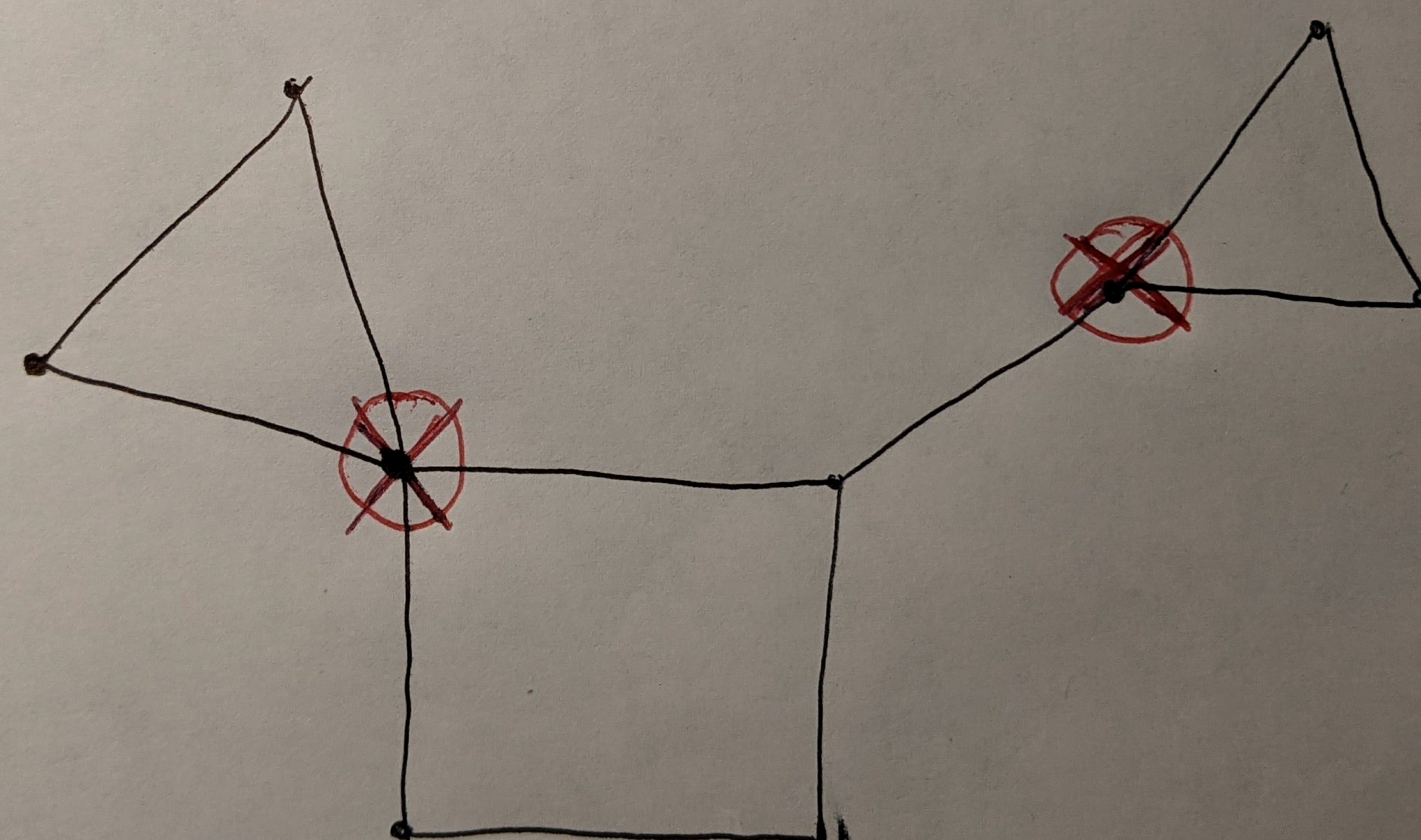
Lecture 5: Alterations



3. Alterations

Recall:

