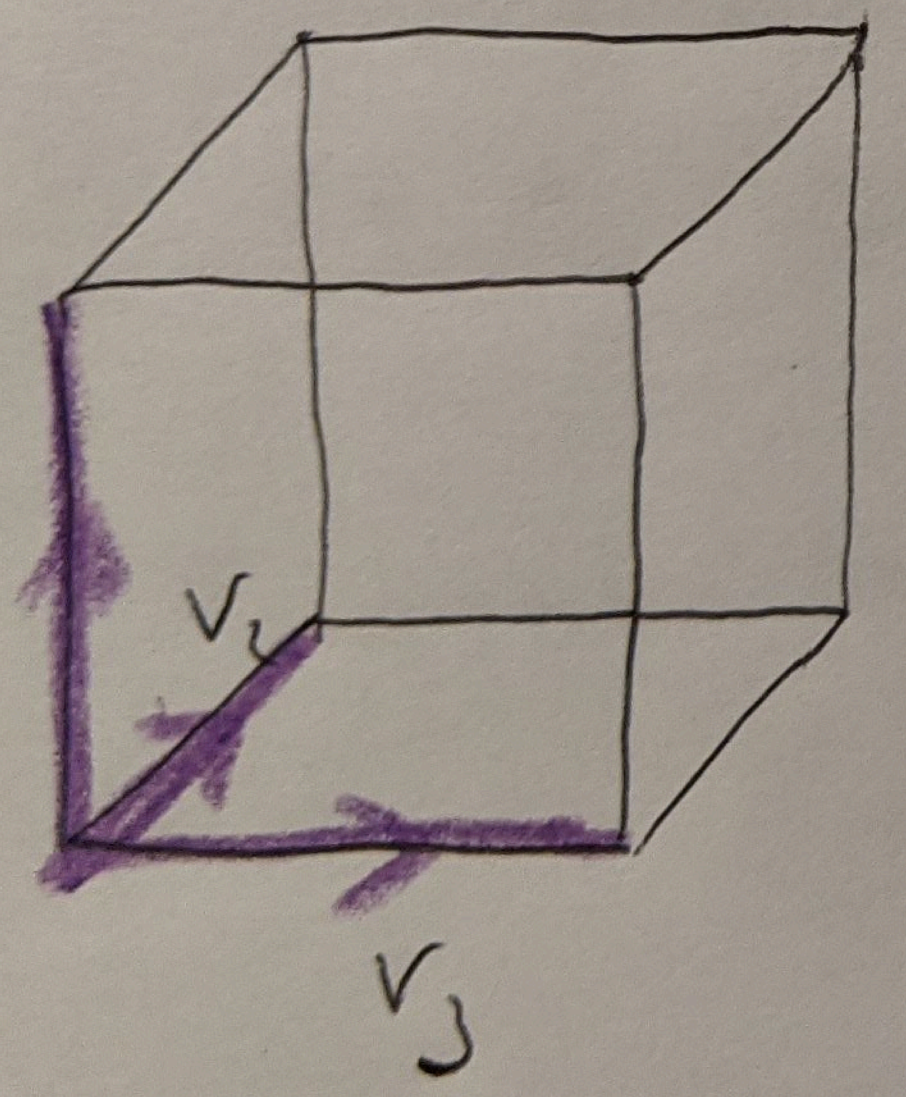
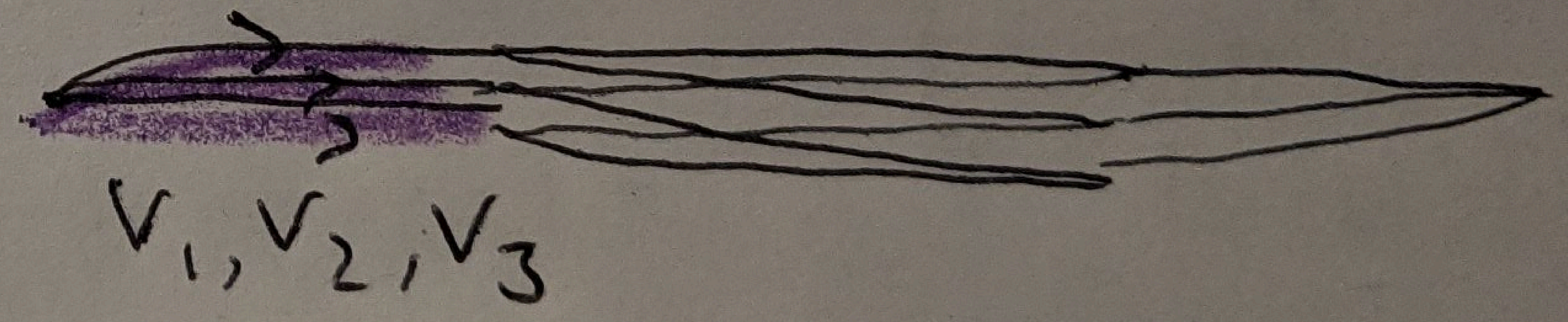
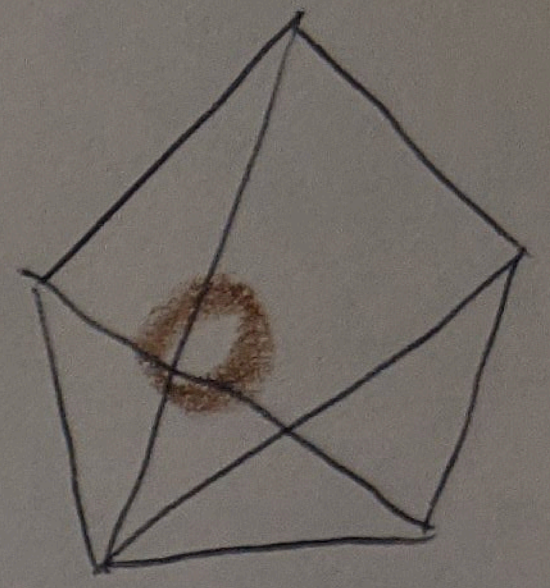
Lecture 4:
Linearity of Expectation: Crossings, Vectors & Cycles.





