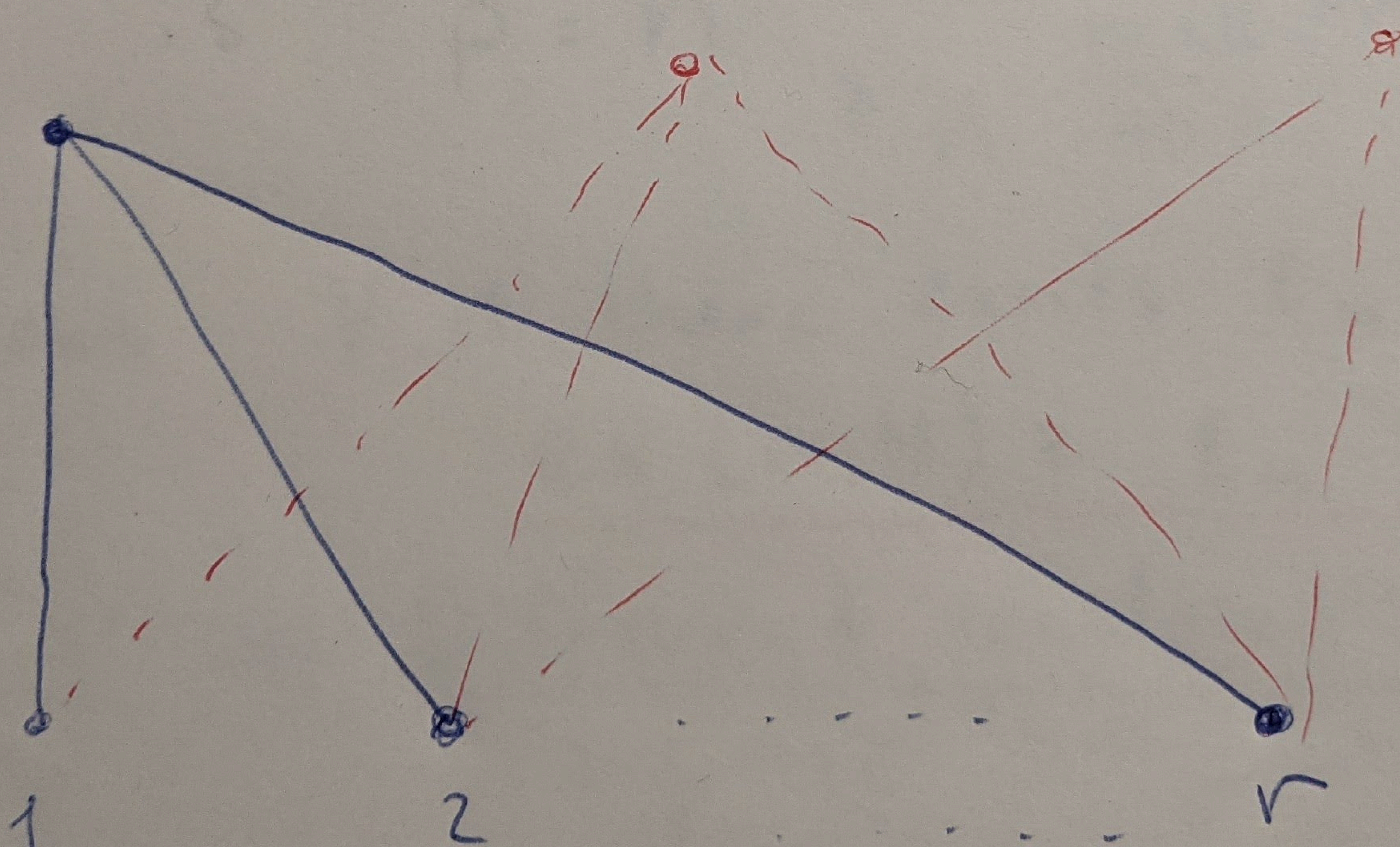
Lecture 21: Dependent random choice



11. Dependent random choice

Fox, Sudakov + Colon

<u>Turán</u> type <u>problems</u>:

Given graph H, Determine maximum number of edges in a graph on n vertices with no copy of H (no H subgraph).