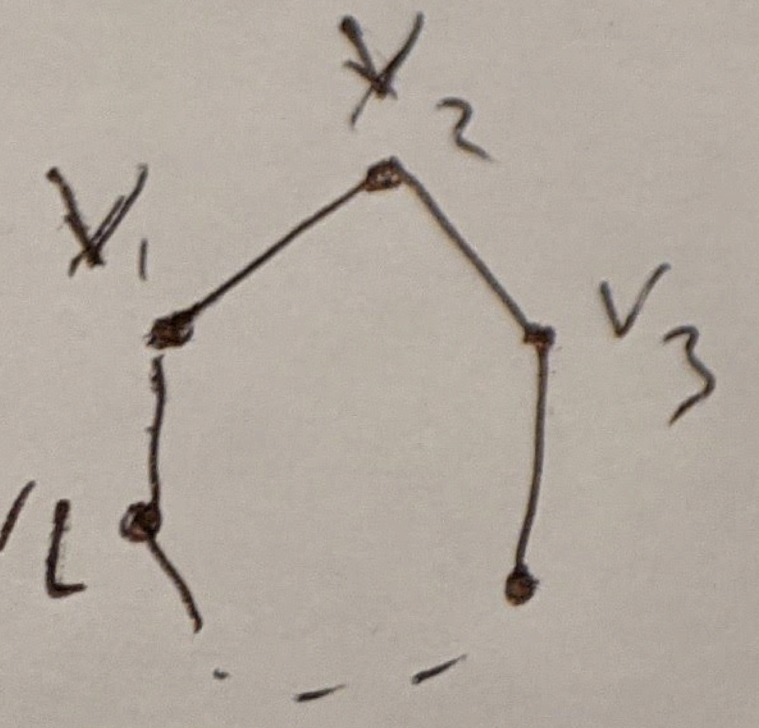
k-coloring of a graph G → $c: V(G) \to [k] = \{1,2,\dots,k\}$
$c(u) \neq c(v)$ if u & v are adjacent

The **chromatic number** of G $\chi(G)$ - minimum k s.t.
there exists a k-coloring of G.

Goal: show that there exist graphs with $\chi(G) \geq k$
and every subgraph on $\leq L$ vertices 2-colorable.

$G(n,p)$ is a random graph with n vertices, where
For every pair of vertices we join them by an edge
with probability p.

Theorem 3.1: For ~~every~~ all $k, L \in \mathbb{N}$ there exists
a graph G with $\chi(G) \geq k$
and no cycles of length $\leq L$



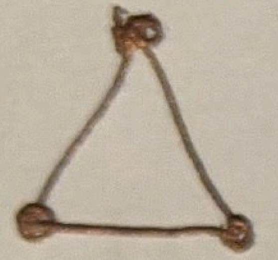
Proof: Consider $G(n,p)$ $\left(c\frac{\log n}{n} \ll p \ll \varepsilon \frac{n^{1/\ell}}{n} \right)$

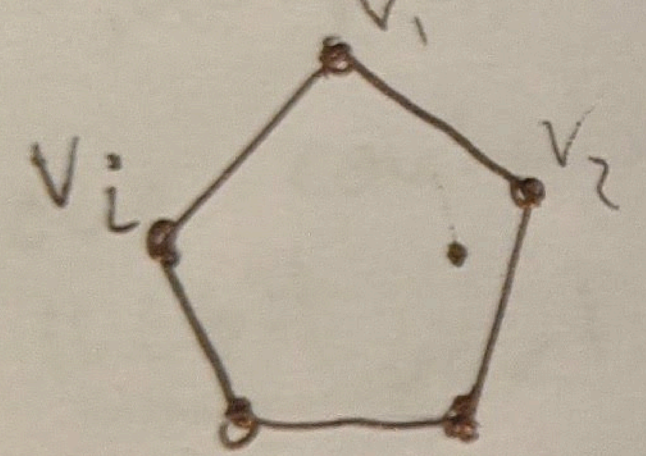
Let X be the number of cycles of length $\leq \ell$ in $G(n,p)$.

$$E[X] = \sum_{i=3}^{\ell} \frac{(i-1)!}{2}\binom{n}{i}p^i \leq \sum_{i=3}^{\ell} n^i p^i \leq \ell(pn)^\ell \leq \varepsilon \ell n$$

for any ε we want.

(?) expected number of cycles of length i.