Recall:

Theorem 2.5: (Crossing Lemma) Let G be a graph with n vertices & m edges, $m \geq 4n$.
Then $cr(G) \geq \frac{1}{64} \frac{m^3}{n^2}$



Proof: From Euler's formula $m - 3n$

$cr(H) \geq |E(H)| - 3|V(H)|$.

Select $X \subseteq V(G)$ by choosing each vertex independently with probability p

Let H be a random subgraph of G with $V(H) = X$

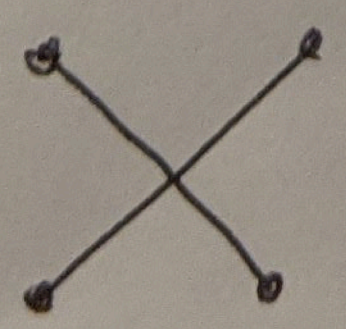
$E(H)$ consists of all edges with both ends in X.

(H is induced by X).

$$\mathbb{E}[cr(H)] \geq \mathbb{E}[|E(H)|] - 3\,\boxed{\mathbb{E}[|V(H)|]}$$

$$\mathbb{E}[|V(H)|] = pn$$

$$\mathbb{E}[cr(H)] = p^4 \cdot cr(G) \qquad \mathbb{E}[|E(H)|] = p^2 m \qquad \mathbb{E}[|V(H)|] \overset{\text{linearity}}{=} \sum_{v \in V(G)} \mathbb{P}(v \in X) = pn$$



$$p^4 \, cr(G) \geq p^2 m - 3pn$$

$$cr(G) \geq \underbrace{m p^{-2} - 3 n p^{-3}}$$ $\rightarrow$ choose p to maximize

$$\frac{\partial}{\partial p} = \underbrace{-2mp^{-3} + 9np^{-4}} \qquad \times p^4$$

$$p = \frac{9n}{2m}. \qquad \boxed{p = \frac{4n}{m}}$$

$$cr(G) \geq m\left(\frac{m}{4n}\right)^2 - 3n\left(\frac{m}{4n}\right)^3 = \left(\frac{1}{16} - \frac{3}{64}\right)\frac{m^3}{n^2} = \frac{1}{64}\frac{m^3}{n^2}.$$