$ex(n, H)$ - Turán number of H.

What bounds come from random graphs?

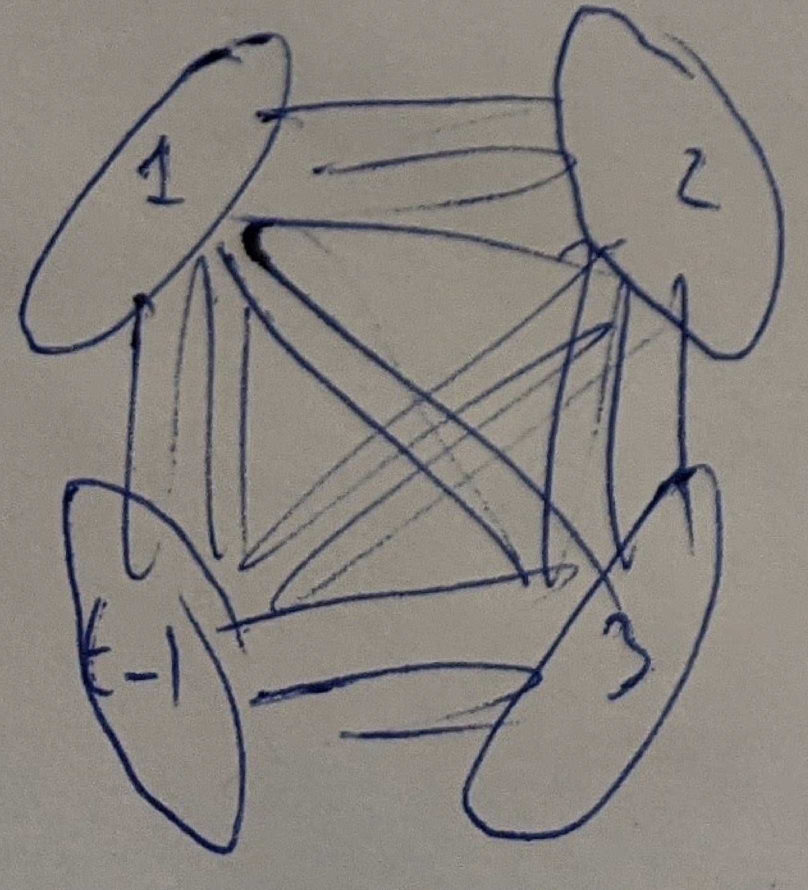
Recall <u>Theorem 4.5 Bollobás</u>:

The threshold for containing H in $G(n,p)$ is $p = n^{-\frac{1}{m(H)}}$ $m(H) = \max_{H' \subseteq H} \frac{|E(H')|}{|V(H')|}$.

$\geq n^{-\frac{|V(H)|}{|E(H)|}}$.

It follows: $\forall$ H there exists $\varepsilon_H > 0$ s.t.

$$ex(n, H) \geq \varepsilon_H \, n^{2 - \frac{|V(H)|}{|E(H)|}}.$$



$\rightarrow$ <u>Different construction</u>:

There exist graphs with n vertices $\left(\frac{t-2}{t-1} + o(1)\right)\frac{n^2}{2}$ edges and chromatic number $t-1$.