$i = 3$  $p^3\binom{n}{3}$

i : 

$\rightarrow$ $p^i \cdot \underbrace{\text{\# of cycles of length } i \text{ in } K_n}_{= \frac{i!\binom{n}{i}}{2i}}$

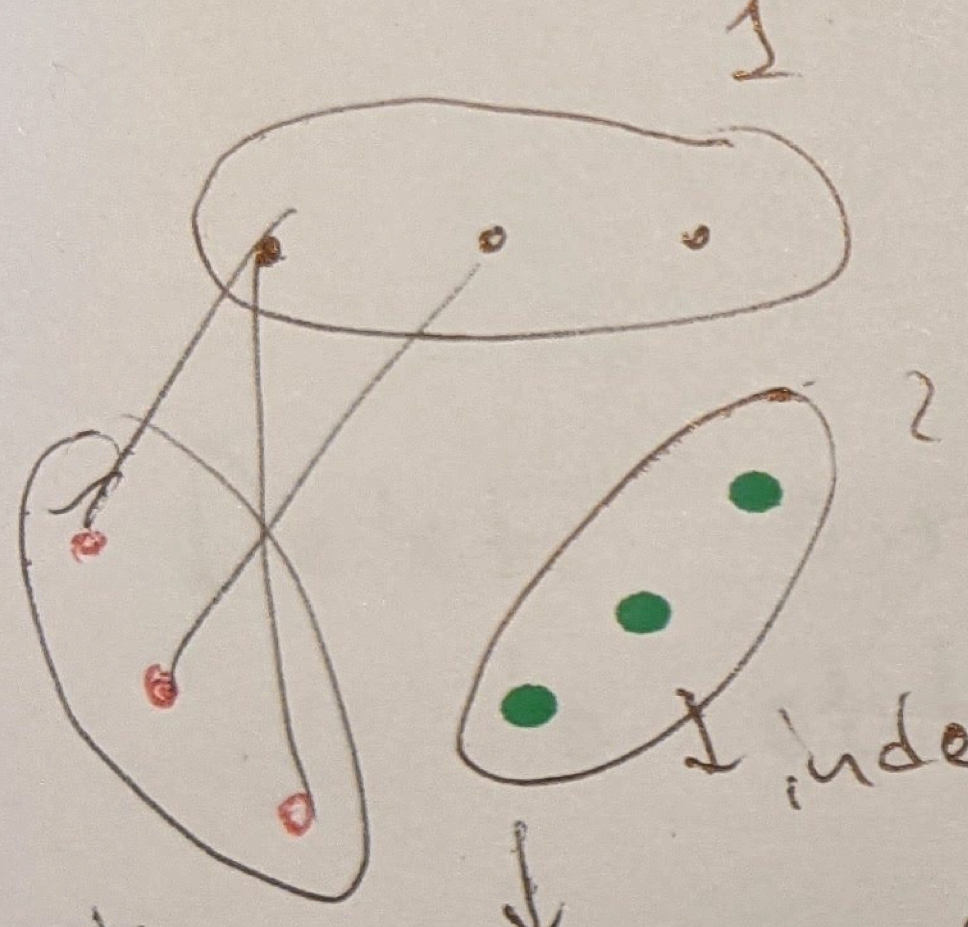
$$\Pr\left[X \geq \frac{n}{2}\right] \leq \frac{E[X]}{n/2} \leq 2\varepsilon\ell \leftarrow \text{can be made small.}$$

by Markov

(So it is enough to show that with probability $> \frac{1}{2}$ no subgraph of $G(n,p)$ on $\frac{n}{2}$ vertices is k-colorable).

Let $\underbrace{\alpha(G)}$ <u>independence number</u> be the maximum size of an independent set in G:
set X s.t. no two vertices in X are adjacent.

Then $\chi(G) \geq \dfrac{|V(G)|}{\underbrace{\alpha(G)}}$ for any G.



independent

$\geq$ one contains $\dfrac{|V(G)|}{k}$ vertices.

