that $|V(T_1)|$ is minimum etc. For each i, let A_i be the sequence of components of $T_i \setminus r_i$, considered as trees rooted at the neighbors of r_i. Let $\mathcal{A}$ denote the union of the sets of elements of all A_i for $i \in \mathbb{Z}_+$.

We claim that $\preceq$ is a well-quasi-order on $\mathcal{A}$. Suppose not and let $T'_0, T'_1, \ldots, T'_n, \ldots$ be the sequence of rooted trees in $\mathcal{A}$. For each i choose $n(i)$ such that $T'_i \in A_{n(i)}$, and suppose without loss of generality that $n(0) = \min_{i\in\mathbb{Z}_+} n(i)$. Consider now a sequence

$$(T_0, r_0), (T_1, r_1), \ldots, (T_{n(0)-1}, r_{n(0)} - 1), T'_0, T'_1, \ldots, T'_n, \ldots.$$

As $T'_0 \in A_{n(0)}$, we have $|V(T'_0)| < |V(T_{n(0)})|$, and so by our choice of the initial bad sequence the above sequence is not bad, but as in the proof of Theorem 7.4 this yields a contradiction, implying the claim.

By Theorem 7.4 the set $\mathcal{A}^{(<\omega)}$ is well-quasi-ordered, and so there exist $i < j$ such that $A_i \preceq A_j$. It follows that $(T_i, r_i) \preceq (T_j, r_j)$ contradicting our choice of a bad sequence. $\square$

7.3 Well-quasi-ordering graphs of bounded treewidth.

TO BE INSERTED

8 The graph minor structure theorem

The central result in the graph minor theory is an approximate topological characterization of graphs which do not contain a given graph as a minor. This description is the main structural result underlying the proof of Theorem 7.1 and has numerous other applications.

Fix a graph H. Informally, every graph G that does not contain H as a minor can be obtained by gluing graphs, which can be "almost" embedded in some surface in which H cannot be embedded, in a "tree-like fashion". Clarifying the notions of surface and "almost" is our next goal.

8.1 Surfaces

A *surface* is a compact connected Hausdorff topological space in which a neighborhood of every point is homeomorphic to $\mathbb{R}^2$. Rather than using this abstract definition of surfaces we will be using a more constructive one, i.e. we will consider surfaces as being obtained from the sphere by adding "handles" and "crosscaps".

Given a surface Σ cut out to disjoint disks from it and identify their boundaries with the two boundary circles of a cylinder. We say that the resulting surface Σ' is obtained from Σ by *adding a handle*. If we cut out a single disk and identify the opposite points of this disk, then we say that the resulting surface Σ' is obtained from Σ by *adding a crosscap*crosscap.

The surface obtained from a sphere by adding one handle is called a *torus*, by adding one crosscap - a *projective plane*, and the surface obtained by adding two crosscaps is a *Klein bottle*.

The following theorem classifies the surfaces.

Theorem 8.1. *Every surface can be obtained from a sphere by adding some number of handles and zero, one or two crosscaps.*

We define *the Euler genus* $\varepsilon(\Sigma)$ of a surface sigma obtained from the sphere by adding k handles and l crosscaps by $\varepsilon(\Sigma) = 2k + l$.

An *embedding* $\sigma : G \hookrightarrow \Sigma$ of a graph G in the surface Σ is a map that maps vertices of G to distinct points on Σ and the edges to curves joining the corresponding points, so that no inner point of such a curve belongs to any other curve or is an image of a vertex. A *face* of σ is a component of $\Sigma - \sigma(G)$, where $\sigma(G)$ denotes the union of the curves and points of the embedding. An embedding is *cellular* if every face is a disc.

Note that the class of graphs which can be embedded on any fixed surface Σ is closed under taking minors.

The following result generalizes Euler's formula for planar graphs.

Theorem 8.2 (Euler's formula for general surfaces)**.** *If $\sigma : G \hookrightarrow \Sigma$ is a cellular embedding of a graph G in a surface Σ with f faces then*

$$|V(G)| - |E(G)| + f = 2 - \varepsilon(\Sigma)$$

Corollary 8.3. *If a graph G with $|V(G)| \geq 3$ can be embedded in a surface Σ then $|E(G)| \leq 3|V(G)| - 6 + 3\varepsilon(\Sigma)$.*

An sample application of Corollary 8.3 includes tight bounds on chromatic number of graphs embedded on surface. The following corollary is the easiest case of such a bound.

Corollary 8.4. *If a graph G can be embedded on a torus then $\chi(G) \leq 7$. (This bound is tight as K_7 can be embedded on a torus.)*

Proof. By induction on $V(G)$. For the induction step, assume that $|V(G)| \geq 3$. By Corollary 8.3 we have $|E(G)| \leq 3|V(G)|$. Thus $\deg(v) \leq 6$ for some $v \in V(G)$. The corollary follows, by applying the induction hypothesis to $G \setminus v$. $\square$

8.2 Vortices

A *society* is a cyclic permutation of some set of vertices G, which we denote by $V(\Omega)$. A *vortex* is a pair (G, Ω), where G is a graph and Ω is a society in G. For $x, y \in V(\Omega)$ we denote by $\Omega[x, y]$ and $\Omega[y, x]$ the two intervals in Ω with ends x and y.

A *vortical decomposition* $\mathcal{V}$ of (G, Ω) is a notion closely related to the path decomposition, defined as follows. The set $\mathcal{V}$ is a family of vertex sets $\{V_x \,|\, x \in V(\Omega)\}$ such that the following four conditions hold:

(V1) $\cup_{x \in V(\Omega)} V_x = V(G)$,

(V2) $x \in V_x$,

(V3) every edge of G has both ends in some V_x,

(V4) For $x, y \in V(\Omega)$ every vertex of $V_x \cap V_y$ either lies in $\cap_{z \in \Omega[x,y]} V_z$ or $\cap_{z \in \Omega[y,x]} V_z$.

The *depth* of $\mathcal{V}$ is $\max_{x \in \Omega} |V_x|$, and, naturally, the *depth* of a vortex is the minimum width of its vortical decomposition. One can similarly define the *adhesion* of $\mathcal{V}$, as $\max_{x,y \in \Omega, x \neq y} |V_x \cap V_y|$, and the *adhesion* of a vortex as the minimum adhesion of its vortical decomposition.

