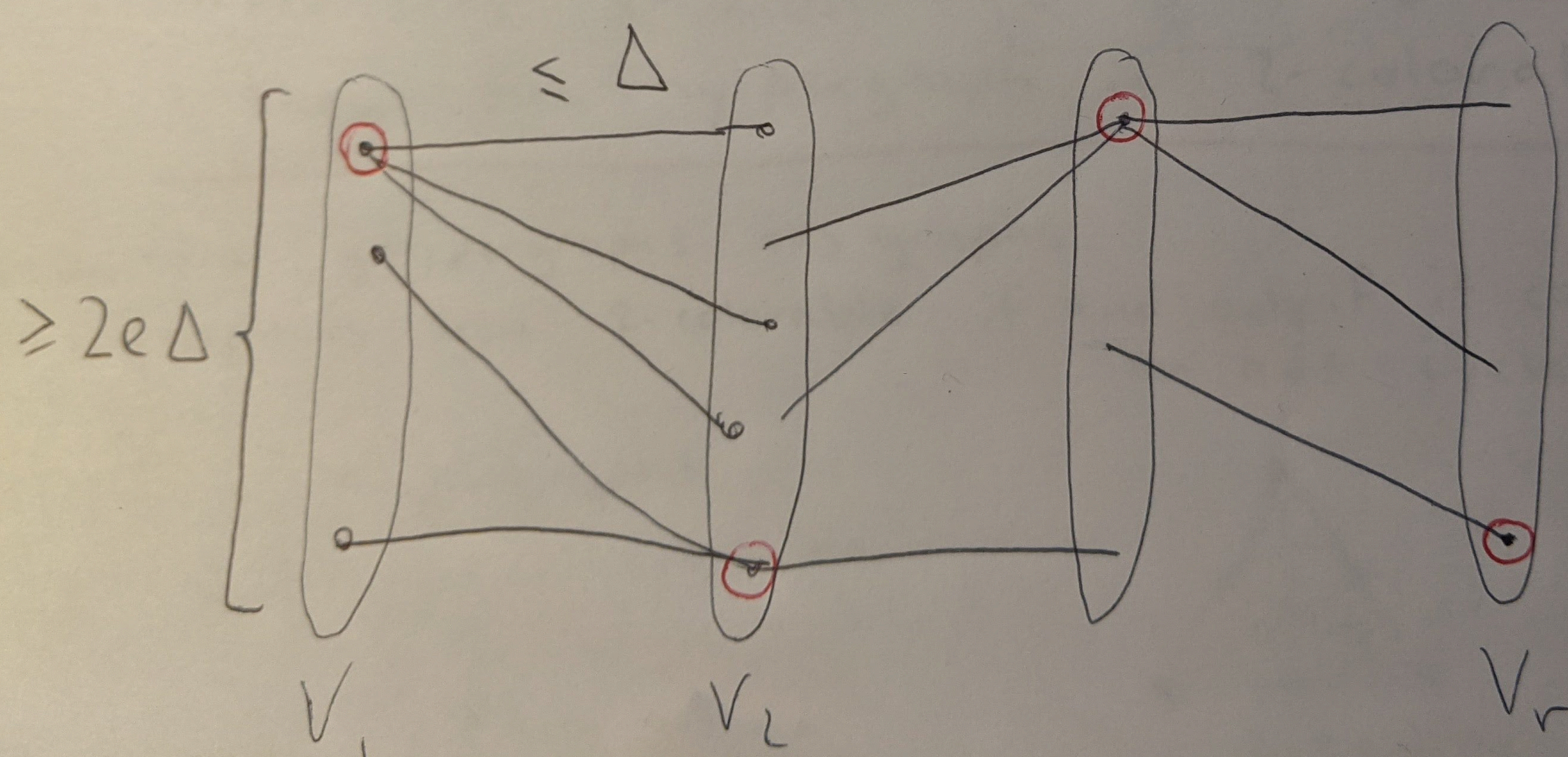
Lecture 12: Applications of LLL



Recall:

6.1. Lovász Local Lemma (LLL)

Symmetric form

$A_1, \ldots, A_n$ events $\mathbb{P}[A_i] \le p$ for every i

Each A_i is independent of all but at most d other events.