$G = G(n,p)$

$$\Pr(\alpha(G) \geq x) \leq \sum_{\substack{X \subseteq V(G) \\ |X| = x}} \Pr[X \text{ is independent}]$$

$$= \underbrace{(1-p)}^{\binom{x}{2}} \binom{n}{x} \leq n^x \cdot e^{-p\frac{x^2}{2}}$$

$$= e^{\log n \, x - p \frac{x^2}{2}}$$

$$= e^{\frac{3\log^2 n}{p} - 4.5\frac{\log^2 n}{p}}$$

$$= e^{-\frac{1.5\log^2 n}{p}} < \frac{1}{2}.$$

$1 - p \leq e^{-p}$

$\boxed{x = \dfrac{3\log n}{p}}$

$\Pr\left(\alpha(G) \geq \dfrac{n}{2k}\right) < \dfrac{1}{2}$ as long as $\dfrac{n}{2k} > \dfrac{3\log n}{p}$ $\quad p > \dfrac{6k\log n}{n}.$

So probability that $G(n,p)$ has $\geq \frac{n}{2}$ cycles of length $\leq L$ $< \frac{1}{2}$

and probability $\alpha(G(n,p)) \geq \frac{n}{2k}$ $< \frac{1}{2}$.

So there exists G with n vertices

$\not\geq \frac{n}{2}$ cycles of length $\leq L$

and $\alpha(G) < \frac{n}{2k}$.

Delete one vertex from each short cycle of G to obtain G' with

$|V(G')| \geq \frac{n}{2}$ $\quad \alpha(G') < \frac{n}{2k}$

then $\chi(G') > \dfrac{n/2}{n/2k} = k$. ✓.

Erdös ($G(n,p)$) has shown that there exists $\varepsilon > 0$ and for any k a graph G with $\chi(G) \geq k$ on n vertices s.t. every subgraph of G on $\leq \varepsilon n$ vertices is 3-colorable.

Question: can this be true with 2-colorable?

Reminder: $m(k)$ - minimum # of edges in a non 2-colorable k-graph. (k-uniform hypergraph).

minimum size of a collection S of k-element sets s.t. for every "coloring"

$$c: \bigcup_{X \in S} X \to \{\text{red}, \text{blue}\}.$$

there exists $X \in S$ with all element of the same color

$$\boxed{2^{k-1}} \leq m(k) \leq C k^2 2^k.$$

Theorem 3.2 (Radhakrishnan & Srinivasan, 2000) Proof by Cherkashin & Kozik

$$m(k) = \Omega\left(\sqrt{\frac{k}{\log k}}\, 2^k\right)$$

(i.e. there exists $\varepsilon > 0$ s.t. $m(k) \geq \varepsilon \sqrt{\frac{k}{\log k}}\, 2^k$

Proof: Let S be a collection of m k-element sets.

We want to show that if $m < \varepsilon \sqrt{\frac{k}{\log k}}\, 2^k$ then S is 2-colorable.

Let X be the union of sets in S.

