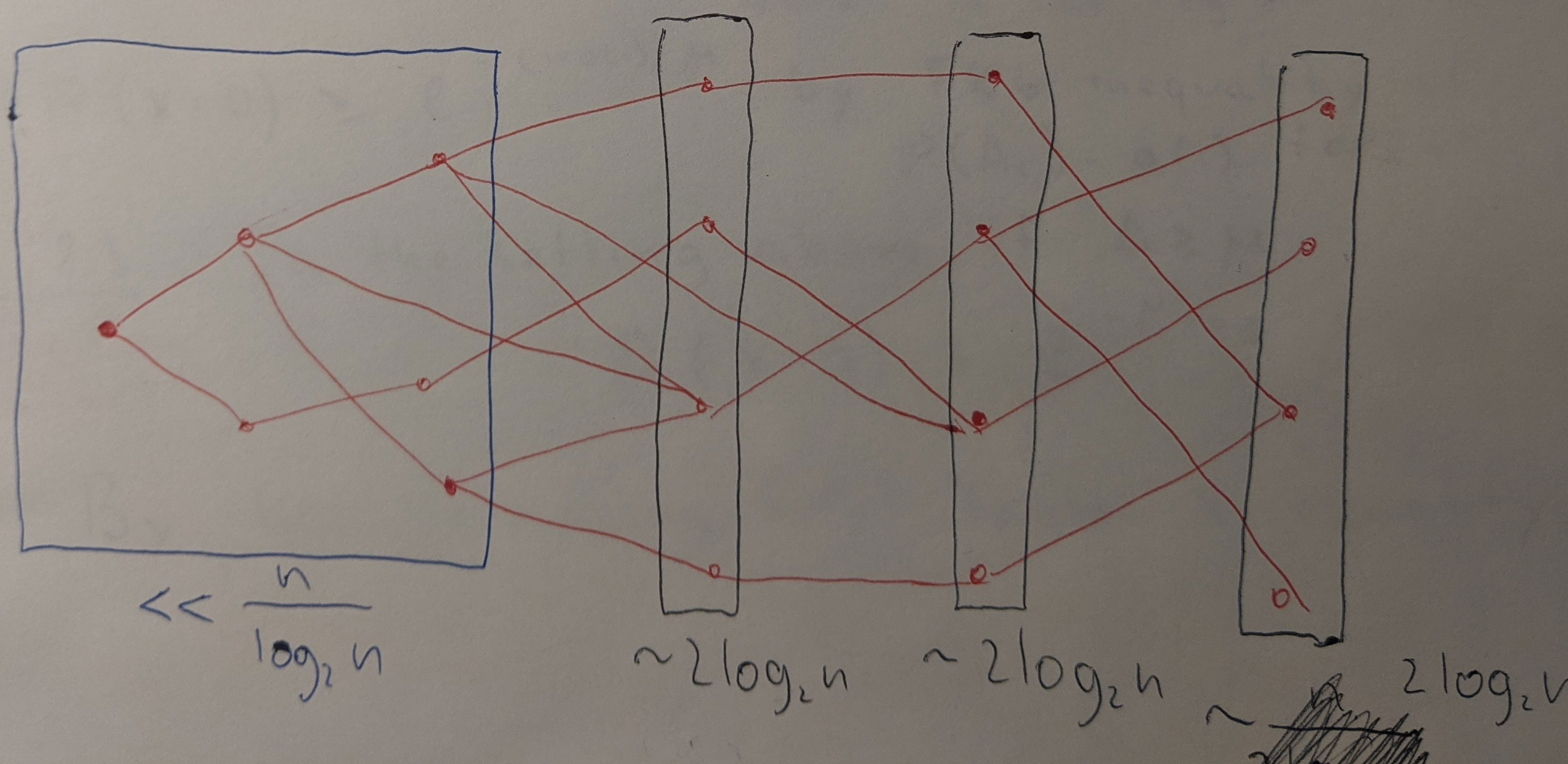
Lecture 15:

