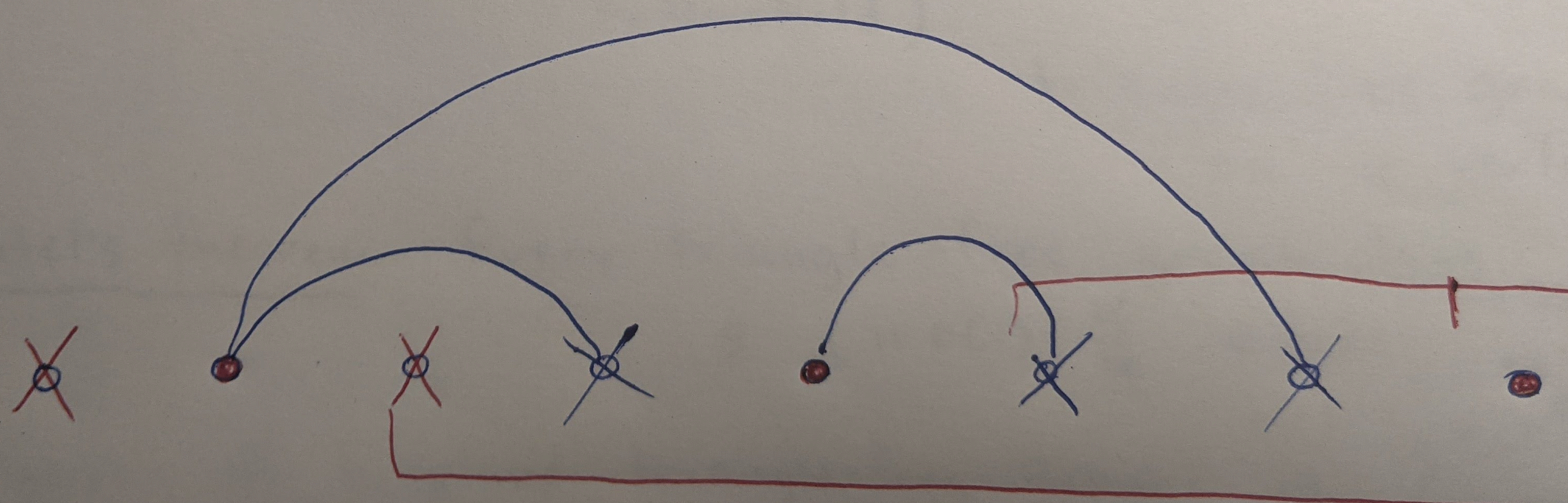
Lecture 22

Hypergraph Containers



12. Hypergraph Containers

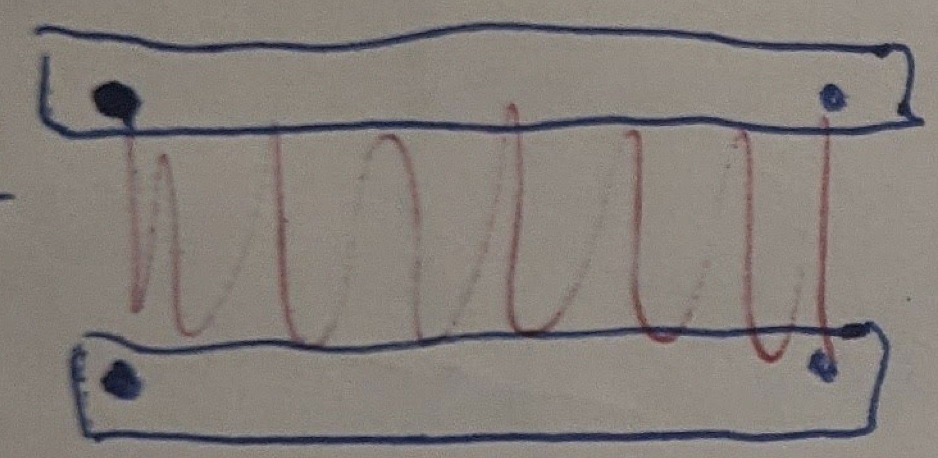
(Balogh, Morris, Samotij & Saxton, Thomason, 2015)

"Good" approximations of independent sets in Hypergraphs.