Choose an Order vertices in X uniformly at random $v_1, v_2, \ldots, v_n$

and greedily color S: in this order

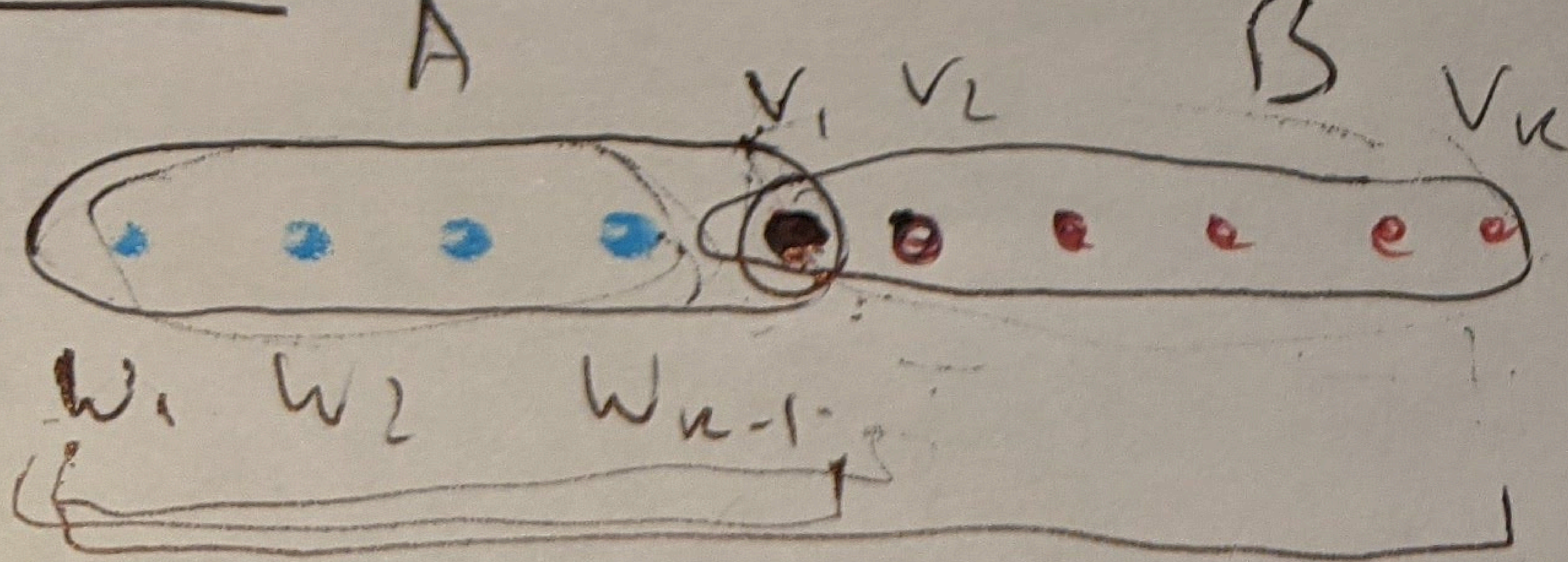
- color v_i $\boxed{\text{blue}}$ unless this creates a monochromatic edge $A \subseteq \{v_1, v_2, \ldots, v_i\}$, otherwise color v_i $\boxed{\text{red}}$

— Choose a map $\underbrace{X \to [0,1]}$ mapping vertices independently uniformly at random.

$x \to \underbrace{r(x)} \in [0,1]$

Color vertices from smallest to largest according to r.

— Want: probability that there is no monochromatic edge is > 0.



If there is a monochromatic edge then there edges $A, B \in S$ s.t. $A \cap B = \{v\}$ and v is the last vertex of A first vertex of B.

Call this a conflicting pair

__First attempt:__ (?) probability that a given pair (A, B) with $|A \cap B| = 1$, is conflicting.

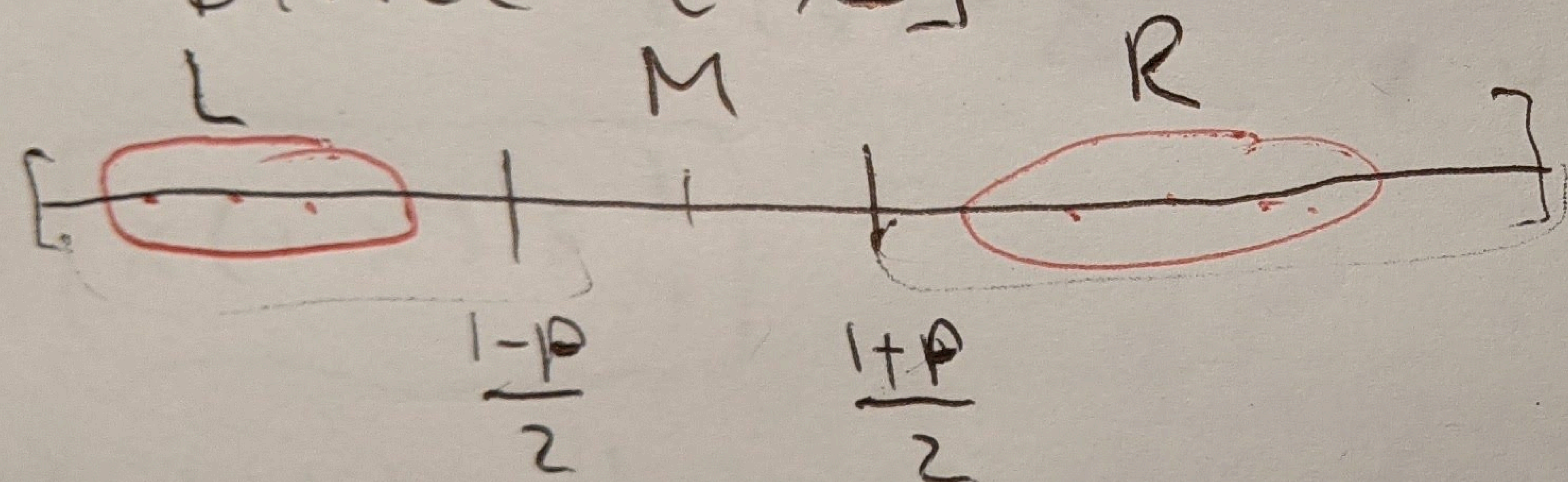
$$\frac{((k-1)!)^2}{(2k-1)!} = \frac{1}{2k-1}\binom{2k-2}{k-1}^{-1} = \Theta\left(\frac{1}{\sqrt{k}\,4^k}\right)$$

