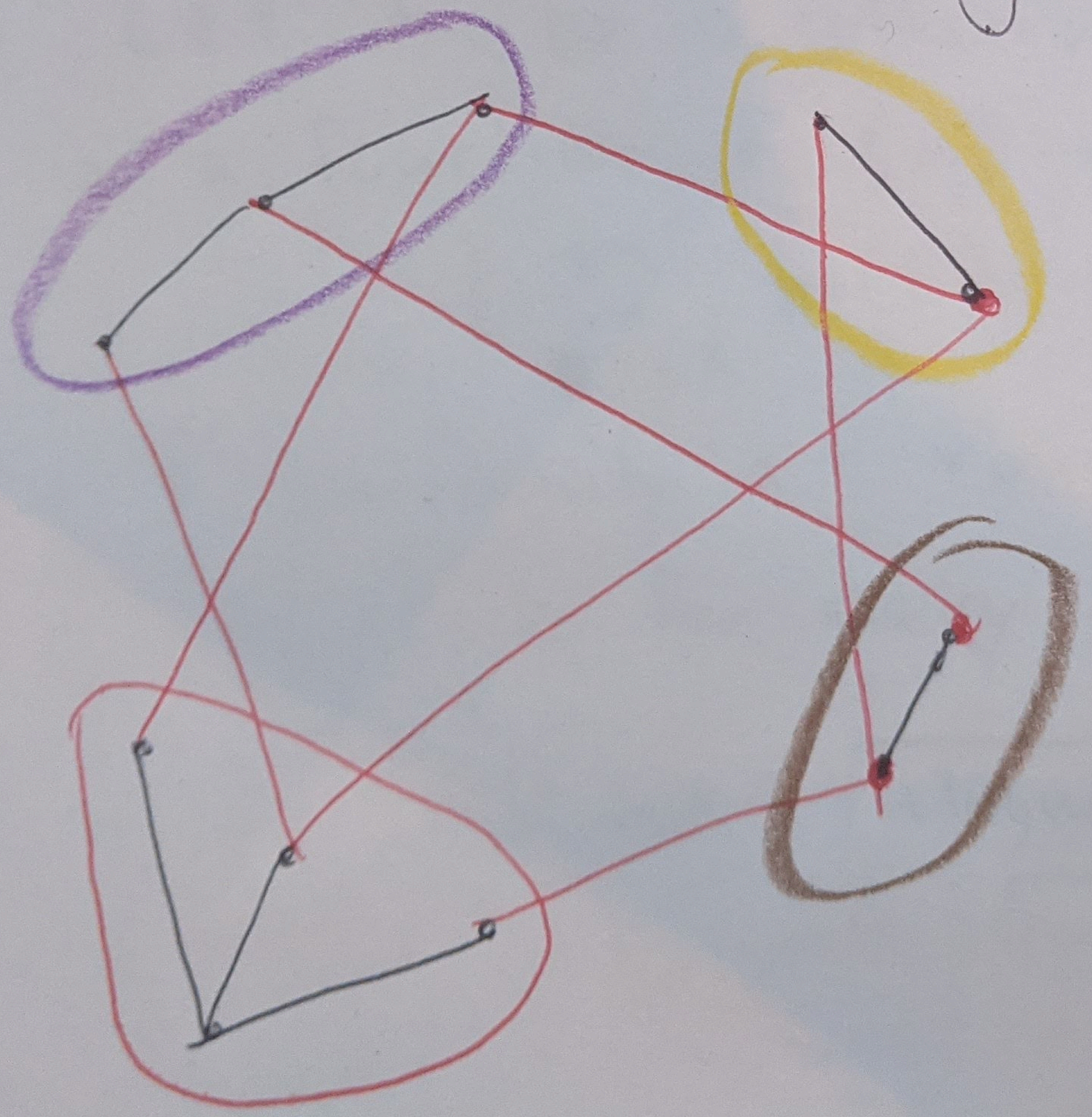
Lecture 9:

Hadwiger & Hajos's Conjectures

for random graphs.



Chernoff Bound:

$X_1, X_2, \ldots, X_n$ independent.