If $\boxed{e\,p\,(d+1) \le 1}$

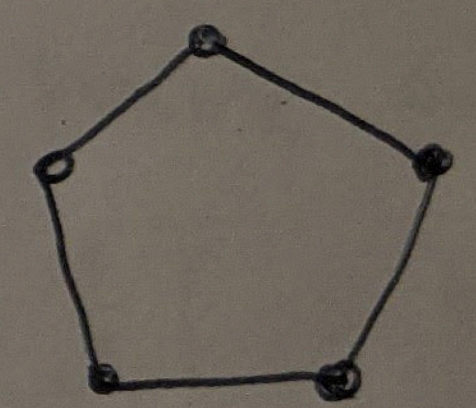
then with positive probability none of A_i occur.

6.4. If every edge in a k-uniform hypergraph intersects at most $\underbrace{2^{k-1}/e - 1}$ other edges.

then the hypergraph is 2-colorable.

2-uniform ~~g~~ hypergraphs $\iff$ graphs.

graph is non 2-colorable if and only if it contains an odd cycle

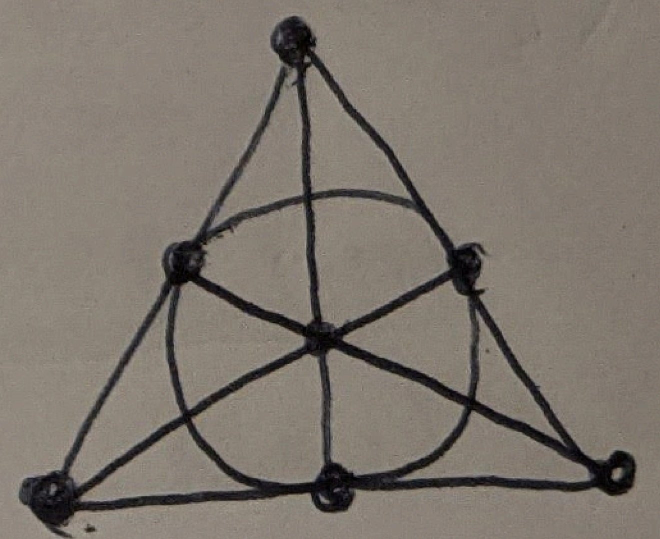


2-regular

3-uniform hypergraphs

minimum non 2-colorable has 7 edges

Fano plane



3-regular.

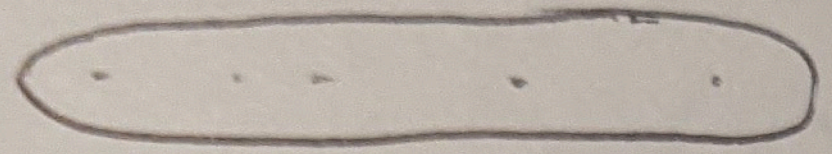
Do there exist k-regular k-uniform non 2-colorable hypergraphs? (for given k).

$\boxed{\text{every vertex is in exactly edges}}$

Yes. $k=2,3$.

No for large k

in a k-regular k-uniform hypergraph every edge intersects $\leq k(k-1)$ others.



So if $k(k-1) \leq 2^{k-1}/e - 1$ then

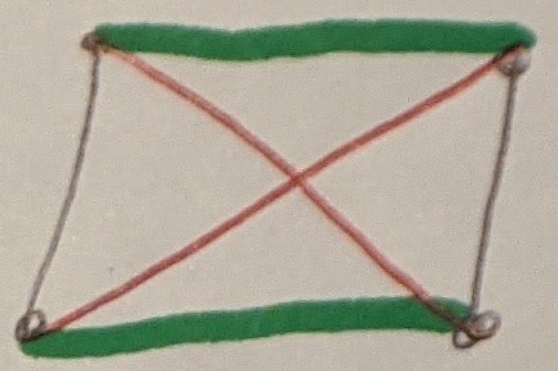
$\boxed{k \geq 9}$

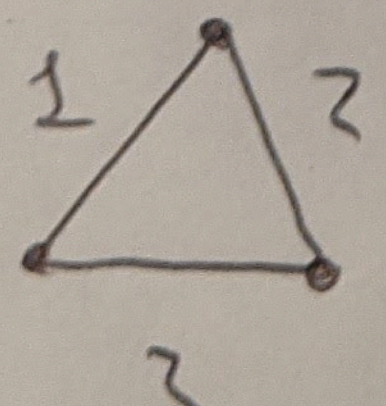
Thomassen '92: No for $k \geq 4$. (non-probabilistic proof).

Linear arboricity conjecture

Goal: color edges of a d-regular graph ~~in a~~ using as few colors as possible s.t. every color class "is nice".

1. Proper coloring: edges of given color don't share ends (matching)





Vizing's theorem

[$3\lceil \frac{d}{2} \rceil$ is ~~6~~ is not simple]

Shannon

d or (d+1) colors is optimum (depending on a graph).