Vortices of small depth and of small adhesion are considered in different versions of the graph minor structure theorem. While vortices of small depth do not seem to allow a dual characterization along the lines of Theorem 6.1, vortices of small adhesion do.

A *multivortex* is a tuple $(G, \Omega_1, \ldots, \Omega_r)$ such that (G, Ω_i) is a vortex for every $i = 1, \ldots, r$ and $V(\Omega_i) \cap V(\Omega_j) = \emptyset$ for $i \neq j$. An *embedding of a multivortex* $(G, \Omega_1, \ldots, \Omega_r)$ *in a surface* Σ *with cuffs* $\Delta_1, \ldots, \Delta_r$ is an embedding $\sigma : G \hookrightarrow \Sigma - \cup_{i=1}^{r} \Delta_i$, where $\Delta_1, \ldots, \Delta_r$ are pairwise disjoint interiors of disks in Σ and for every $1 \leq i \leq r$ we have

- $\partial \Delta_i \cap \sigma(G) = \sigma(V(\Omega_i))$, and

- the clockwise cyclic order of the vertices of $\sigma(V(\Omega_i))$ on Δ_i corresponds to Ω_i if Σ is orientable, and is Ω_i or its reverse, if Σ is not orientable.

Vortices and multivortices are central to the definition of graphs almost embeddable on a surface, as seen in the next subsection.

8.3 The clique sum structure

We are now ready to define graphs almost embeddable on a surface. A *segregation of a graph* G is a tuple $(G_0, V_1, V_2, \ldots, V_r)$, such that

(S1) $V_i = (G_i, \Omega_i)$ is a vortex for $i = 1, \ldots, r$,

(S2) $G_0, G_1, \ldots, G_r$ are subgraphs of G,

(S3) $G = G_0 \cup G_1 \ldots \cup G_r$,

(S4) $V(\Omega_i) = V(G_i) \cap V(G_0)$, and

(S5) $G_1, \ldots, G_r$ are pairwise vertex disjoint.

One can consider a segregation as a partition of a graph into a "central part" G_0 and disjoint "attachments" $G_1, \ldots, G_r$, where a cyclic order is prescribed on the set of vertices each attachment shares with the central part.

Let Σ be a surface and k a positive integer. A *near embedding of* G *in* Σ is a tuple $(G_0, X, \mathcal{V}, \sigma)$, such that

(E1) $\mathcal{V} = (V_1, \ldots, V_r)$ for some positive integer r, where $V_i = (G_i, \Omega_i)$ is a vortex for every $1 \leq i \leq r$,

(E2) $(G_0, V_1, \ldots, V_r)$ is a segregation of $G - X$, and

(E3) σ is an embedding of the multivortex $(G_0, \Omega_1, \ldots, \Omega_r)$ in Σ.

Essentially, a near-embedding describes an embedding of the central part of a segregation of a graph, after first deleting a specified set of vertices. We say that a near embedding *has depth* $\leq k$, if $r \leq k$, $|X| \leq k$, and V_i has depth at most k for every $1 \leq i \leq k$.

The graph minor structure theorem can now be precisely stated as follows.

Theorem 8.5. *For every graph H there exists an integer k such that every graph not containing H as a minor can be obtained by $\leq k$-sums from graphs which allow a near embedding of depth $\leq k$ in some surface, in which H cannot be embedded.*

In the next two subsection we explore the structure of vortices.

8.4 The two paths theorem

A *bump* in a vortex (G, Ω) is a path P in G with both ends in $V(\Omega)$ and otherwise disjoint from $V(\Omega)$. A *cross* in a vortex (G, Ω) is a pair of vertex disjoint bumps (P, Q) with ends x_1, y_1 and x_2, y_2, respectively, so that the set of vertices (x_1, x_2, y_1, y_2) is in this or reverse order in Ω.

The first main result of this subsection characterizes vortices which contain no cross. We precede the statement with necessary definitions. A vortex (G, Ω) is *planar* if (G, Ω) can be embedded into a disk so that $V(\Omega)$ embedded in its boundary in an order corresponding to Ω, i.e. if (G, Ω) can be embedded in the sphere with one cuff according to the definition of a multivortex embedding.

We say that a separation (A, B) of G is a *separation of* (G, Ω) if $V(\Omega) \subseteq A$. We say that (G, Ω) is *k-connected* if $|V(\Omega)| \geq k$ and every separation (A, B) of (G, Ω) of order less than k is trivial.

Let (A, B) be a non-trivial separation of a vortex (G, Ω) of order at most 3. Let G' be obtained from $G \setminus (B - A)$ by adding edges with both ends in $A \cap B$ so that $A \cap B$ is a clique in G'. Then we say that a vortex (G', Ω) is an *elementary reduction* of (G, Ω). We say that a vortex (G'', Ω) is a *reduction* of (G, Ω) if (G'', Ω) can be obtained from (G, Ω) by repeatedly taking elementary reductions. We say that a vortex is *rural* if some reduction of it is planar.

The first main result of this subsection characterizes vortices which contain no cross.

Theorem 8.6. *A vortex contains no cross if and only if it is rural. In particular,* 4*-connected vortex contains no cross if and only if it is planar.*

The proof of Theorem 8.6 will require additional preparation, but first let us present a motivating corollary. We say that a graph G is *k-linked* if $|V(G)| \geq 2k$ and for every sequence $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$ of distinct vertices of G, there exists a linkage $\{P_1, P_2, \ldots, P_k\}$ in G such that the path P_i has ends s_i and t_i for every $1 \leq i \leq k$. Theorem 8.6 implies the following.

Corollary 8.7. *Let G be a* 4*-connected graph with $|V(G)| \geq 4$ and $|E(G)| \geq 3|V(G)| - 6$. Then G is* 2*-linked.*

Proof. Let s_1, t_1, s_2, t_2 be distinct vertices of G. Our goal is two show that G contains a pair of vertex disjoint paths P_1 and P_2 sch that P_i has ends s_i and t_i for $i = 1, 2$. Let $\Omega = (s_1, s_2, t_1, t_2)$. Then a cross in the vortex (G, Ω) is the required pair of paths. By Theorem 8.6 such a cross exists as otherwise G can be drawn in the plane with s_1, s_2, t_1, t_2 lying on the boundary of some face, implying $|E(G)| \leq 3|V(G)| - 7$. □

For general k the best result in the spirit of Corollary 8.7 is given by the following theorem of Thomas and Wollan.