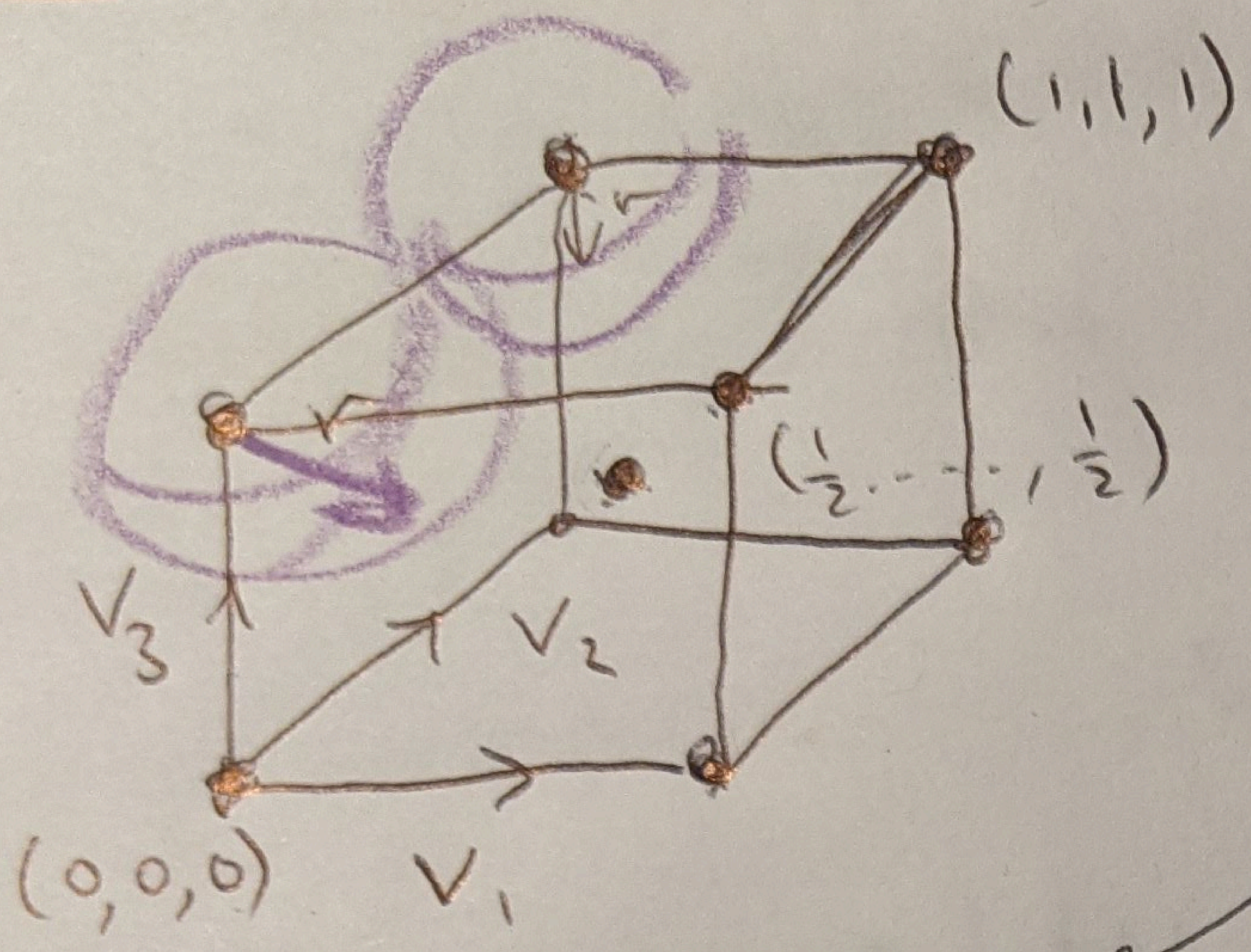
Crossing Lemma has many nice unexpected applications.