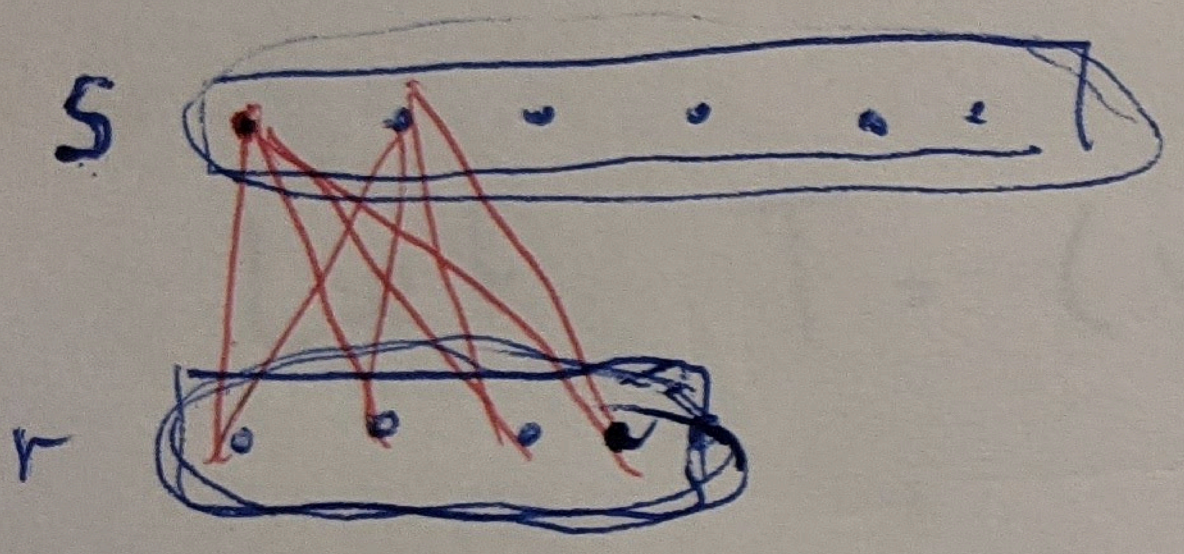
So $ex(n, H) \geq \left(\frac{t-2}{t-1} + o(1)\right)\frac{n^2}{2}$

For every graph H with $\chi(H) = t$.

Turán's theorem : Tight for all H

Erdős- Stone. $$ex(n,H) = \left(\frac{t-2}{t-1} + o(1)\right)\frac{n^2}{2}$$ where $t = \chi(H)$

The above theorem gives asymptotics for $ex(n,H)$ except when H is bipartite.



$K_{s,r}$ $\rightarrow$ $ex(n, K_{s,r}) \geq \varepsilon_{s,r}\, n^{2-\frac{s+r}{sr}}$

$s \geq r$.

$\downarrow s \rightarrow \infty$

$n^{2-\frac{1}{r}}$ *

Conjecture: $ex(n, K_{s,r}) = \Theta\left(n^{2-\frac{1}{r}}\right)$ for all $s \geq r$.

Kovari-Sós Turán

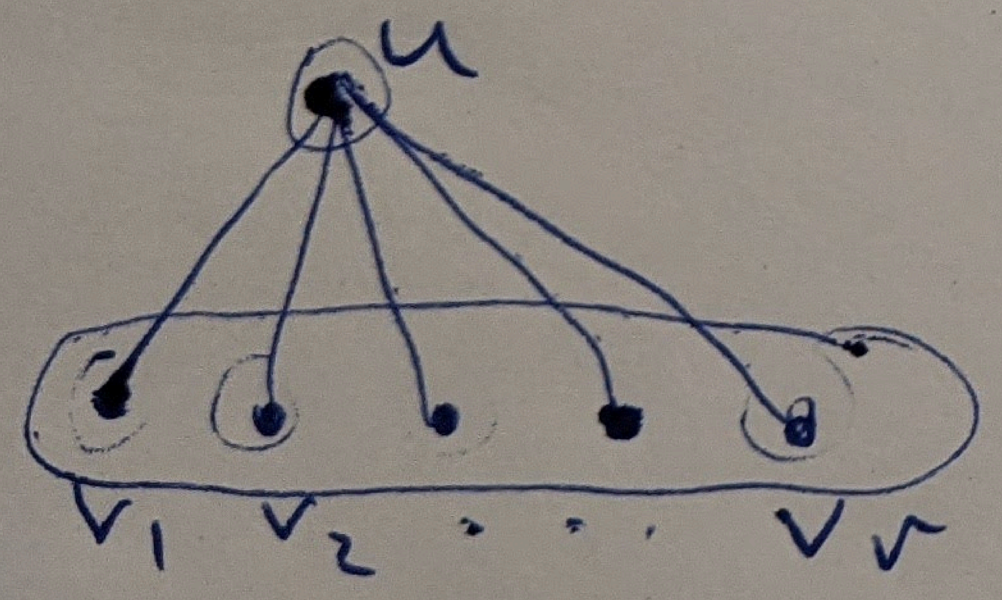
Theorem 11.1: ~~There~~ For all s, r there exists $C = C(s,r)$ such that $ex(n, K_{s,r}) \leq \underbrace{C \cdot n}^{2-\frac{1}{r}}$.