Theorem 8.8 (Thomas and Wollan). *If G is a $2k$-connected graph with $|E(G)| \geq 5k|V(G)|$ then G is k-linked.*

Thomas and Wollan conjecture that the direct analogue of Corollary 8.7 holds.

Conjecture 8.9 (Thomas and Wollan)**.** *If G is a $2k$-connected graph with $|E(G)| \geq (2k-1)|V(G)| - (3k+1)k/2 + 1$ then G is k-linked.*

Note that for $k = 2$ Conjecture 8.9 corresponds exactly to Corollary 8.7.

Returning to the proof of Theorem 8.6 let us introduce additional notation.

A *weak linkage* $\mathcal{P}$ in a graph G is a collection of (possibly trivial) paths disjoint except for their ends. An *$\mathcal{P}$-bridge* in G is a connected subgraph B of G such that $E(B) \cap E(\mathcal{P}) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in S, or for some component C of $G \backslash V(\mathcal{P})$ the set $E(B)$ consists of all edges of G with at least one end in $V(\mathcal{P})$. The vertices in $V(B) \cap V(\mathcal{P})$ are called the *attachments* of B. We say that an $\mathcal{P}$-bridge B *attaches to* a subgraph H of S if $V(H) \cap V(B) \neq \emptyset$. if B is an $\mathcal{P}$-bridge of G, then we say that B is *unstable* if some path of $\mathcal{P}$ includes all the attachments of B, and otherwise we say that B is *stable*.

For a path P, and $x, y \in V(P)$ we denote by xPy the subpath of P with ends x and y. Consider $P \in \mathcal{P}$, and let Q be a path in G with ends $x, y \in V(P)$ and otherwise disjoint from S. Let P' be obtained from P by replacing the path xPy by Q, and let $\mathcal{P}' = (\mathcal{P} - \{P\}) \cup P'$ then we say that $\mathcal{P}'$ was obtained from $\mathcal{P}$ by *rerouting* P along Q, or simply *rerouting*.

Lemma 8.10. *Let G be a graph, and let $\mathcal{P}$ be a weak linkage in G. Then there exists a weak linkage $\mathcal{P}'$ obtained from $\mathcal{P}$ by a sequence of reroutings such that if an $\mathcal{P}'$-bridge B of G is unstable, say all its attachments belong to a path $P \in \mathcal{P}'$, then there exist vertices $x, y \in V(P)$ such that some component of $G \backslash \{x, y\}$ includes a vertex of B and is disjoint from $V(\mathcal{P}') \backslash V(P)$.*

Proof. We may choose a weak linkage $\mathcal{P}'$ of G obtained from $\mathcal{P}$ by a sequence of reroutings such that the number of vertices that belong to stable $\mathcal{P}'$-bridges is maximum, and, subject to that, $|V(\mathcal{P}')|$ is minimum. We will show that $\mathcal{P}'$ is as desired. Assume for a contradiction that B is an $\mathcal{P}'$-bridge with all attachments on $P \in \mathcal{P}'$.

Let $v_0, v_1, \dots, v_k$ be distinct vertices of P, listed in order of occurrence on P such that v_0 and v_k are the ends of P and $\{v_1, \dots, v_{k-1}\}$ is the set of all internal vertices of P that are attachments of a stable $\mathcal{P}'$-bridge. We claim that if u, v are two attachments of B, then no v_i belongs to the interior of uPv. Suppose that v_i is an internal vertex of uPv. Replacing uPv by a subpath of B with ends u, v and otherwise disjoint from $\mathcal{P}'$ is a rerouting that produces a weak linkage $\mathcal{P}''$ with strictly more vertices belonging to stable $\mathcal{P}''$-bridges, contrary to the choice of $\mathcal{P}'$.

Thus for some $1 \leq i \leq k$ the path $v_{i-1}Pv_i$ includes all attachments of B. By the minimality of $|V(\mathcal{P}')|$, we further have $V(B) - \{v_{i-1}, v_i\} \neq \emptyset$. Consequently some component J of $G \backslash \{v_{i-1}, v_i\}$ includes a vertex of B. It follows that $B \backslash \{v_{i-1}, v_i\}$ is a subgraph of J. As B has all its attachments in $v_{i-1}Pv_i$, the interior of $v_{i-1}Pv_i$ includes no attachment of a stable $\mathcal{P}'$-bridge, and every unstable $\mathcal{P}'$-bridge with an attachment in the interior of $v_{i-1}Pv_i$ has all its attachments in $v_{i-1}Pv_i$. It follows that J is disjoint from $V(\mathcal{P}') \backslash V(P)$, as desired. □

Let (G, Ω) be a vortex. Let $u_1, u_2, x_1, x_2, x_3 \in V(G)$ be distinct, such that $u_1, u_2 \notin V(\Omega)$, and let $y_1, y_2, y_3 \in V(\Omega)$ also be distinct. Let $\mathcal{T} = \{P_1^1, P_2^1, P_3^1, P_1^2, P_2^2, P_3^2, Q_1, Q_2, Q_3\}$ be

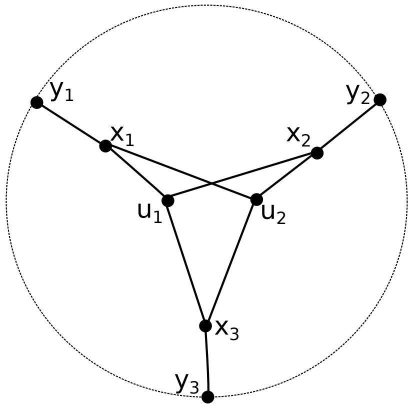

Figure 3: A tripod in a vortex.

a weak linkage in G, where the path P_j^i has ends u_i and x_j, and the path Q_j has ends x_j and y_j for $i = 1, 2,\ j = 1, 2, 3$. Suppose further that the paths in $\mathcal{T}$ are disjoint from $V(\Omega) - \{y_1, y_2, y_3\}$. Then we say that $\mathcal{T}$ is a *tripod* in (G, Ω). See Figure 3

Lemma 8.11. *Let (G, Ω) be a 4-connected vortex. If (G, Ω) contains a tripod then it contains a cross.*