see notes for MATH 550,

Yufei Zhao's notes.

(i.e. sum-product inequalities)

Sums of vectors



$$H_n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \{0,1\} \}$$

n-dimensional hypercube $\quad \parallel$

$$\{ \varepsilon_1 v_1 + \varepsilon_2 v_2 + \dots + \varepsilon_n v_n \mid \varepsilon_i \in \{0,1\} \}.$$

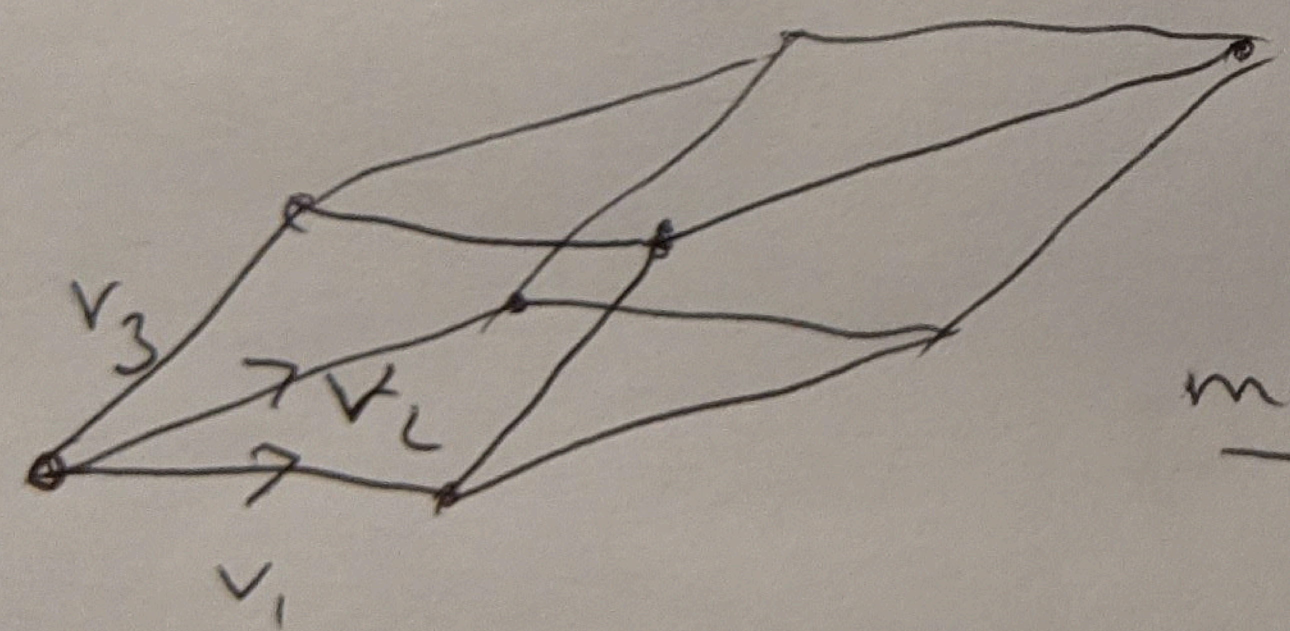
$$\boxed{V_I = \sum_{i \in I} v_i} \qquad \overset{\parallel}{\{ V_I \mid I \subseteq [n] \}}.$$

$V_{\{1,2\}} = v_1 + v_2$

radius of $H_n = \sqrt{n \cdot (\frac{1}{2})^2} = \frac{\sqrt{n}}{2}$.