Second Janson Inequality & Applications



Setup: We select random subset R of $[N]$ by selecting its elements independently (possibly with different probabilities)

$S_i \subseteq [N]$ $\quad A_i$ - event that $S_i \subseteq R$

$$X = \sum \mathbb{1}_{A_i} = \# S_i\text{'s in } R. \qquad \mu = E[X]$$

$$\Delta = \sum_{\substack{i \sim j \\ (i,j)}} \mathbb{P}(A_i \wedge A_j)$$

(i.e. $S_i \cap S_j \neq \emptyset$)

8.2. $\mathbb{P}(X=0) \leq e^{-\mu + \frac{\Delta}{2}}$ First Janson inequality.

Works in the regime $\mu \geq \Delta$.

($\mathbb{P}(X=0) \geq e^{-(1-o(1))\mu}$ by FKG inequality if $\mathbb{P}(A_i) = o(1)$ for all i).

Theorem 8.3: In the setting above if $\Delta \geq \mu$

Second Janson inequality

$$\mathbb{P}(X=0) \leq e^{-\mu^2/2\Delta}.$$

Proof: By bootstrapping First Janson inequality.

Let $S_1, S_2, \dots, S_k$ be the sets in the setup
$A_1, A_2, \dots, A_k$ corresponding events.
Select $I \subseteq [k]$ by choosing each $i \in [k]$ independently with probability q. (to be chosen)

Apply 8.2. to sets $\{S_i\}_{i \in I}$.

$$\mathbb{P}(X=0) \le \mathbb{P}(X_I = 0) \underset{8.2.}{\le} e^{-\mu_I + \frac{\Delta_I}{2}}.$$

For fixed I: $\mu_I = \mathbb{E}_q\left[\sum_{i \in I} \mathbb{1}_{A_i}\right]$ | $\mu = \mathbb{E}\left[\sum_{i \in [k]} \mathbb{1}_{A_i}\right]$

$\Delta_I = \sum_{\substack{i \sim j \\ i,j \in I}} \mathbb{P}(A_i \wedge A_j)$. $\Delta = \sum_{\substack{i \sim j \\ i,j \in [k]}} \mathbb{P}(A_i \wedge A_j)$

$q = \frac{\mu}{\Delta}$

$$\mathbb{E}_I\left[-\mu_I + \frac{\Delta_I}{2}\right] = -q\mu + \frac{q^2 \Delta}{2} = -\frac{\mu^2}{2\Delta}$$

$\mathbb{E}[\mu_I] \overset{?}{=} q\mu$. $\mathbb{E}[\Delta_I] = q^2 \Delta$

$$\mathbb{P}(X=0) \le e^{-\frac{\mu^2}{2\Delta}} \checkmark.$$

8.2 & 8.3

$$\mathbb{P}(X=0) \le \begin{cases} e^{-\mu + \frac{\Delta}{2}} & \text{if } \mu \ge \Delta \\ e^{-\mu^2/2\Delta} & \text{if } \mu \le \Delta. \end{cases}$$

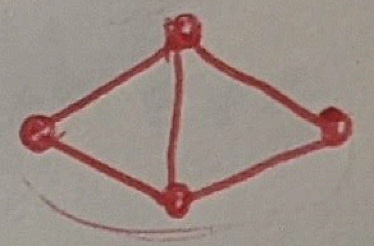
Probability that $G(n,p)$ has no K_3 subgraph (triangle-free).