Let $X = X_1 + X_2 + \ldots + X_n$, $X_i \in [0,1]$, $\mu = \mathbb{E}[X]$

then $\mathbb{P}(X \geq \mu + \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$

$\mathbb{P}(X \leq \mu - \lambda\sqrt{n}) \leq e^{-\lambda^2/2}$

Discrepancy

Theorem 5.3: Let $\mathcal{F}$ be a collection of $m \geq 2$ subsets of $[n]$.

Then there is a function $f : [n] \to \{-1, 1\}$ s.t

$$\left|\sum_{i \in A} f(i)\right| \leq 2\sqrt{n \log m}$$

If $\underbrace{m = 2^n}$ the bound is within Factor 4 of optimal

Proof: Choose f uniformly at random then

$\mathbb{E}[f(i)] = 0$ for every $i \in [n]$

$$\mathbb{P}\Big(\underbrace{\Big|\sum_{i \in A} f(i)\Big| \geq 2\sqrt{n \log m}}_{\lambda = 2\sqrt{\log m}}\Big) \leq 2e^{-2\log m} = \frac{2}{m^2}$$

– If $m = n$

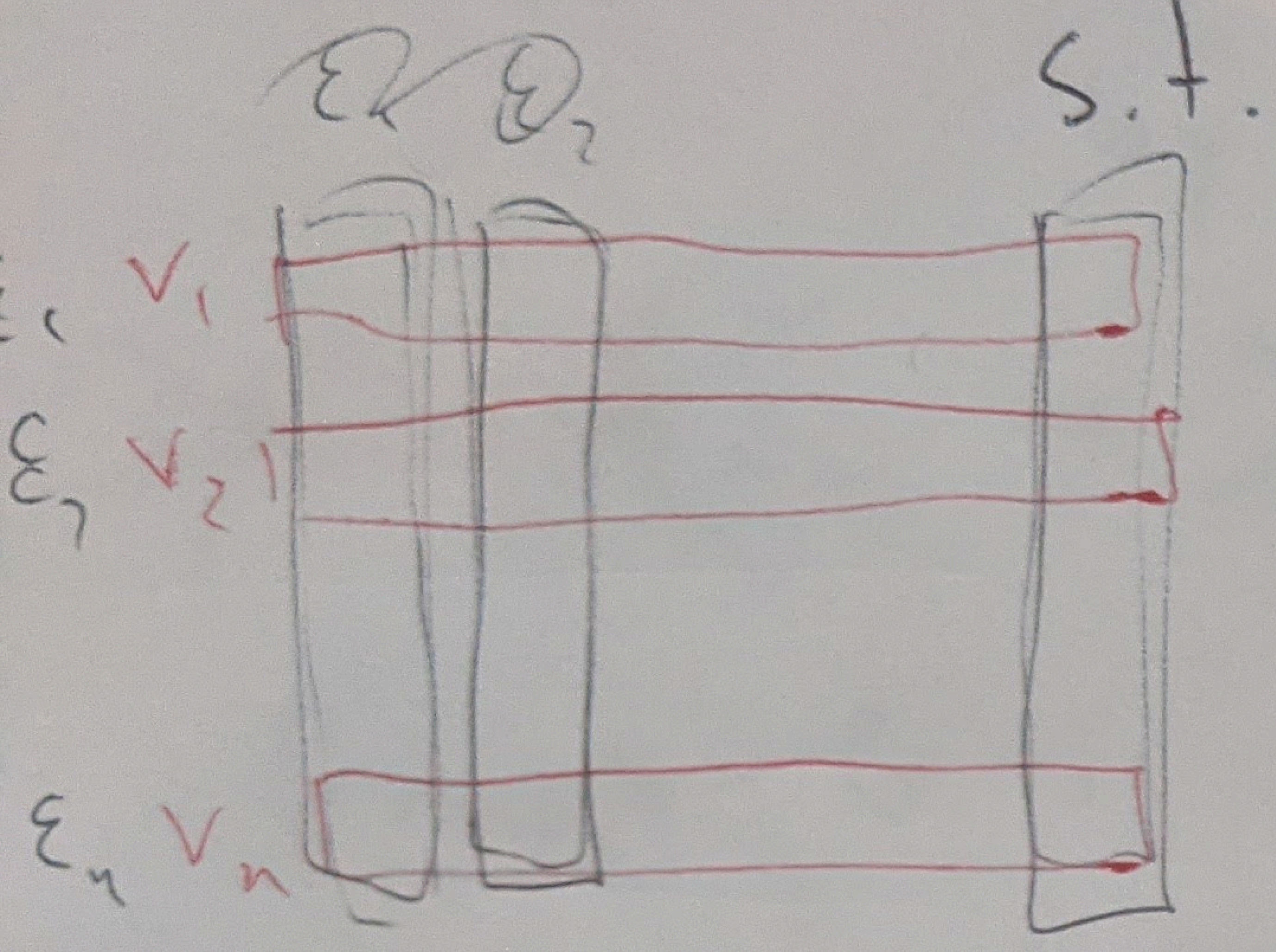
Spencer 1985

we can guarantee $6\sqrt{n}$

By union bound the assignment violates the bound for some $A \in \mathcal{F}$

$\leq m \cdot \frac{2}{m^2} = \frac{2}{m} = 1$. So there exists f as required.