2. **Arboricity**: edges of given color form forests
(There are no monochromatic cycles).

min # of colors ~~[illegible]~~ arboricity $a(G)$.

G is d-regular has n vertices

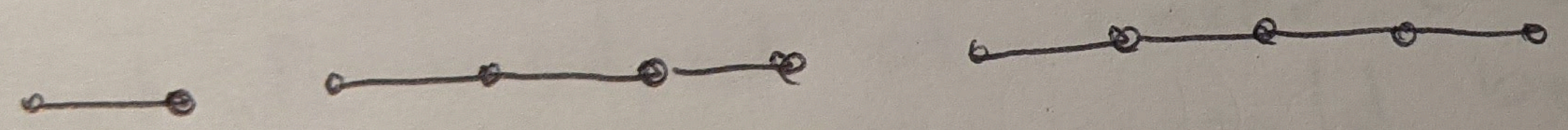
$\frac{dn}{2}$ edges every color has $\leq n-1$ edges

$$\underbrace{a(G) \geq \frac{dn}{2(n-1)} > \frac{d}{2}}.$$

$$a(G) \geq \left\lceil \frac{d+1}{2} \right\rceil \leftarrow$$

Nash-Williams: $a(G) = \left\lceil \frac{d+1}{2} \right\rceil$ for every d-regular G. $d \geq 1$.

3. **Linear arboricity** $la(G)$
minimum # of colors s.t. every color class is a linear forest. $\rightarrow$ union of disjoint paths.



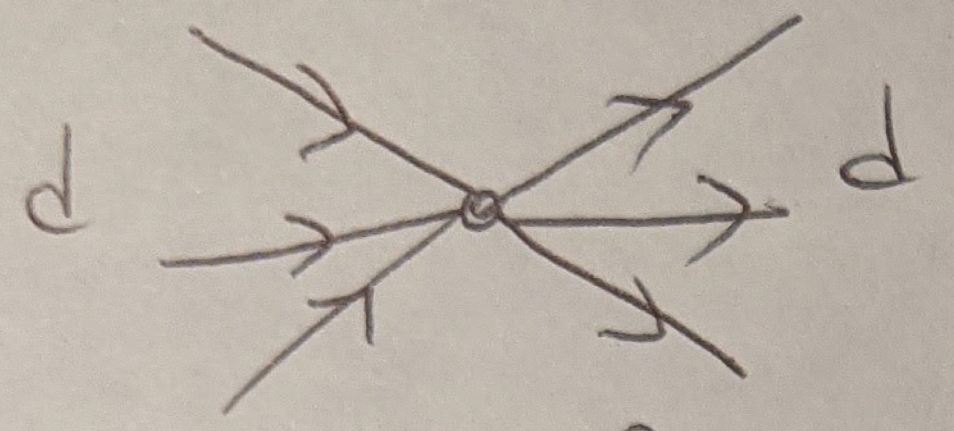
$$\boxed{\text{Linear arboricity conjecture (Akiyama, Exoo, Harary, 1981)} \quad La(G) = \left\lceil \frac{d+1}{2} \right\rceil \text{ for every } d\text{-regular } G.}$$

Theorem (Alon, 88): $la(G) = (1+o(1))\frac{d}{2}$ — $o(1) \to 0$ as $d \to \infty$

the conjecture is asymptotically true.

We can work with directed graphs instead:

(di)graph is d-regular if every vertex has d incoming & d outgoing edges.

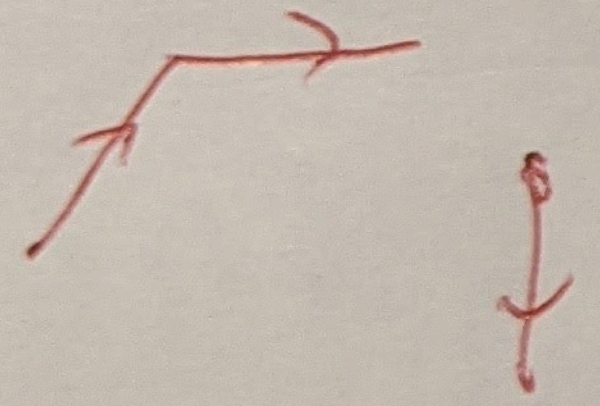


Linear (di) forest - disjoint union of directed paths

Directed linear arboricity conjecture

Edges of any directed d-regular graph can be partitioned into $\underline{d+1}$ Linear directed forests.

(Implies linear arboricity conjecture)



What if instead of linear directed forests we would be happy with directed disjoint union of cycles as color classes (1-regular digraphs)?