Proof: Let G be a graph with $|V(G)| = n$ and $|E(G)| \geq C \cdot n^{2-\frac{1}{r}}$ (C to be chosen).

Let $T = (v_1, v_2, \ldots, v_r)$ be a sequence of vertices of G chosen uniformly at random (with repitition.

Let $N(T)$ be the common neighborhood of T.

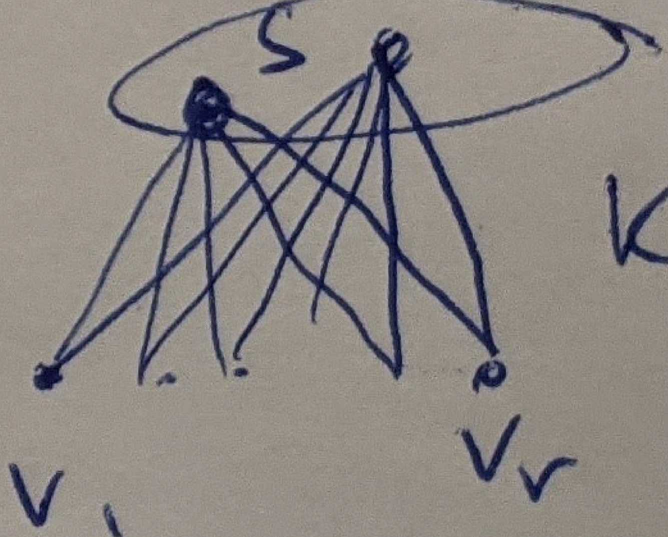
$$\mathbb{E}[\,|N(T)|\,] = n\,\mathbb{P}[u \in N(T)] = n \cdot \mathbb{E}\left[\left(\frac{\deg(u)}{n}\right)^r\right]$$



$$\geq n \cdot \left(\frac{\overset{\geq C n^{1-\frac{1}{r}}}{\mathbb{E}[\deg(u)]}}{n}\right)^r$$

$$\geq n \left(\cancel{2}\, C \cdot n^{\cancel{1}-\frac{1}{r}}\right)^r$$

$$\geq \boxed{C^r} \overset{(2)}{\geq} \underline{s+r} \rightarrow$$



$K_{s,r}$.

An issue: $v_1, \dots, v_r$ are not necessarily distinct.

Fix: $\underset{|T|=r}{\mathbb{E}}[|N(T)|] \geq c^r - \underbrace{n}_{\text{max size of } |N(T)|} \cdot \mathbb{P}[\underbrace{\text{size of } T \text{ is less than } r}]$

vertices in T are distinct

$$\geq c^r - n \cdot \underbrace{\binom{r}{2}}_{\text{choices of } \{i,j\}.} \cdot \underbrace{\frac{1}{n}}_{\text{probability } v_i = v_j} = c^r - \binom{r}{2}$$

In conclusion if $c^r - \binom{r}{2} \geq s+r$. then

there exist distinct $v_1, \dots, v_r \in V(G)$ s.t.