How many triangle-free graphs are there on n vertices?

<u>Lower bound:</u> $\frac{n^2}{4}$ edges $\to$ $\underbrace{2^{\frac{n^2}{4}}}$ subgraphs.

$K_{\frac{n}{2},\frac{n}{2}}$



$\frac{n}{2}$

$\frac{n}{2}$

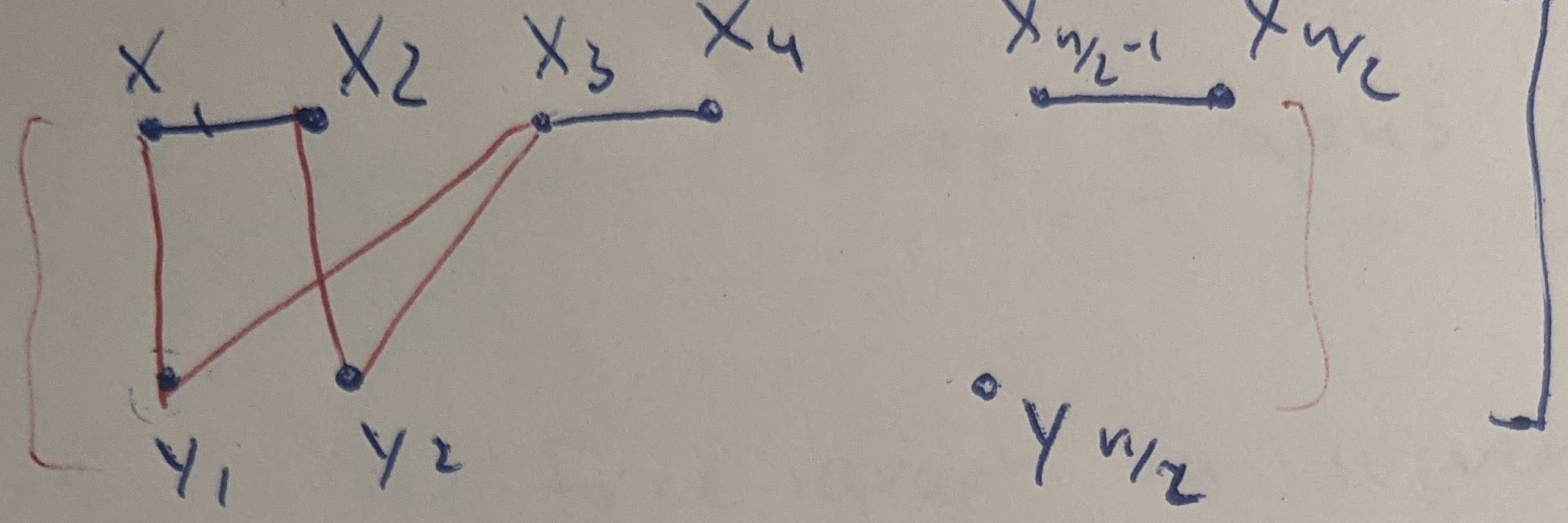
<u>Theorem</u> (Erdös, Kleitman, Rothschild): There are $2^{(1+o(1))\frac{n^2}{4}}$ triangle-free graphs on n vertices.

<u>Mantel's theorem:</u> Every triangle-free graph on n vertices has $\leq (1+o(1))\frac{n^2}{4}$ edges

If number of maximal triangle-free graphs is $2^{o(n^2)}$ then the theorem follows from Mantel by taking union.

Unfortunately, it's not true:

Example:



join each y_i to exactly one of vertices x_{2j-1} x_{2j} for every j

There are $2^{(1-o(1))\frac{n}{2}\cdot\frac{n}{4}}$ maximal triangle free graphs of this form.