Proof. Let $\mathcal{T}$ be a tripod in (G, Ω), and let $u_i, x_j, y_j, P_j^i, Q_j$ for $i = 1, 2,\ j = 1, 2, 3$ be as in the definition of the tripod. We assume that the tripod $\mathcal{T}$ is chosen so that $|V(Q_1)| + |V(Q_2)| + |V(Q_3)|$ is minimum among all tripods in (G, Ω). Let $A = \bigcup_{i=1,2,j=1,2,3} |V(P_j^i)|$, and let $B = V(\Omega) \cup \bigcup_{j=1,2,3} |V(Q_j)|$. Note that $A \cap B = \{x_1, x_2, x_3\}$. As (G, Ω) is 4-connected, there does not exist a separation (A', B') of G of order 3 with $A \subseteq A'$, $B \subseteq B'$. Thus there exists a path R in G with one end in $s \in A - B$, the other end in $t \in B - A$, and otherwise disjoint from $V(\mathcal{T}) \cup V(\Omega)$. By contracting a subpath of a path in $\mathcal{T}$ we assume without loss of generality that $s = u_1$. If $t \in V(\Omega) - \{y_1, y_2, y_3\}$, we assume again without loss of generality that (y_1, y_2, y_3, s) are in this clockwise order in Ω. In this case, $(Q_1 \cup P_1^2 \cup P_3^2 \cup Q_3, Q_2 \cup P_2^1 \cup R)$ is a cross in (G, Ω), as desired. See Figure 4 a).

Otherwise, $t \in V(\mathcal{T})$ and we assume without loss of generality that $t \in V(Q_3) - \{y_3\}$. Then $E((\mathcal{T} - \{P_3^1\}) \cup E(R)$ contains the edge set of a tripod $\mathcal{T}'$ in (G, Ω) with the path tQ_3y_3 replacing Q_3. See Figure 4 b). Such a tripod contradicts the choice of $\mathcal{T}$. $\square$

Proof of Theorem 8.6. The "if" direction of the theorem is fairly straightforward and we only present the proof of the "only if" direction. It is by induction on $V(G)$.

Suppose first that the vortex (G, Ω) is not 4-connected and choose a separation (A, B) of G with $V(\Omega) \subseteq A$ and $B - A \neq \emptyset$ of minimum order. Then $|A \cap B| \leq 3$ and $G[B]$ is connected. Let the graphs G_A and G_B be obtained from $G[A]$ and $G[B]$, respectively, by adding edges with both ends in $A \cap B$ so that $A \cap B$ is a clique in G_A and G_B. Then (G_A, Ω) is an elementary reduction of (G, Ω). It contains no cross as such a cross could be modified

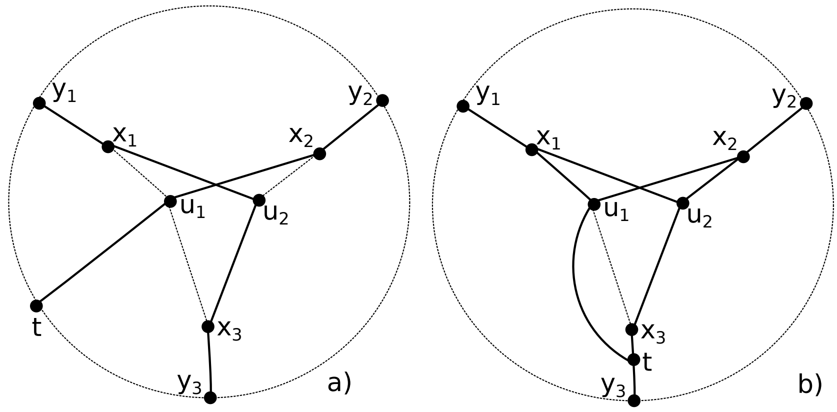

Figure 4: Two cases in the proof of Lemma 8.11.

to form a cross in (G,Ω) by using the edges in $G[B]$ instead of edges in $E(G_A) - E(G)$. Therefore (G_A,Ω) is rural by the induction hypothesis, and thus so does (G,Ω) .

It remains to consider the case when (G,Ω) is 4-connected. If $V(G) - V(\Omega) = \emptyset$ then drawing $V(\Omega)$ on the boundary of a disk in the order given by Ω and drawing the edges of G as straight lines joining the corresponding vertices produces the desired drawing of (G,Ω). Thus we assume that there exists $v \in V(G) - V(\Omega)$. By considering the four paths from v to $V(\Omega)$ disjoint except for v, which exist by 4-connectivity of (G,Ω) we find a bump P in (G,Ω) with ends x and y, not consecutive in Ω. Let $\mathcal{P} = \{P\} \cup V(\Omega)$ be a weak linkage, where every vertex in $V(\Omega)$ is considered as a trivial path. By Lemma 8.10 we may assume that P is chosen so that every $\mathcal{P}$-bridge is stable.

As (G,Ω) contains no cross there exists a separation (A,B) of G such that $A \cap B = V(P)$, $\Omega[x,y] \in A$ and $\Omega[y,x] \in B$. Let Ω_1 be a cyclic order formed by the vertices in $\Omega[x,y]$ in order from x to y followed by internal vertices of P in order from y to x, and let Ω_2 be defined symmetrically with the roles of x and y switched. Then $(G[A],\Omega_1)$ and $(G[B],\Omega_2)$ are 4-connected vortices. If neither contains a cross then by the induction hypothesis both of them ar planar and by combining the planar drawings of $(G[A],\Omega_1)$ and $(G[B],\Omega_2)$ we obtain the desired drawing of G.

Thus we assume without loss of generality that $(G[A],\Omega_1)$ contains a cross (Q_1,Q_2) with ends s_1,t_1 and s_2,t_2 and suppose that such a cross is chosen so that $k = |V(P) \cap \{s_1,t_1,s_2,t_2\}|$ is minimum. If $k \leq 2$ then extending Q_1 and/or Q_2 using subpaths of P we obtain a cross in (G,Ω). Suppose now that $k = 4$. As the $\mathcal{P}$-bridge B containing Q_1 is stable, there exists a path R in G with one end in $u \in (V(Q_1) \cup V(Q_2)) - V(P)$ and the other end in $v \in V(\Omega) - V(P)$, otherwise disjoint from $V(Q_1) \cup V(Q_2) \cup V(\Omega) \cup V(P)$. If $u \in V(Q_i)$ then replacing a subpath of Q_1 by R we obtain a cross in $(G[A],\Omega_1)$ with at least one attachment in $V(\Omega) - V(P)$ contradicting our assumption that $k = 4$.