with $\geq s+r$ common neighbors

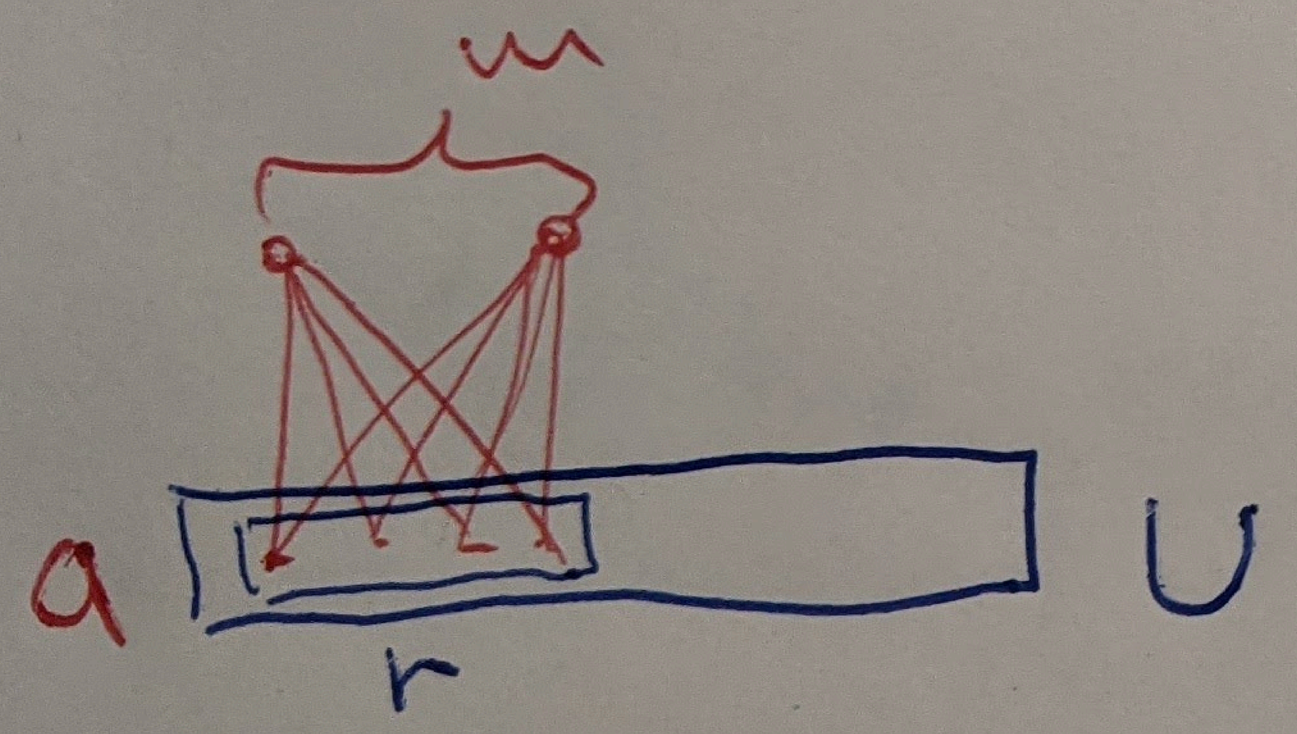
$K_{s,t}$ is a subgraph of G.

Lemma 11.2 (Fox, Sudakov): Let a, d, m, n, r be positive integers. Let G be a graph with n vertices and average degree $\frac{2|E(G)|}{|V(G)|} \geq d$.

If

$$\boxed{\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a \quad \text{for some } t \geq 1. \qquad (*)}$$

then $V(G)$ contains a subset U with $|U| = a$ s.t. every r vertices of U have $\geq m$ common neighbors.



(we needed such result for $a = r$ $m = s+r$ $d = Cn^{1-1/r}$ in 11.1)

Proof: Let t be a positive integer. Select $T = (v_1, v_2, \dots, v_t)$ uniformly at random with repetition.

$$\mathbb{E}[|N(T)|] \underbrace{\geq}_{\text{as in 11.1}} n \cdot \left(\frac{d}{n}\right)^t = \frac{d^t}{n^{t-1}}.$$

Let $X = N(T)$. Let Y count the number of subsets S of X of size r s.t.

$|N(S)| \leq m$

Let S be a subset of $V(G)$ with $|N(S)| \leq m$.

Given S

(?) $$\mathbb{P}[S \subseteq N(T)] = \left(\frac{|N(S)|}{n}\right)^t \leq \left(\frac{m}{n}\right)^t$$

By union bound.

$$\mathbb{E}[Y] \leq \binom{n}{r} \cdot \left(\frac{m}{n}\right)^t$$

If $$\mathbb{E}[|X| - Y] \geq a$$

Then there exists $X \subseteq V(G)$ s.t. ~~at most~~
deleting a vertex of X for each $S \subseteq X$ with at most m common neighbors, we have a set U of $\geq a$ vertices left, which satisfies the lemma.

So the required condition is

$$\frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq a$$

for some $t \geq 1$ integer.