Balls of radius $\frac{\sqrt{n}}{2}$ centered at vertices cover $\underbrace{H_n.}_{\text{interior of}}$

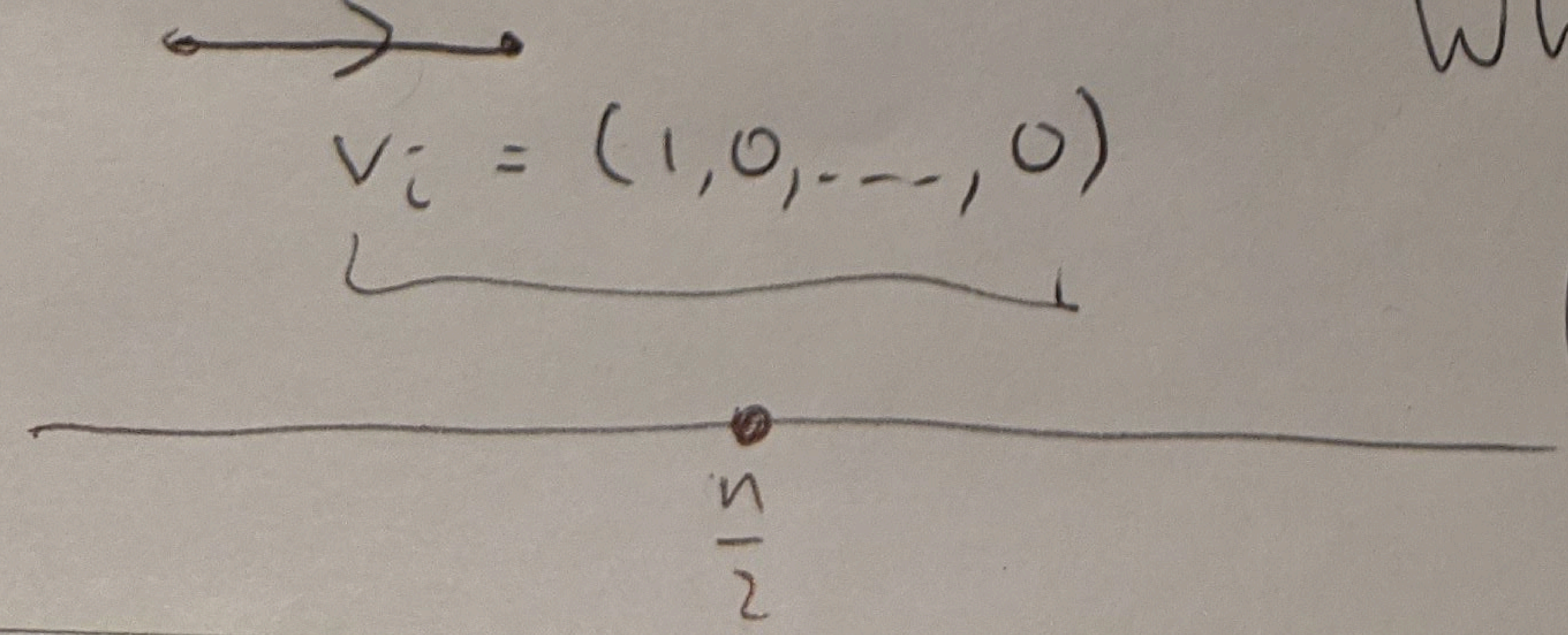
— Now relax the assumption that $v_1, v_2, \dots, v_n$ is orthonormal basis, only assume $|v_i| = 1$ for every i



$$S = \{ V_I \mid I \subseteq [n] \}$$

maximum radius of $S = \frac{n}{2}$ ($\frac{n}{2}$ always suffices exercise)

What about covering half of S?



$$\binom{n}{\frac{n}{2} - r} + \dots + \binom{n}{\frac{n}{2}-1} + \underbrace{\binom{n}{\frac{n}{2}}}_{\leq c \frac{2^n}{\sqrt{n}}} + \binom{n}{\frac{n}{2}+1} + \dots + \binom{n}{\frac{n}{2}+r}$$

Theorem 2.6: Let $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ s.t. $|v_i| = 1$.