It remains to consider the case $k = 3$. Suppose without loss of generality that $s_1 \in V(\Omega) - V(P)$ and that x, s_2, t_1, t_2, y occur on P in this order (where possibly $x = s_2$ and $y = t_2$). As the $\mathcal{P}$-bridge B containing Q_2 is stable there exists a path R in G with one end

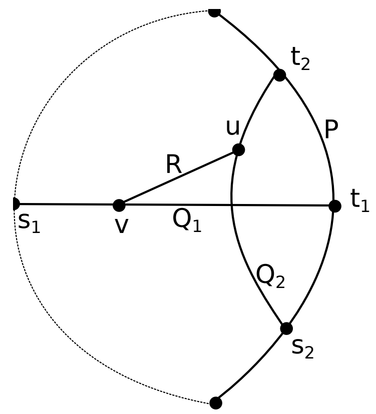

Figure 5: A tripod in the proof of Theorem 8.6.

in $u \in V(Q_2) - \{s_2, t_2\}$ and the other end in $v \in (V(\Omega) \cup V(Q_1)) - V(P)$, otherwise disjoint from $V(Q_1) \cup V(Q_2) \cup V(\Omega) \cup V(P)$. If $v \notin V(Q_1)$ then replacing a subpath of Q_2 by R we obtain a cross with two attachments in $V(\Omega) - V(P)$, once again contradicting the choice of k. Finally, if $v \in V(Q_1)$, then $E(Q_1 \cup Q_2 \cup P \cup R$ contains the edge set of a tripod in (G, Ω). See Figure 5 It follows from Lemma 8.11 that (G, Ω) contains a cross, a contradiction.

□

8.5 Transactions in vortices

A *transaction* in a vortex (G, Ω) is a linkage $\mathcal{P}$ in G such that every path in $\mathcal{P}$ is a bump in (G, Ω) and there exist $x, y \in V(\Omega)$ such that every $P \in \mathcal{P}$ has one end in $\Omega[x, y]$ and the other end in $\Omega[y, x] - \{x, y\}$. Informally, a transaction in a vortex links one half of the society to the other. Note that if $\mathcal{V}$ is a vortical decomposition of $V(\Omega)$, y' is the vertex following y in Ω and x' is the vertex preceding x then $(V_x \cap V_{x'}) \cup (V_y \cap V'_y)$ separates $\Omega[x, y]$ from $\Omega[y, x] - \{x, y\}$. Thus if (G, Ω) admits a vortical decomposition of adhesion at most k then the order of every transaction in (G, Ω) is at most $2k$. The next theorem dues to Robertson and Seymour is a partial converse of this statement.

Theorem 8.12. *If every transaction in a vortex (G, Ω) has order at most k then (G, Ω) admits a vortical decomposition of adhesion at most k.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the vertices in Ω listed in order. Let $X_i = \{x_1, \ldots, x_i\}$, $Y_i = \{x_{i+1}, \ldots, x_n\}$ for $1 \leq i \leq n$, and let k_i be the maximum order of an (X_i, Y_i)-linkage in G. By our assumption $k_i \leq k$ for every i. Choose a separation (A_i, B_i) of G of order k_i such that $X_i \subseteq A_i, Y_i \subseteq B_i$ with $|A_i|$ minimum.

We claim that $A_i \subseteq A_j$ and $B_i \supseteq B_j$ for all $1 \leq i \leq j \leq n$. Indeed, the separation $(A_i \cap A_j, B_i \cup B_j)$ has order at least k_i, and the separation $(A_i \cup A_j, B_i \cap B_j)$ has order at least k_j. It follows from (1) that the order of $(A_i \cap A_j, B_i \cup B_j)$ is exactly k_i and the order of $(A_i \cup A_j, B_i \cap B_j)$ is exactly k_j. Thus by the choice of (A_i, B_i) we have $A_i \subseteq A_j$. Further, we have

$$|B_i \cap B_j| = n + k_j - |A_i \cup A_j| = n + k_j - |A_j| = |B_j|,$$

and therefore $B_i \supseteq B_j$, as claimed.

We now define the vortical decomposition $\mathcal{V} = \{V_x | x \in V(\Omega)\}$ of (G, Ω) by setting $V_{x_i} = A_i \cap B_{i-1}$, where $B_0 = \emptyset$. It is easy to check that $\mathcal{V}$ is indeed a vortical decomposition. Moreover, $V_{x_i} \cap V_{x_{i+1}} \subseteq A_i \cap B_i$ and so $|V_{x_i} \cap V_{x_{i+1}}| \leq k_i \leq k$. It follows that the adhesion of $\mathcal{V}$ is at most k. □

We would like to combine Theorem 8.12 with Theorem 8.6 to show that a vortex that does not contain a large transaction which is "substantially crossed" can be "almost embedded in a disk". Formalizing such a result requires additional definitions.

Generalizing the definition of a planar vortex, we say that a multivortex (G, Ω_1, Ω_2) is a *planar* if there exists an embedding of(G, Ω_1, Ω_2) in the plane with two cuffs. We define reductions of multivortices analogously to the reductions of vortices and say that (G, Ω_1, Ω_2) is *rural* if some reduction of it is planar. We say that a vortex (G, Ω) is a *composition* of a vortex (G_0, Ω_0) and a multivortex (G_1, Ω, Ω_0) if $(G_1, (G_0, \Omega_0))$ is a segregation of G that is $G = G_0 \cup G_1$, and $V(\Omega_0) = V(G_1) \cap V(G_0)$.

Now let us define several types of "substantially crossed" transactions. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be a transaction in a vortex (G, Ω), and let s_i, t_i be the ends of P_i. We say that $\mathcal{P}$ is a *crosscap* of order k if $(s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k)$ appear in Ω in this clockwise order. If $(s_1, s_2, \ldots, s_k, t_{k-1}, t_{k-2}, \ldots, t_1, t_k)$ appear in Ω in this clockwise order, then we say that $\mathcal{P}$ is a *leap* of order k. Finally if $(s_1, s_2, \ldots, s_k, t_{k-1}, t_k, t_{k-2}, t_{k-3}, \ldots, t_3, t_1, t_2)$ appear in Ω in this clockwise order, then we say that $\mathcal{P}$ is a *double cross* of order k. We are now ready to state the promised combination of Theorems 8.6 and 8.12, presented without proof.