<u>Corollary 11.3:</u> Let H be a bipartite graph with bipartition (A,B) s.t. $\deg(v) \le r$ for every $v \in B$. Then there exists $C = C_H$

~~Then~~ $ex(n,H) \le C n^{2-\frac{1}{r}}$.

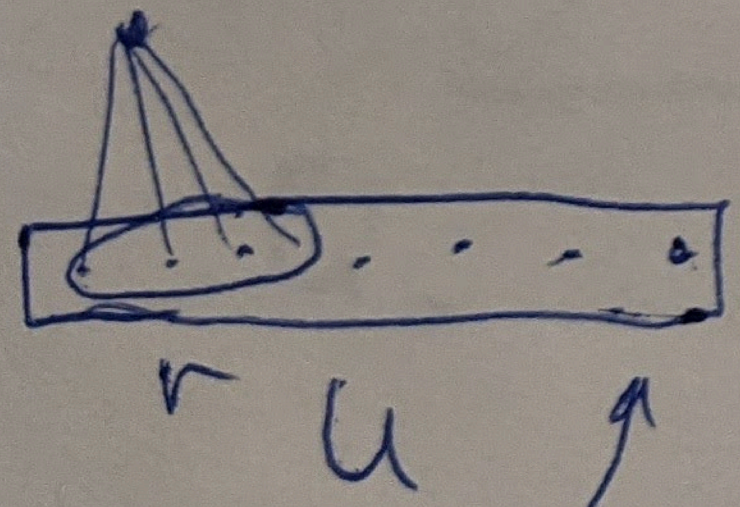
- generalizes 11.1.
- ~~by~~ random graphs show that the exponent $2-1/r$ is optimal.

<u>Proof:</u> We apply 11.2. to a graph G with $|V(G)| = n$, $|E(G)| \ge C n^{2-\frac{1}{r}}$

$$\left.\begin{array}{l} d = C n^{1-\frac{1}{r}} \\ a = |A|,\ m = |V(H)|. \end{array}\right]$$

If the conclusion of 11.2 holds with these parameters then G contains H subgraph.

Let $t = r$. Remains to check that $(*)$ holds, for some C.



$$\frac{(C n^{\frac{r-1}{r}})^r}{n^{r-1}} - \binom{n}{r}\cdot\left(\frac{m}{n}\right)^r \ge a$$

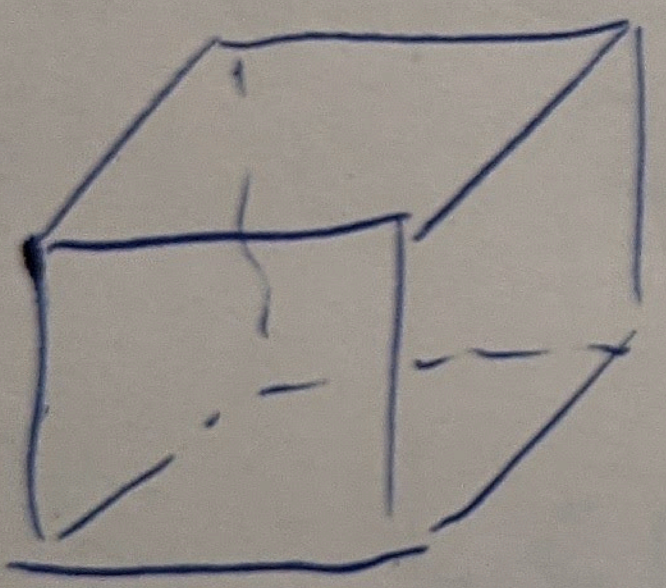
$$C^r - \frac{n^r}{r!}\,\frac{m^r}{n^r} \ge a$$

$$C^r \ge |A| + \cancel{a}\ \frac{|V(H)|^r}{r!}$$ suffices.

Ramsey numbers: For a graph H the ramsey number $r(H)$ is the minimum n s.t. for every two coloring of edges of K_n there exists a copy of H in K_n with all edges of the same color.

We considered $r(K_s)$ and shown: $r(K_s) \geq \underline{\ \ldots\ }\, 2^{s/2}$

Hypercube Q_r is a graph with $V(Q_r) = \{0,1\}^r$ and edges join pairs of vertices differing in one coordinate.