Conjecture (Komlós): There exists $\underline{K > 0}$ (independent on k & n) such that for $\underline{\text{any } d}$ and any $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ $\lfloor$ s.t. $\|v_i\| \leq 1 \rfloor$ there exist signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$



s.t. $\varepsilon_1 v_1 + \varepsilon_2 v_2 + \ldots + \varepsilon_n v_n \in [-K, K]^d$

If $v_i = \frac{1}{\sqrt{n}} (0,1)$-vector

this problem ~~is equivalent~~ is related to the previous problem.

Banaszczyk 98 $K = O(\sqrt{\log d})$ suffices.

In the above special case one should be able to recover this bound from Theorem 5.3.

The Four Color Theorem: (Appel, Haken, 1977)

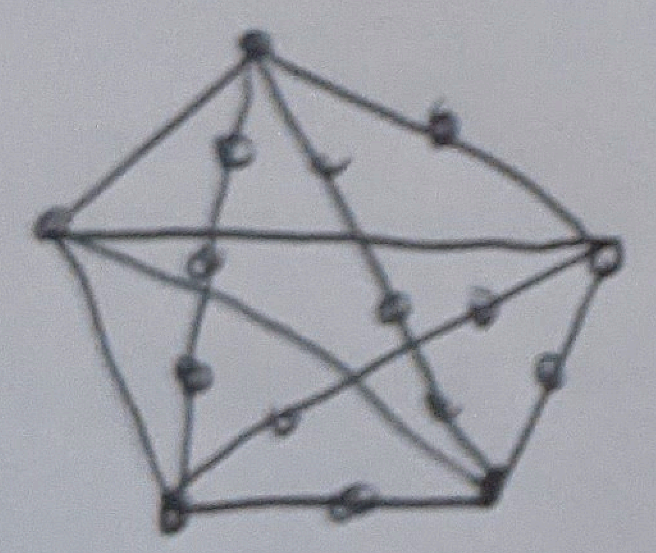
Planar graphs are four-colorable.

Graph H is a *minor* of a graph G if we can obtain H from a subgraph of G by *contracting* edges.

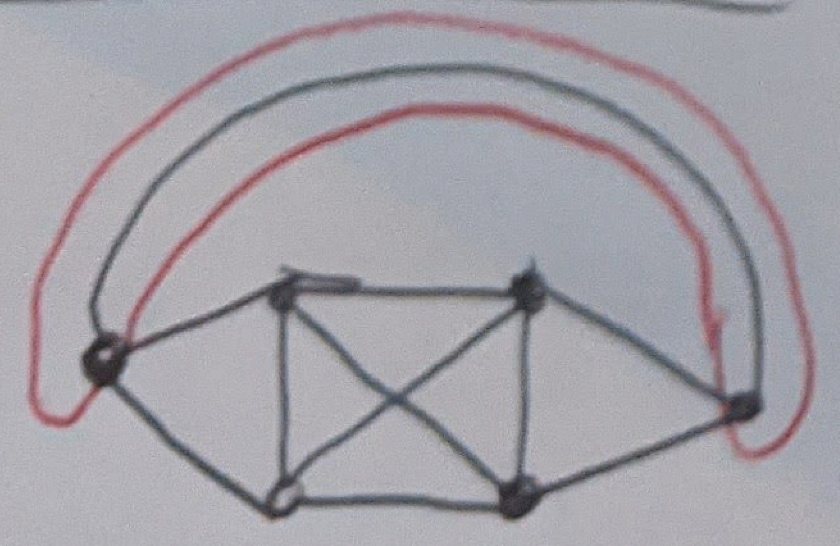
Graph G is a *subdivision* of H if G is obtained from H by replacing edges with paths, disjoint except for their ends

If G is a subdivision of H then H is a minor of G.