Theorem 8.13. *Let (G, Ω) be a vortex and let $k \geq 4$ be an integer. Then*

- (G, Ω) *contains a crosscap, a leap or a doublecross of order k, or*
- (G, Ω) *is a* composition *of a vortex (G_0, Ω_0), which admits a vortical decomposition of adhesion at most $3k + 9$ and a rural multivortex (G_1, Ω, Ω_0).*

8.6 Towards the proof of the graph minor structure theorem

In this subsection we give an extremely informal sketch of the proof of Theorem 8.5, stating a few key auxiliary results along the way.

Let H be a fixed graph, and let G be a graph such that $G \not\geq H$. Our goal is to show that can be obtained by $\leq k$-sums from graphs which allow a near embedding of depth $\leq k$ in some surface, in which H cannot be embedded for some k depending on H by not on G. Thus we may assume that the treewidth of G is large, and so by By Theorem 6.1 G contains a large grid as a minor. As a first step we would like to show that a large subgrid of this grid can be embedded in the plane, so that the rest of the graph, except for the bounded number of vertices only attaches to its boundary.

It will be more convenient to work with subdivisions rather than models, and so we replace the grid by the following graph with maximum degree three. An *elementary h-wall* has vertex-set

$$\{(x,y) : 0 \leq x \leq 2h+1, 0 \leq y \leq h\} - \{(0,0),(2h+1,h)\}$$

and an edge between any vertices (x,y) and (x',y') if either

- $|x-x'| = 1$ and $y = y'$, or
- $x = x'$, $|y-y'| = 1$ and x and $\max\{y,y'\}$ have the same parity.

An *h-wall* W is a subdivision of an elementary h-wall W_0. The *outer cycle* of a W is a cycle which forms the boundary of the outer face in the natural planar drawing of W. The *pegs* of W are the vertices corresponding to the vertices of W_0 of degree two. Note that the pegs of W are not uniquely determined.

Let G be a graph and let W be a wall in G with the outer cycle C. We say that W is *flat* in G if there exists a separation (A,B) of G with the following properties:

- $A \cap B \subseteq V(D)$,
- $V(W) \subseteq B$,
- the vortex $(G[B],\Omega)$ is rural,
- there exists a choice of pegs of W such that every peg belongs to A.

The following theorem is a weakening of the Flat Wall Theorem of Robertson and Seymour.

Theorem 8.14. *For every graph H there exists a such that for every integer h there exists w satisfying the following. If G is a graph such that $G \not\geq H$ and $\mathrm{tw}(G) \geq w$ then $G \setminus A$ contains a flat h-wall for some $A \subseteq V(G)$ with $|A| \leq a$.*

Very informally, in the proof of Theorem 8.5 ,one constructs a sequence of near embeddings $(G_0^i, X^i, \mathcal{V}^i, \sigma_i)$ of G for $i = 1,2,\ldots$ on surfaces $\Sigma_1, \Sigma_2, \ldots$ such that $\varepsilon(\Sigma_1) \leq \varepsilon(\Sigma_2) \leq \ldots$. Let $n(\mathcal{V}^i)$ be the number of non-rural vortices in $\mathcal{V}_i$. We further require that if at some step $\varepsilon(\Sigma_i) = \varepsilon(\Sigma_{i+1})$ then $n(\mathcal{V}^{i+1}) > n(\mathcal{V}^i)$. Moreover, $|X^{i+1}| - |X^i| \leq f(i)$ for some function f independent on G. Further, the non-rural vortices in $\mathcal{V}^i$ are "far apart" in σ_i, in a sense which we are not making precise here, and these vortices are "well connected" to G_0^i. Finally, we require that the embedding of the multivortex corresponding G_0^i in Σ_i has "high representativity". While making this notion precise for multivortices is slightly technical let us define it for graph embeddings.

Let $\sigma : G \hookrightarrow \Sigma$ be a cellular embedding of a graph G in a surface Σ which is not a plane. Then the *representativity* of σ is the minimum number of facial walks of σ whose union contains a cycle non-contractible in Σ. In other words the representativity of σ is a minimum positive integer θ such that there exists a non-contractible closed curve C in Σ

intersecting $\sigma(G)$ only in vertices and at most θ points. The following theorem of Robertson and Seymour is another key ingredient in the proof of Theorem 8.5 and motivates the above definition.

Theorem 8.15. *For every graph H and an integer $g > 0$ there exists θ satisfying the following. If Σ is a surface with $\varepsilon(\Sigma) = g$, H can be embedded in Σ, and a graph G admits a cellular embedding in Σ with representativity at least θ then $H \leq G$.*

Theorem 8.15 ensures that if we maintain high representativity of the embeddings discussed above then H can not be embedded in the surfaces in the sequence and so the genus of these surfaces is bounded. But the sequence $n(\mathcal{V}^i)$ is also bounded as a collection of many non-rural vortices produces many crosses which could be connected to build a model of H. Thus the sequence of near-embeddings mentioned above has bounded length. We'd like it to terminate in a near-embedding $(G_0^n, X^n, \mathcal{V}^n, \sigma_n)$ of a part of G satisfying the conditions of Theorem 8.5. If every non-rural vortex is a composition of a rural multivortex and a vortex of bounded depth then $(G_0^n, X^n, \mathcal{V}^n, \sigma_n)$ can be extended to the desired near-embedding. (Note that the total number of vortices in $\mathcal{V}^n$ is not necessarily bounded, but rural vortices can be embedded in Σ_n up to 3-separations, which are taken care of by the k-sum global structure.) Therefore by Theorem 8.13, we may assume that some vortex in $(G', \Omega) \in \mathcal{V}^n$ contains a leap, crosscap or a double-cross $\mathcal{T}$ of large order. If (G', Ω) contains a crosscap we can add a crosscap to Σ_n as a next step in the sequence of near-embeddings. If (G', Ω) contains a double-cross we can increase the number of non-rural vortices. Finally, a leap in (G', Ω) is a first step to creating a handle to be added to Σ_n, but here even more care is required. Note that in each of these cases we need to extend the partial embedding of G to the new surface. Here Lemma 8.10, 8.11 and Theorem 8.6 are used to analyze the attachments of $\mathcal{T}$-bridges in (G', Ω).