Reminder: $f = \Omega(g)$ if $f \ge c \cdot g$ for $c>0$
$f = \Theta g$ if $\underbrace{cg \le f \le C \cdot g}$

We already proved that for $p = o(n^{-1/2})$

$$\mathbb{P}(G(n,p) \text{ is triangle-free}) = e^{\underbrace{-(1-o(1))\frac{n^3p^3}{6}}}$$

$\underline{p \gg n^{-1/2}}$. $\quad \mu \sim n^3p^3 \quad \Delta \sim n^4p^5 \quad \mu < \Delta$



By 8.3. $\quad \mathbb{P}(G(n,p) \text{ is triangle free}) \le e^{\underbrace{-\Omega(n^2p)}}$

Is there a "matching lower bound.

$$\mathbb{P}(G(n,p) \text{ is edgeless}) = (1-p)^{\binom{n}{2}} \underset{\underbrace{p<0.99}}{=} e^{-\Theta(p)\frac{n^2}{2}} \ge e^{\underbrace{-\Theta(n^2p)}}.$$

Theorem 8.4: For $p < 0.99$

$$\mathbb{P}(G(n,p) \text{ is triangle free}) = \begin{cases} e^{-\Theta(n^3p^3)} & p < n^{-1/2} \\ e^{-\Theta(n^2p)} & p > n^{-1/2}. \end{cases}$$

For $p = 1/2$ $\quad G(n, 1/2)$ uniformly samples graphs G with $V(G) = [n]$.

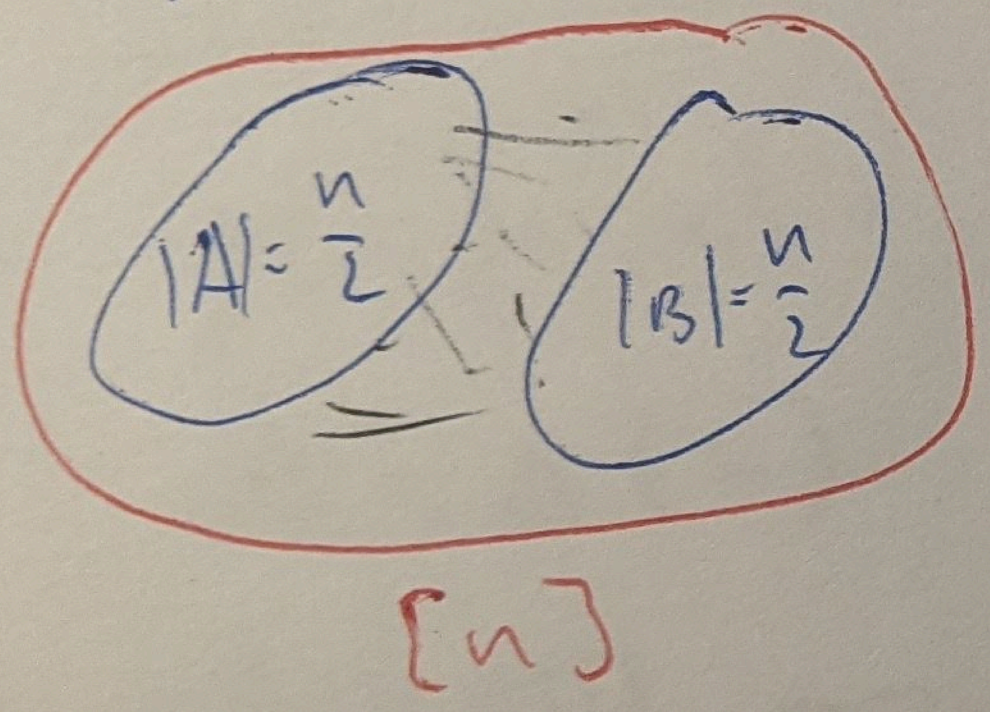
So estimating $\mathbb{P}(G(n, 1/2)$ is triangle free$)$

$\sim$ calculating # of triangle free graphs. $\leq 2^{c\frac{n^2}{2}}$ $\quad$ 8.4 $\quad \boxed{c<1}$

Total # of graph $\quad 2^{(1+o(1))\frac{n^2}{2}}$

Erdös, Kleitman & Rothschild 1976:

Almost all triangle-free graphs are bipartite



$\sim$ there are $2^{(1+o(1))\frac{n^2}{4}}$ triangle free graphs

$$\mathbb{P}(G(n, 1/2) \text{ is } \Delta\text{-free}) = 2^{-(1-o(1))\frac{n^2}{4}}$$

In fact, the same logic applies for $p \geq n^{-1/2+\varepsilon}$.