~~Let $S = \{V_I \mid I \subseteq [n]\}$ (considered with multiplicities)~~

Then $|V_I - W| \leq \sqrt{n/2}$ for at least 2^{n-1} $I \subseteq [n]$ where

$$W = \frac{V_1 + V_2 + \dots + V_n}{2}.$$

Markov's inequality

Let $X \geq 0$ random variable. Then

$$P[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \quad \text{for every } a > 0.$$

Proof: $\mathbb{E}[X] = \underbrace{\mathbb{E}[X \mid X < a]}_{\geq 0} P[X < a] + \underbrace{\mathbb{E}[X \mid X \geq a]}_{\geq a} P[X \geq a]$

$$\geq a P[X \geq a].$$

$$\mathbb{P}[X \leq a] \geq 1 - \frac{\mathbb{E}[X]}{a}.$$

Example: $a = 2\mathbb{E}[X]$

$$\mathbb{P}[X \leq 2\mathbb{E}[X]] \geq \frac{1}{2}.$$

Proof: $V_I - W = \sum_{i\in[n]}' \varepsilon_i v_i$ $\quad\quad$ $\varepsilon_i = \begin{cases} \frac{1}{2} & \text{if } i \in I \\ -\frac{1}{2} & \text{if } i \notin I \end{cases}$

So we assume that ε_i are independently chosen so that

$$\mathbb{P}[\varepsilon_i = \tfrac{1}{2}] = \mathbb{P}[\varepsilon_i = -\tfrac{1}{2}] = \tfrac{1}{2}.$$

$$\mathbb{E}\Big[\big|\sum_{i\in[n]} \varepsilon_i v_i\big|^2\Big] = \mathbb{E}\Big[\big\langle \sum_{i\in[n]}' \varepsilon_i v_i , \sum_{j\in[n]}' \varepsilon_j v_j \big\rangle\Big]$$

$$= \sum_{\substack{i,\in[n] \\ j\in[n]}} \underset{\overset{\shortparallel}{?}}{\Big[\mathbb{E}[\varepsilon_i \varepsilon_j]\Big]} \langle v_i, v_j\rangle = \sum_{i\in[n]}' \tfrac{1}{4} |v_i|^2 = \frac{n}{4}.$$

If $i \neq j$ then $\mathbb{E}[\varepsilon_i \varepsilon_j] = \mathbb{E}[\varepsilon_i]\,\mathbb{E}[\varepsilon_j] = 0$

By Markov

$$\mathbb{P}\Big[\big|\sum_{i\in[n]} \varepsilon_i v_i\big|^2 \leq \frac{n}{2}\Big] \geq \frac{1}{2}$$

$$\shortparallel$$

$$\mathbb{P}\Big[|V_I - W| \leq \sqrt{\frac{n}{2}}\Big] \geq \frac{1}{2}$$

$\uparrow$

I is chosen independently random.