9 Balanced separations

The main result of this section is a theorem of Alon, Seymour and Thomas stating that for fixed t the treewidth of a graph G with no K_t minor is $O_t(\sqrt{|V(G)|})$. A typical application of this theorem uses only the fact that such a graph G has a balanced separation of order $O_t(\sqrt{|V(G)|})$, hence the name of the section. We will discuss a few such applications in the later part of the section.

9.1 Alon-Seymour-Thomas theorem

Theorem 9.1 (Alon, Seymour, Thomas). *Let $t \geq 1$ be an integer, and let G be a graph on n vertices with $\mathrm{tn}(G) \geq t^{3/2}n^{1/2}$. Then $G \geq K_t$.*

We precede the proof of Theorem 9.1 by a technical lemma.

Lemma 9.2. *let G be a graph on n vertices, $A_1, A_2, \ldots, A_k \subseteq V(G)$, and let $r \geq 1$ be real. Then one of the following holds*

(i) *There exists a connected subgraph $T \subseteq G$ such that $|V(T)| \leq r$ and $V(T) \cap A_i \neq \emptyset$ for every $1 \leq i \leq k$.*

(ii) *There exists $Z \subseteq V(G)$ such that $|Z| \leq (k-1)n/r$ and no component of $G \setminus Z$ intersects all of $A_1, A_2, \ldots, A_k$.*

Proof. We assume $k \geq 2$ and construct the graph J as follows. Let $G^1, G^2, \ldots, G^{k-1}$ be isomorphic vertex disjoint copies of G, where for $v \in V(G)$ we denote its copy in G^i by v^i. The graph J is obtained from $G^1 \cup G^2 \cup \ldots \cup G^{k-1}$ by adding an edge with one end in v^{i-1} and another in v^i for all $2 \leq i \leq k-1$ and all $v \in A_i$. Let $X = \{v^1 | v \in A_1\}$ and $Y = \{v^{k-1} | v \in A_k\}$. For a set of vertices $Z \subseteq V(J)$ let

$$\pi(Z) = \{v \in V(G) \mid v^i \in Z \text{ for some } 1 \leq i \leq k-1\}.$$

Finally, for each $u \in V(J)$ let $d(u)$ be equal the number of vertices in the shortest path from u to X, or $+\infty$ if no such path exist.

Suppose first that $d(u) \leq r$ for some $u \in Y$. Let P be a path on at most r vertices. Then $\pi(Z)$ induces a connected subgraph of G, satisfying (i).

Thus we assume that $d(u) > r$ for every $u \in Y$. For $j = 1, 2, \ldots, \lceil r \rceil$ let $Z_j = \{u \in V(J) \mid d(u) = j\}$. As these sets are mutually disjoint there exists j such that $|Z_j| \leq |V(G)|/r = (k-1)n/r$. Let $Z = \pi(Z_j)$. We claim that Z satisfies (ii). Suppose not, and there exists a component C of $G \setminus Z$ intersects which intersect all of $A_1, A_2, \ldots, A_k$. Choose $a_i \in V(A_i) \cap V(Z)$ for every $1 \leq i \leq k$. Then there exists a path in $J \setminus Z_j$ with ends a_i^i and a_{i+1}^{i+1} for $1 \leq i \leq k-2$, as well as the path with ends a_{k-1}^{k-1} and a_k^{k-1}. Concatenating these paths yields a path from X to Y in $J \setminus Z_j$, which is impossible. $\square$

Proof of Theorem 9.1. Let $\mathcal{T}$ be a tangle in G of order at least $t^{3/2}n^{1/2}$. We choose a separation $(A, B) \in \mathcal{T}$ such that there exists a model μ of K_k for some $k \leq t$ in $G[A]$ with the following properties:

- $A \cap B \subseteq \cup_{v \in V(K_k)} \mu(v)$,
- $|(A \cap B) \cap \mu(v)| \leq t^{1/2}n^{1/2}$,
- subject to (a) and (b) $k + |B| + 2|B - A|$ is minimum.

If $s = t$ the theorem holds and so we suppose for a contradiction that $k < t$. By Lemma **??**, $G[B - A]$ is connected. We assume that $V(K_k) = [k]$ and let A_i be the set of all the vertices in $B - A$ with a neighbor in $\mu(i)$ for $1 \leq i \leq k$. We apply Lemma 9.2 to $G' = G[B - A]$ and $A_1, A_2, \ldots, A_k$ with $r = t^{1/2}n^{1/2}$

Suppose first that there exists $T \subseteq G'$ satisfying outcome (i) of Lemma 9.2. Then extending μ to a model μ' of K_{k+1} by setting $\mu'(k+1) = T$, we see that $(A \cup V(T), B)$ and μ' contradict the choice of (A, B).

Thus there exist $Z \subseteq B - A$ satisfying outcome (ii) of Lemma 9.2, that is $|Z| \leq (k-1)n/r < t^{1/2}n^{1/2}$and no component of $G \setminus Z$ intersects all of $A_1, A_2, \ldots, A_k$. Then $(A \cup Z, B) \in \mathcal{T}$ and by Lemma 4.13 there exists $(A', B') \in \mathcal{T}$ and $G[B' - A']$ is connected, and

$B' - A' \subseteq B - A - Z$. By the choice of Z there exists i such that A_i is disjoint from $B' - A'$. Let C be the maximum connected subgraph of $G[A']$ containing $\mu(i)$ and vertex disjoint from $\mu(j)$ for $j \neq i$. Let $Z' = (Z \cap V(C)) \cup ((A \cap B) - \mu(i))$. Note that $Z' \cap (B' - A') = \emptyset$. Let W be the vertex set of the component of $G \setminus Z'$ containing $B' - A'$. Then $W \cap V(C) = \emptyset$, as otherwise there exists a path from a vertex in $\mu(i)$ to a vertex in $B' - A'$ disjoint from Z. Let $A'' = V(G) - W$, $B'' = W \cup Z'$. Then (A'', B'') is a separation of G, $A'' \cap B'' = Z'$ and $|Z'| \leq |Z| + |A \cap B| < t^{3/2}n^{1/2}$. Thus either $(A'', B'') \in \mathcal{T}$ or $(B'', A'') \in \mathcal{T}$. The second outcome is impossible as $B' \subseteq B''$ and $(A', B') \in \mathcal{T}$. Therefore, $(A'', B'') \in \mathcal{T}$.