Subgraphs of a universal graph which is not triangle-free but still has $(1+o(1))\frac{n^2}{4}$ edges (and is "close" to being triangle-free).

Container

Theorem 12.1: For every $\varepsilon > 0$ there exist $C > 0$. Such that for every n there exists a collection $\boxed{\mathcal{C}}$ of n vertex graphs such that.

containers for triangle free graphs.

- every triangle free graph on n vertices is a subgraph of a graph in $\mathcal{C}$
- every graph in $\mathcal{C}$ has at most $(\frac{1}{4}+\varepsilon)n^2$ edges
- $|\mathcal{C}| \leq n^{Cn^{3/2}}$

Corollary 12.2: There are ~~[illegible]~~ $2^{(1+o(1))\frac{n^2}{4}}$ triangle free graphs on n vertices

EKR

Proof: Every trianglee free graph on n vertices is a subgraph of some graph in $\mathcal{C}$, and each $G \in \mathcal{C}$ has

$$\leq 2^{(\frac{1}{4}+\varepsilon)n^2} \text{ subgraphs}$$

By union bound there are at most

$$2^{(\frac{1}{4}+\varepsilon)n^2} |\mathcal{C}| \leq 2^{(\frac{1}{4}+\varepsilon)n^2 + Cn^{3/2}\log n}$$

triangle free graphs.

Corollary 12.3: For $p \gg \frac{\log n}{\sqrt{n}}$ with high probabilty every triangle free subgraph of a graph in $G(n,p)$ has $\leq (\frac{1}{4}+o(1))pn^2$ edges.

Mantel's theorem for random graphs.

Proof: Let $G \in \mathcal{C}$

By Chernoff bound

$$\mathbb{P}\big[\, |\underbrace{E(G) \cap E(G(n,p))}_{\substack{\text{expected size} \\ = p|E(G)|}}| \geq (p+\varepsilon)|E(G)| \,\big] \leq e^{-C_\varepsilon n^2 p}$$

$$(p+\varepsilon)|E(G)| \leq (p+\varepsilon)\left(\tfrac{1}{4}+\varepsilon\right)n^2 \leq \left(\tfrac{p}{4}+3\varepsilon\right)n^2$$

Taking union over $G \in \mathcal{C}$

$$\mathbb{P}\left[\begin{array}{l} G(n,p) \text{ contains a triangle-free} \\ \text{subgraph with } \geq \left(\frac{p}{4}+3\varepsilon\right)n^2 \text{ edges} \end{array}\right] \leq e^{-C_\varepsilon n^2 p} \cdot e^{c n^{3/2}\log n}$$

$\boxed{(1+3\varepsilon)\frac{pn^2}{4} \text{ instead}}$

$n^2 p \gg n^{3/2}\log n$

$p \gg \frac{\log n}{\sqrt{n}}$ ✓

Construct a 3-uniform hypergraph H by setting $V(H) = [n]^{(2)}$

set of all edges of ~~complete~~ K_n

$E(H) = \{ e_1 e_2 e_3 , \ e_1, e_2, e_3 \text{ form a triangle} \}$.

Graph on n vertices $\longleftrightarrow$ subsets of $V(H)$

triangle free $\longleftrightarrow$ subset of $V(H)$ is independent

does not contain all vertices of any edge.

Theorem 12.1 follows from a more general container theorem for independent sets in 3-uniform hypergraphs.

Theorem 12.4: For every $c>0$ there exist $\delta>0$. satisfying the following.

for initial H
for 12.1
$\underbrace{d=n-1}$

Let H be a 3-uniform hypergraph with average degree $\underbrace{d \geq 1/\delta}$

s.t. no vertex has degree $> \underbrace{c \cdot d}$
no pair of vertices belongs to more than $c\sqrt{d}$ edges together.