$\{p_1 v_1 + p_2 v_2 + \dots + p_n v_n \mid p_i \in [0,1]\}$

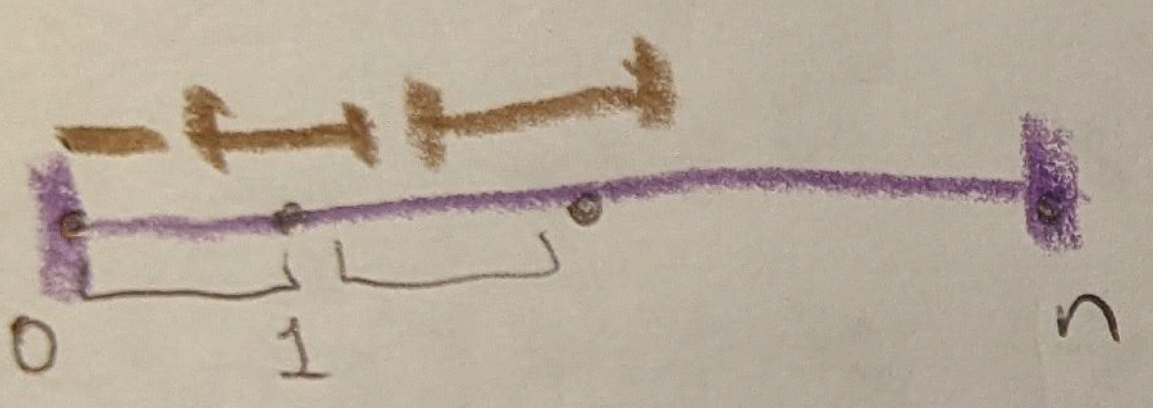
cover it with balls centered at "vertices"

$$v_I, \quad I \subseteq [n]$$

How big a radius do we need? Given $|v_i| = 1$.

? $\boxed{1/2}$

What if $v_1 = v_2 = \dots = v_n$?



Theorem 2.7: Let $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ be s.t. $|v_i| = 1$.

Then for every $w = p_1 v_1 + p_2 v_2 + \dots + p_n v_n$ s.t. $p_i \in [0,1]$ there exists $I \subseteq [n]$ s.t.

$$|w - v_I| \le \frac{\sqrt{n}}{2}.$$

(i.e. balls of radius $\frac{\sqrt{n}}{2}$ ~~cover~~ centered at vertices cover interior).

Proof sketch

$$w - v_I = \sum_{i \in [n]} \varepsilon_i v_i \qquad \varepsilon_i = \begin{cases} p_i & \text{if } i \notin I \\ p_i - 1 & \text{if } i \in I. \end{cases}$$

Select I at random. Let $\mathbb{P}[i \in I] = p_i$ for each $i \in [n]$, independently for each i.

$$\mathbb{E}[\varepsilon_i] = p_i(p_i - 1) + (1 - p_i)p_i = 0.$$

Calculations as in Theorem 2.6 yield the result.

Girth & chromatic number of graphs

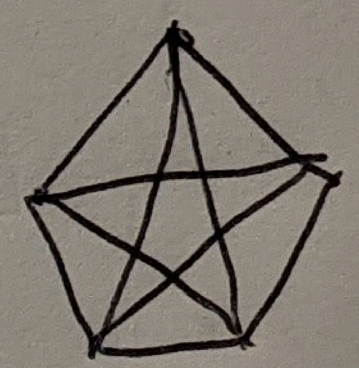
$\chi(G)$ the chromatic number is the minimum number of colors needed to color vertices s.t. adjacent vertices receive different colors.

$\chi(G)$ is hard to determine.

- Showing $\chi(G) \leq k$ is not hard $\rightarrow$ show coloring

What about showing $\chi(G) > k$ $\rightarrow$ Maybe we can find a small obstruction to k-coloring?

i.e. K_{k+1} $\rightarrow$ complete subgraph on $(k+1)$ vertices.



$(\chi(K_{k+1}) = k+1)$.

We will show that there exist graphs with $\chi(G) \geq k$ and every subgraph on $\leq L$ vertices is two-colorable for any pair k & L.

Examples come from random graphs.

~~A random~~

Erdös-Renyi <u>random graph</u> $G(n,p)$

is a graph with n vertices where every pair of vertices is chosen to be adjacent independently with probability p.

Expected number of <u>triangles</u> in $G(n,p)$

$\binom{n}{3} p^3 < 1$

If expected number < 1 then $p < \frac{2}{n}$

But then average degree < 2, the whole graph is likely two-colorable

To be continued.