Chromatic number of $G(n, 1/2)$.

<u>Recall:</u> The clique number $\omega(G)$ of G
is the size of the largest complete subgraph.

Behaviour of $\omega(G(n, 1/2))$.

Estimate probability $\omega(G(n, 1/2)) \leq k$.

To setup things s.t. Janson inequalities are applicable

Let S_i correspond to edge set of complete subgraphs on k vertices in $[n]^{(2)}$.

$$\mu = \mu(k) = \binom{n}{k} \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\Delta = \Delta(k)$$

$\mu(k) = o(1)$ then by Markov
$\omega(G(n, 1/2)) < k$ w.h.p.

If $\Delta = o(\mu^2)$ and $\mu(k) \to \infty$
then $\omega(G(n, 1/2)) \geq k$ w.h.p.

$\cancel{\frac{\mu(k+1)}{\mu(k)} = \frac{1}{n^{1-o(1)}}}$ ~~for k~~

$$\frac{\mu(k)}{\mu(k+1)} = n^{(1-o(1))} \text{ for } k = (1-o(1)) \cdot 2\log_2 n.$$

$$\mu(k) \approx \frac{n^k}{k!} \cdot \left(\frac{1}{2}\right)^{\frac{k^2}{2}} \sim \left(\frac{en}{k}\right)^k \left(\frac{1}{2}\right)^{\frac{k^2}{2}}$$

$$\log_2 \mu(k) \sim k \cdot \log_2 n - \frac{k^2}{2} = k\left(\log_2 n - \frac{k}{2}\right)$$

$\mu(k) \geq 1$ when $k \leq (1+o(1))\, 2\log_2 n$

$$\frac{\partial}{\partial k} \log_2 \mu(k) \sim \log_2 n - k \sim -\log_2 n$$

Theorem 8.5: Let k_0 be maximum s.t. (k₀ depends on n)

$$\mu(k_0) = \binom{n}{k_0}\left(\frac{1}{2}\right)^{\binom{k_0}{2}} \geq 1.$$

Then

$$\mathbb{P}\left(\omega(\underbrace{G(n,\tfrac{1}{2})}) < k_0 - 3\right) \leq \underbrace{e^{-n^{\boxed{2}-o(1)}}}.$$