Q_3

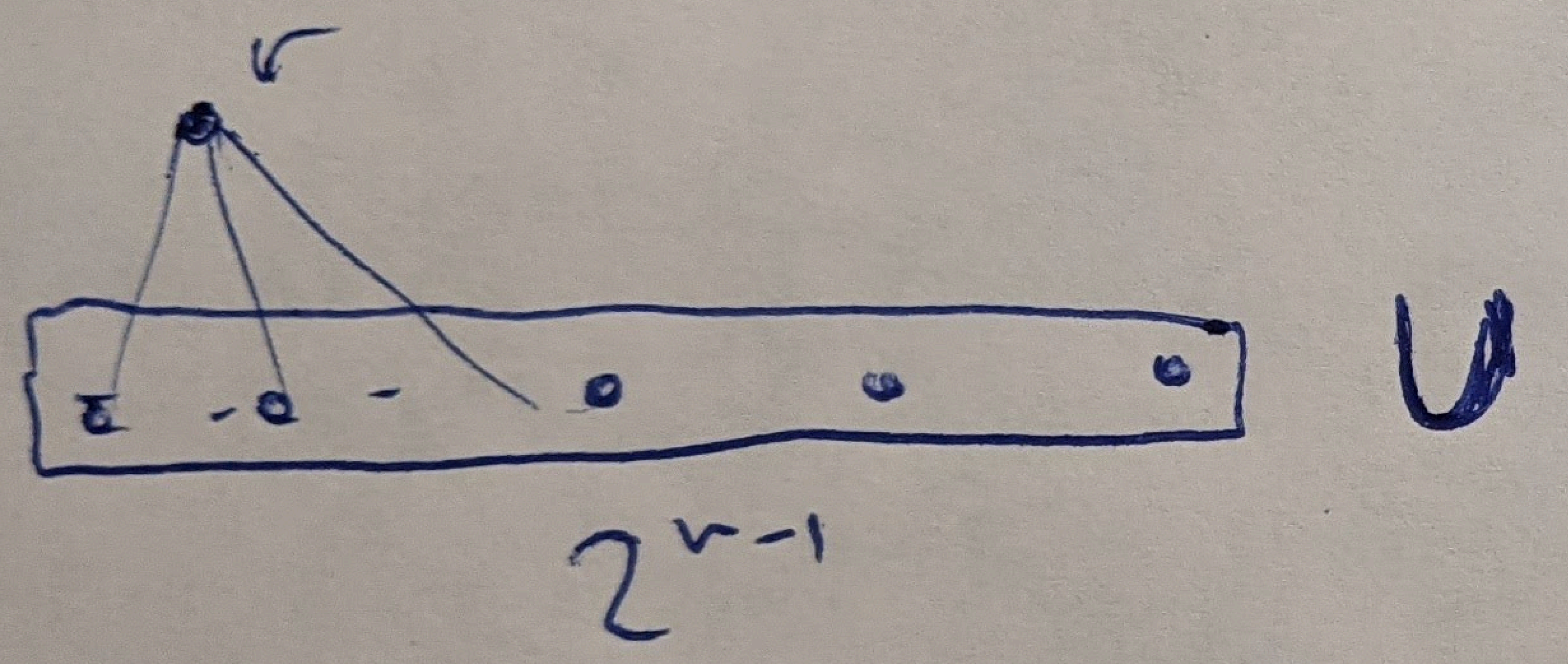
Conjecture (Burr-Erdös): $r(Q_r) \leq C 2^r$ for some C.

Corollary 11.4: $r(Q_r) \le 2^{3r}$.

Proof: Let G be a graph with $|V(G)| = \boxed{n = 2^{3r}}$
and average degree $\boxed{d \ge \frac{n-1}{2}}$.

It suffices to show that G contains Q_r subgraph.

We again apply 11.2. with $\boxed{a = 2^{r-1}}$, $\boxed{m = 2^{r}}$



Turns out (*) holds with $t = \frac{3}{2} r$.

Lee: $r(Q_r) \le 2^{2r + o(r)}$