Converse does not hold

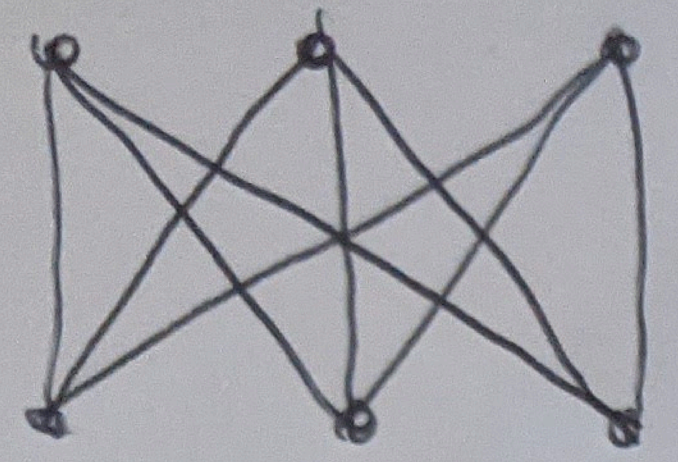


Subdivision of K_5



has a K_5 minor, but no subdivision

Kuratowski's theorem: A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$



$K_{3,3}$.

Wagner's theorem: G is planar if and only if G has no K_5 minor or $K_{3,3}$ minor.

Four color theorem $\iff$ $\boxed{\text{Every graph with no } K_5 \text{ minor is 4-colorable}}$

Wagner 1937

Conjecture (Hadwiger, 1943): Every graph with no K_{t+1} minor is t-colorable.

$t=1$

$t=2$ $\iff$ trees are 2-colorable

$t=3$ 'a Graph~~s~~ with no K_4 minor contain vertex of degree ≤ 2.

$t=4$ $\iff$ 4CT.

$t=5$. Robertson, Seymour, Thomas 94 uses 4CT.

$t \geq 6$ open. [no K_{t+1} minor $\chi(G) \leq O(t\sqrt{\log t})$ Thomason, Kostochka 1980's.

2020+

Best bound: Postle $\chi(G) = O(t(\log\log t)^6)$

Conjecture (Hajos, 1961): Every graph with no subgraph isomorphic to K_{t+1} subdivision is t colorable.

True for $t \leq 3$ <=> Hadwiger's conjecture.

Open For $t = 4, 5$

$\boxed{\text{False for } t \geq 6}$

We explore validity of Hajos' & Hadwiger's conjecture for random graphs. $G(n, \frac{1}{2})$.

$\alpha(G)$ is the independence number, size of the largest independent set in G.

set of vertices, pairwise non-adjacent.

$$\mathbb{P}[(A \text{ is independent})] = (2)^{-\binom{|A|}{2}}.$$

$$\mathbb{E}[\#\text{ independent sets of size } k] = \binom{n}{k} 2^{-\binom{k}{2}}$$

$$= 2^{(1+o(1))\left(k \log_2 n - \frac{k^2}{2}\right)} < 1$$

$$\text{if} \quad k = (1+o(1))\underbrace{2\log_2 n}$$

So with high probability $G(n, \frac{1}{2})$ has no independent sets of size $c\log_2 n$ for any $c > 2$.

$$\alpha(G(n, \tfrac{1}{2})) \boxed{\leq} (1+o(1))\, 2\log_2 n$$

in Fact we have equality (Exercise in 2^{nd} moment).

$$\chi(G) \leq \frac{n}{\alpha(G)}$$

So w.h.p. $\chi(G(n, \frac{1}{2})) \boxed{\geq} (1-o(1)) \dfrac{n}{2\log_2 n}$

equality also holds here.

proving this requires one to show that w.h.p any subgraph of $G(n, \frac{1}{2})$ on $\geq \frac{n}{(\log n)^2}$ vertices has independent set of size $\geq (1-o(1)) 2\log_2 n$.