Proof: Let $k = k_0 - 3$

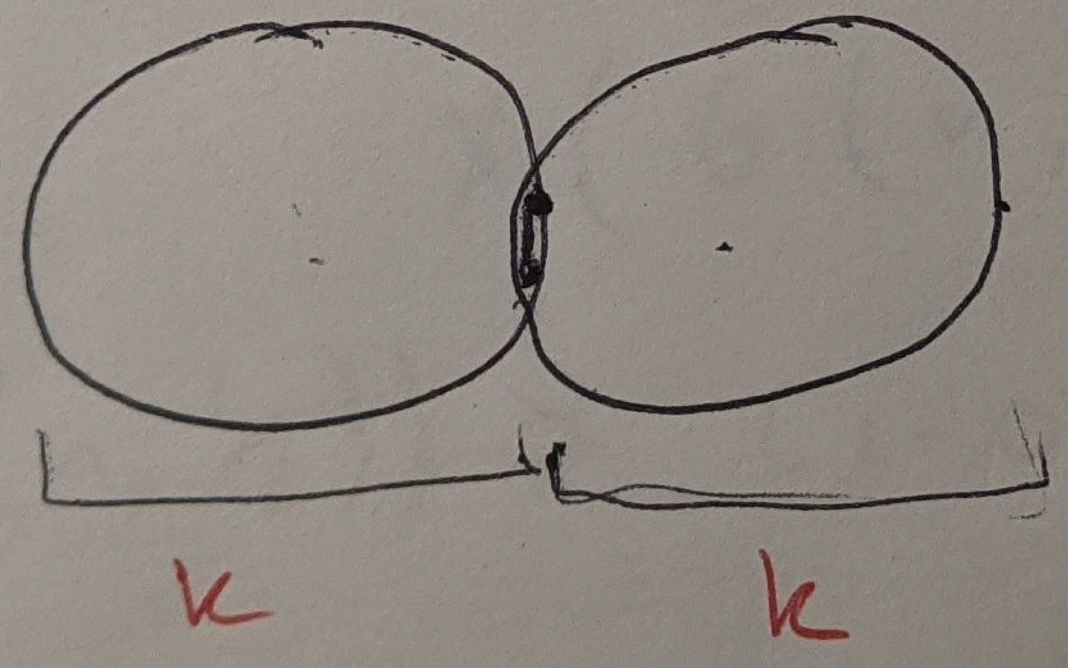
By calculations above

$$\mu(k) \geq n^{3-o(1)}.$$

Let us estimate $\Delta(k)$

Intuition (details can be routinely checked)

the highest order term comes from pairs $S_i\ S_j$ where corresponding cliques have 2 vertices in common.



$$\sum \underbrace{\mathbb{P}(A_i \wedge A_j)}$$

$$\sum \mathbb{P}(A_i) \cdot \sum \mathbb{P}(A_j) = \mu^2$$

$$\underbrace{2 \cdot \binom{k}{2} \cdot \left(\frac{k}{n}\right)^2}$$

$\sim$ proportion of pairs that have 2 vertices in common

We have

$$\Delta(k) = \Theta\left(\frac{k^4}{n^2}\mu^2\right) = o(\mu^2).$$

$$\gg \mu.$$

By 2^{nd} Janson's inequality

$$\mathbb{P}(\omega(G(n,\tfrac{1}{2})) < k) = \mathbb{P}(X=0) \le e^{-\frac{\mu^2}{2\Delta}}$$

$$= e^{-\Theta\left(\frac{n^2}{k^4}\right)} = e^{-n^{2-o(1)}}. \checkmark$$

$k \sim 2\log_2 n$

— Let $\alpha(G)$ denote the independence number of G

maximum size of set of pairwise non-adjacent vertices.

By 8.5. $\boxed{\mathbb{P}(\alpha(G(n,\tfrac{1}{2})) < k_0 - 3) \le e^{-n^{2-o(1)}}}$.

Theorem 8.6: With high probability.

Bollobás, 1988

$$\chi(G(n,\tfrac{1}{2})) = (1+o(1))\frac{n}{2\log_2 n}.$$

Proof: $\chi(G(n,\tfrac{1}{2})) \ge (1-o(1))\frac{n}{2\log_2 n}$.

with high probability by Markov

$$\underbrace{\alpha(G(n,\tfrac{1}{2})) \le k_0 + 1}_{k_0 = (1+o(1))2\log_2 n} = (1+o(1))2\log_2 n.$$

But $\chi(G) \ge \frac{|V(G)|}{\alpha(G)}$ for any G, implying

We will show that for any

$m \geq \boxed{\dfrac{n}{\log_2^2 n}}$ ~~for~~ every ~~any~~ ~~subset $X \subseteq [n]$~~

with high probability

$$\alpha(G[X]) \geq (1-o(1)) 2 \log_2 n.$$

$G[X]$: subgraph induced by X

for every $X \subseteq [n]$ $|X| \geq m$.

where $G = G(n, 1/2)$.

This implies the theorem as we can "pick out" color classes of size $\sim 2 \log_2 n$. until $\leq m$ vertices are left.

implying $\chi(G) \leq (1+o(1)) \dfrac{n}{2\log_2 n} + m = (1+o(1)) \dfrac{n}{2 \log_2 n}$.

Our claim follows from 8.5 (for α instead of ω)

By union bound

(the prob. $\mathbb{P}[\alpha(G[X]) < (1-o(1)) 2 \log_2 n] \leq e^{-n^{2-o(1)}}$

for each such X.)

there are $\leq 2^n$ choices of X.