Change the model μ of K_k to a model μ' by replacing $\mu(i)$ by $\mu'(i) = C$. By the choice of (A, B) and μ, we have $|B''| + 2|B'' - A''| \geq |B| + 2|B - A|$, but $B'' - A'' \subseteq B - A$, and $B'' \subseteq A''$. Thus $B'' - A'' = B - A$, and so $\mu(i) \cap A \cap B = \emptyset$. But then μ can be reduced to a model μ'' of K_{k-1} by removing $\mu(i)$, and μ'' violates the choice of μ. $\square$

9.2 Counting K_t-minor-free graphs

The following theorem due to Norin, Seymour, Thomas and Wollan. We present a proof by Dvořák and Norin.

Theorem 9.3. *Let $N(n) = N(n, t)$ be the number of (unlabelled) K_t-minor free graphs on n vertices. Then there exists $C = C(t)$ such that $N(n) \leq C^n$.*

Proof. Let $s(n) = t^{3/2}n^{1/2}$. Let $c = \frac{6\sqrt{3}}{\sqrt{2}+1-\sqrt{3}} t^{3/2}$. Let $h(n) = c\sqrt{n}\log n$ for $n \geq 3$, and let $n_0 \geq 3$ be an integer such that

- $h(n) < n$ and $s(n) \geq 1$ for all $n \geq n_0$,
- $h(n)$ is non-decreasing and concave on the interval $(n_0, +\infty)$, and
- $2n/3 + s(n) \leq n - 1$ for $n \geq n_0$.

Let $C \geq e$ be a constant such that $N(n) \leq C^{n-h(n)}$ for $n_0 \leq n \leq 3n_0$ and $N(n) \leq C^n$ for $n \leq n_0$. We show by induction that $N(n) \leq C^{n-h(n)}$ for every $n \geq n_0$.

For $n \leq 3n_0$ the claim holds by the choice of C. Assume now that $n > 3n_0$, and that $N(k) \leq C^{k-h(k)}$ for $n_0 \leq k < n$. Let $s = \lfloor s(n) \rfloor$. By Theorem 9.1, $\mathrm{tn}(G) < s(n)$ for every graph G with no K_t minor on n vertices , and so by Theorem 5.5 there exists a balanced separation (A, B) of G of order at most s. We can choose such a separation of order exactly s by adding vertices to A and B. We conclude that

$$N(n) \leq \sum_{a=\lceil n/3 \rceil}^{\lfloor 2n/3 \rfloor + s} \binom{a}{s}\binom{n-a+s}{s} s! N(a) N(n-a+s),$$

since every K_t-minor free graphs on n vertices on n vertices can be constructed in the following way: Choose an integer a such that $\lceil n/3 \rceil \leq a \leq \lfloor 2n/3 \rfloor + s$ and K_t-minor-free graphs G_1, G_2 such that $|V(G_1)| = a$ and $|V(G_2)| = n - a + s$ (for a fixed a, this can be done in $N(a)N(n-a+s)$ ways). Choose subsets $S_1 \subseteq V(G_1)$ and $S_2 \subseteq V(G_2)$ so that $|S_1| = |S_2| = s$ (this can be done in $\binom{a}{s}\binom{n-a+s}{s}$ ways). Choose a perfect matching between the vertices of S_1 and S_2 (in $s!$ ways), and identify the matched vertices in S_1 and S_2.

Note that $\binom{a}{s}s! \le n^s$ and $\binom{n-a+s}{s} \le n^s$. Also, $n_0 \le n/3 \le a < n$ and $n_0 \le n-a+s < n$, thus by the induction hypothesis

$$N(n) \le \sum_{a=\lceil n/3 \rceil}^{\lfloor 2n/3 \rfloor + s} n^{2s} C^{n+s-h(a)-h(n-a+s)}.$$

As h is concave, we get

$$h(a) + h(n-a+s) \ge h(n/3) + h(2n/3+s) \ge h(n/3) + h(2n/3).$$

It follows that

$$\begin{aligned} N(n) &\le n^{2s+1} C^{n+s-h(n/3)-h(2n/3)} \\ &= C^{n+(2s+1)\log_C n + s - h(n/3) - h(2n/3)} \\ &\le C^{n+(2s(n)+2)\log n - h(n/3) - h(2n/3)}. \end{aligned}$$

Moreover,

$$\begin{aligned} h(n/3) \quad &+ h(2n/3) - h(n) \ge \tfrac{\sqrt{2}+1-\sqrt{3}}{\sqrt{3}} c\sqrt{n}\log n - 2c\log 3\sqrt{n} \\ &\ge \tfrac{\sqrt{2}+1-\sqrt{3}}{2\sqrt{3}} c\sqrt{n}\log n \ge 4s(n)\log n \ge (2s(n)+2)\log n. \end{aligned}$$

It follows that $N(n) \le C^{n-h(n)}$, as required. $\square$

Index

k-coloring, 4
k-connected, 3
k-sum, 5

Alon-Seymour-Thomas separator theorem, 38

bramble, 14
bump, 31

chromatic number, 4
contraction, 2
cross, 31

embedding, 28
  cellular, 28
Erdős-Pósa property, 19
Euler genus, 28
Euler's formula, 29

face, 28
Four Color Theorem, the, 4

graph
  apex, 7
grid, 14
Grid theorem, the, 19

Hadwiger's conjecture, 4
handle, 28

Jorgensen's conjecture, 7

Klein bottle, 28
Kruskal's theorem, 27

linkage, 11
  weak, 32

Menger's theorem, 3
minor, 2
model, 2
multivortex
  planar, 36
  rural, 36

order
  quasi-, 25
  well-quasi-, 25

path decomposition, 9
pathwidth, 9
projective plane, 28

separation, 3
  balanced, 20
  order of, 3
society in a graph G, 29
subdivision, 2
surface, 28

tangle, 16
torus, 28
transaction, 35
  crosscap, 36
  doublecross, 36
  leap, 36
tree
  depth of, 8
  order, 8
  decomposition, 12
    adhesion of, 12
treedepth, 8
treewidth, 12
tripod, 33

vortex, 29
  k-connected, 31
  planar, 31
vortical decomposition, 29

Wagner's theorem, 4
wall
  elementary, 37