There are m^2 potential conflicting pairs

so if $\frac{m^2}{\sqrt{k}\,4^k} = o(1)$ then S is 2-colorable

giving $m(k) = \Omega(k^{1/4} 2^k)$

~~Let~~ Divide $[0,1]$



$L = [0, \frac{1-p}{2}]$

$M = [\frac{1-p}{2}, \frac{1+p}{2}]$

$R = [\frac{1+p}{2}, 1]$

p to be chosen

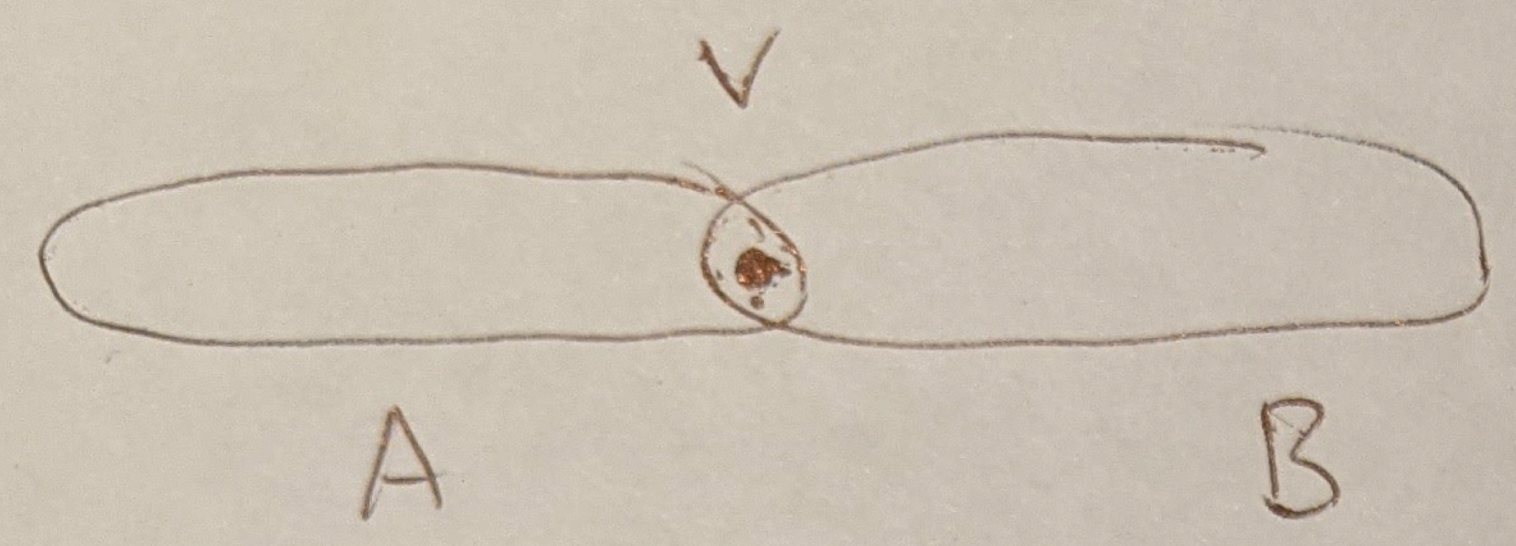
An edge $A \in S$ is __bad__ if $r(x) \in L$ for every $x \in A$

or $r(x) \in R$ for every $x \in A$.