Then there exists $\mathcal{C}$ collection of subsets of $V(H)$

in trianglefree graphs on n vertices.

s.t.

- every indendent set in H is contained in some $C \in \mathcal{C}$
- $|C| \leq (1-\delta)|V(H)|$ where $n = |V(H)|$
- $|\mathcal{C}| \leq n^{n/\sqrt{d}}$ ✓

$\underbrace{(1-\delta)\binom{n}{2}}$ → want $\underbrace{(\frac{1}{4}+\varepsilon)n^2}$

if a container C has $> (\frac{1}{4}+\varepsilon)n^2$ edges
then • $\varepsilon' n^3$ triangles.

apply theorem 12.4 again.

Theorem 12.5: For every $c > 0$ there exists $\delta > 0$ satisfying the following

Let G be a graph with average degree d and maximum degree $\leq cd$

Then there exists $\mathcal{C}$ collection of subsets of $V(G)$

s.t.
- every ind. set of G is contained in some $C \in \mathcal{C}$
- $|C| \leq (1-\delta)|V(G)|$.
- $|\mathcal{C}| \leq n^{\frac{2\delta n}{d}}$ where $n = |V(G)|$.

Proof: ~~Algorithm~~ ~~Input: Graph~~

We will show that for every independent set I in G there exists a Signature $S(I) \subseteq I$

s.t.
- $S(I) \subseteq I \subseteq \underbrace{C(S(I))}$ container for I determined only by its signature $S(I)$
- there are few signatures

$|S(I)| \leq 2\delta n$ $\quad$ $|C(S(I))| \leq (1-\delta) n$

Algorithm for finding $S(I)$ and $C(S(I))$.

Input: I - independent.

Start:
- $S = \emptyset$
- $A = V(G)$
- $X = \emptyset$

Maintain:
- S - current signature
- A - set of available vertices
- X - set of forbidden vertices

$X \cap I = \emptyset$ throughout.
$(S; A, X)$ is a partition of $V(G)$
$S \subseteq I$

~~no~~ ~~greater~~

Steps: While $\boxed{|S| \leq \frac{2\delta n}{d}}$

Let $v \in \underbrace{A \cap I}$ with maximum degree in $\boxed{G[A]}$ (where ties are broken in some canonical order)

- $S := S \cup \{v\}$
- $X := X \cup \{\text{neighbors of } v\} \cup \{\text{vertices of } A \text{ with higher degree than } v \text{ in } G[A]\}$
- $A := V(G) - S - X$.

Output: $S(I) = S$, $C(I) = V(G) - X$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\; = C(S(I))$

Need: $|C(S(I))| = |V(G) - X| \leq C_1 \delta \ldots$ i.e. $|X| \geq \delta \ldots$

We stop when $|S| = \frac{2\delta n}{d}$ we made $\frac{2\delta n}{d}$ step.

Enough to show that X is increased by $\geq \frac{d}{2}$ at each step.

assuming for a contradiction $|X| \leq \underbrace{\delta n}$.

~~Suppose~~ Consider the first step when X is increased by $< \frac{d}{2}$.

So far we have deleted $\leq \underbrace{\left(\frac{2\delta n}{d} + \delta n\right)}_{\text{for } S} \underbrace{cd}_{\text{maximum degree}}$ edges

vertices

then at most $\frac{d}{2}$ vertices precede the last chosen v in its order in $G[A]$

$$|E(G)| \leq \left(\frac{2\delta n}{d} + \underbrace{\delta n}\right) cd$$

$$+ \underbrace{\frac{cd^2}{4}} + \underbrace{\frac{d}{4}} n < \underbrace{\frac{d}{2} n} = |E(G)| \text{ and } \deg_{G[A]}(v) \leq \frac{d}{2}$$

if δ is small enough

$$2E(G[A]) \leq \boxed{\underbrace{cd \cdot \frac{d}{2}}_{\text{degrees before } v} + \underbrace{\frac{d}{2} \cdot n}_{\text{after}}}$$