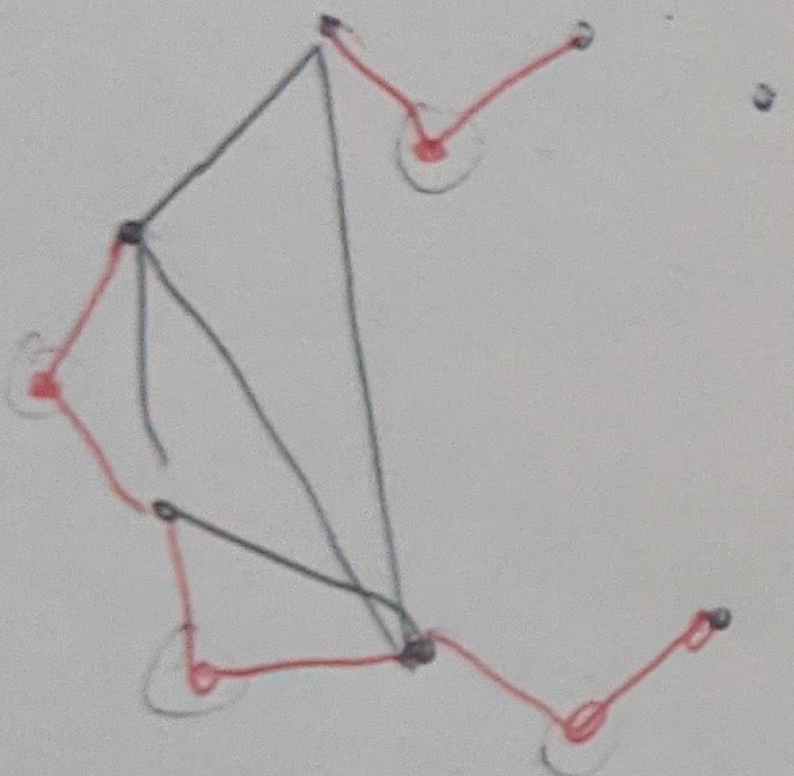
Theorem 5.4: With high probability $G(n, \frac{1}{2})$ has no subdivision of $K_{\lceil 10\sqrt{n}\rceil}$ as a subgraph.

~~Bollobá~~ (Erdős Fajtlowicz, 1981).

(So there exist graphs G $\chi(G) \geq (1-o(1))\frac{n}{2\log_2 n}$, no $K_{\lceil 10\sqrt{n}\rceil}$ subdivision

Proof: Let $t = \lceil 10\sqrt{n}\rceil$. At most n edges of K_t can be subdivided in a subdivision with $\leq n$ vertices.

K_t



So $\geq \binom{t}{2} - n$ edges are not subdivided

$\binom{t}{2} - n \geq \frac{3}{4}\binom{t}{2}$

$n \leq \frac{1}{4}\binom{t}{2}$

If $G(n, \frac{1}{2})$ has a K_t subdivision then there exists a subgraph of $G(n, \frac{1}{2})$ with t vertices and $\geq \underbrace{\frac{3}{4}\binom{t}{2}}$ edges.

$$\mathbb{P}\left(G(t, \tfrac{1}{2}) \text{ has } \geq \underbrace{\tfrac{3}{4}\tbinom{t}{2}}\right) \underset{\text{by chernoff}}{\leq} e^{-\frac{1}{16}\binom{t}{2}\frac{1}{2}} \leq e^{-\frac{t^2}{64}}$$

$n' = \binom{t}{2}$ $\quad \lambda = \frac{1}{4}\sqrt{\binom{t}{2}}$

Summing over all subgraphs on t vertices

$$\mathbb{P}(G(n,\tfrac{1}{2}) \text{ has a } K_t \text{ subdivision}) \leq n^t e^{-t^2/64} = \underbrace{e^{\log n\, t - \frac{t^2}{64}}} \ll 1.$$

— Bollobás, Thomason: If G has no K_t subdivision then $\chi(G) \leq c\,t^2$ for some c independent on t.