$$\Pr[A \text{ is bad}] = 2\left(\frac{1-p}{2}\right)^k$$

$$\Pr[\text{there } \cancel{\text{are no}} \text{ is a bad edge}] \le 2m\left(\frac{1-p}{2}\right)^k$$

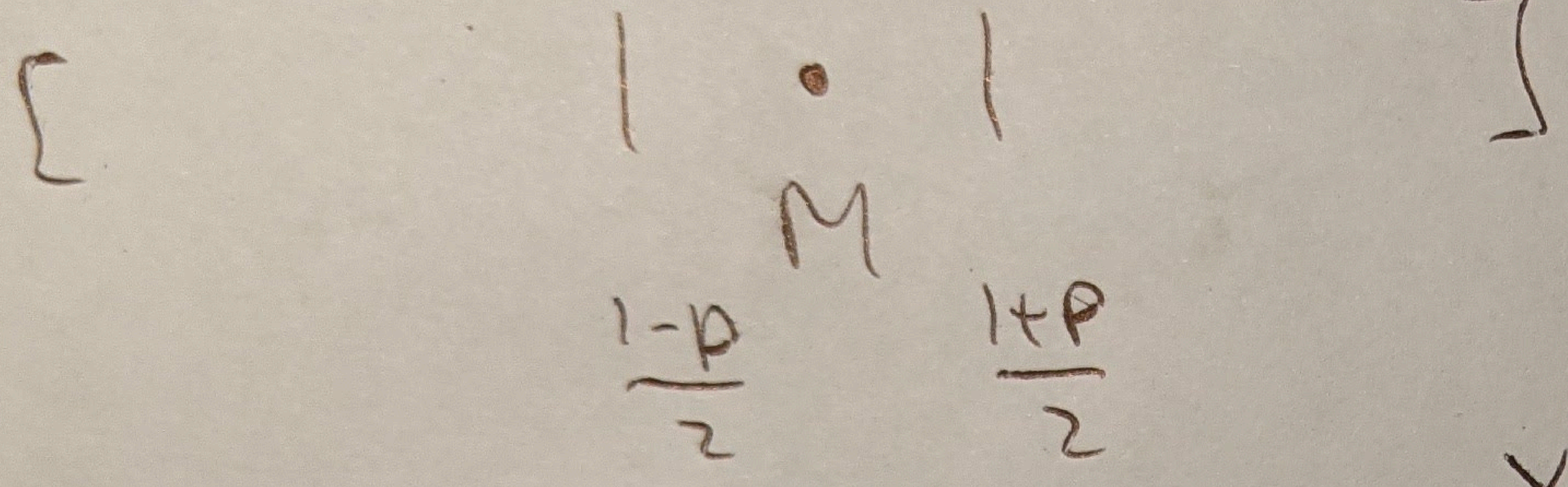
Probability that a pair (A,B) is conflicting,
but neither A nor B are bad.



$$\le |M| \cdot \Pr[(A,B) \text{ is conflicting} \mid r(v) \in M]$$

$$\le |M| \cdot \max_x x^{k-1} \cdot (1-x)^{k-1}$$

$$\le \frac{|M|}{4^{k-1}} = \frac{p}{4^{k-1}}.$$



x^{k-1} ↓ v is maximum in A

$(1-x)^{k-1}$ ↓ v is mininimum in B

$r(v) = x$

$$\boxed{x(1-x) \le \frac{1}{4}}$$

Probability there is a conflicting pair

$\le$ probability of a bad edge + expected number of conflicting pairs with no bad edges

$$= \underbrace{2m\left(\frac{1-p}{2}\right)^k + m^2 \cdot \frac{p}{4^{k-1}}} < 1 \quad \text{if} \quad m = O\left(\sqrt{\frac{k}{\log k}}\, 2^k\right)$$

$$p = \frac{\log(k2^{-k}/m)}{k}.$$