Then d colors suffice.

$\Uparrow$ König's theorem

$$\boxed{\text{Every d-regular digraph decomposes into union of d 1-regular digraphs.}}$$

So it is enough to remove 1 edge from each cycle ... forest

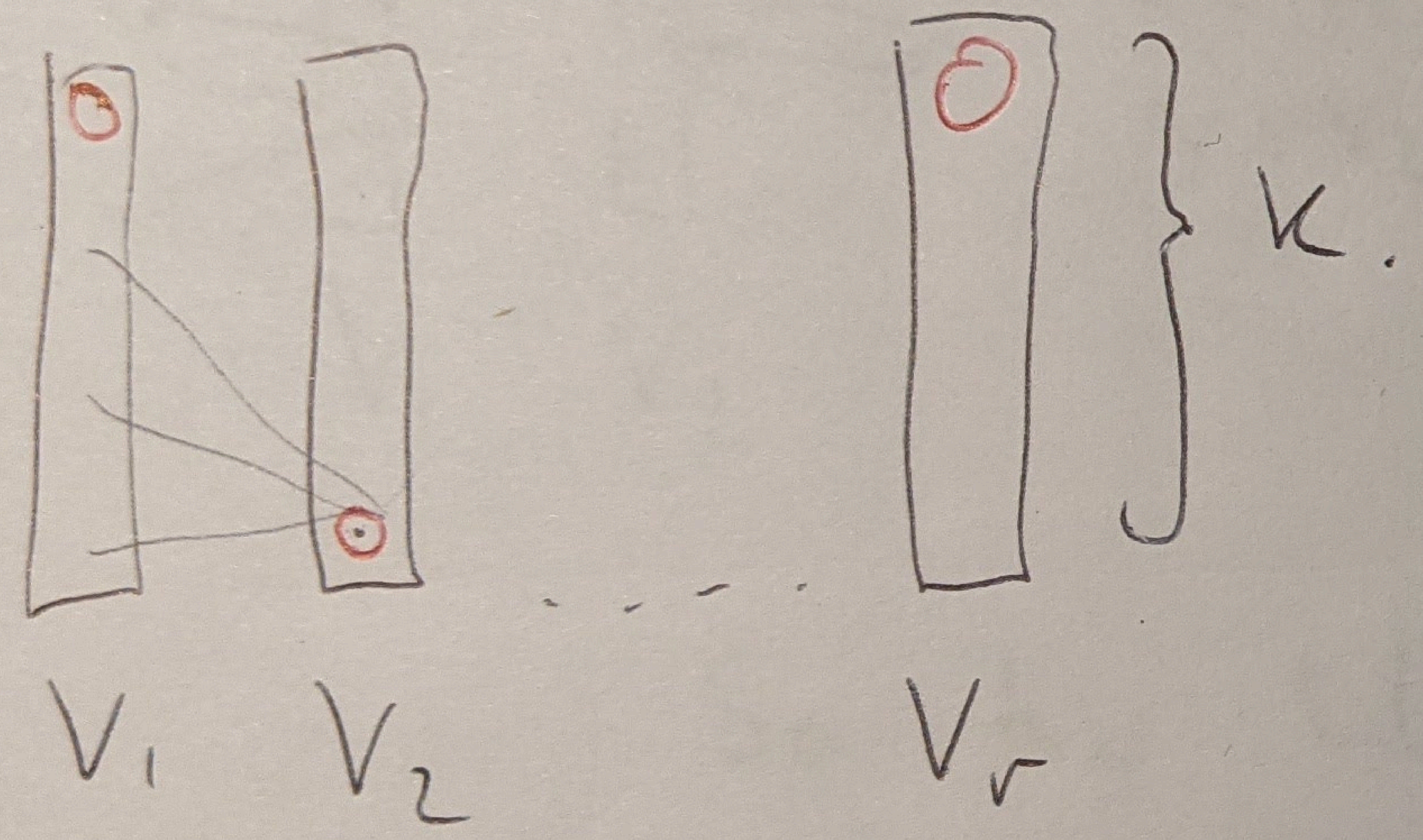
Theorem 6.5: (Independent transversal)

Let G be a graph with maximum degree $\boxed{\Delta}$ and let $(V_1, V_2, \dots, V_r)$ be a partition of $V(G)$ s.t. $|V_i| \geq 2e\Delta$.

Then there exists an independent set $X \subseteq V(G)$ s.t. X contains a vertex of each V_i.

Proof: Let $k = \lceil 2e\Delta \rceil$. Assume $|V_i| = k$ for each i, w.l.o.g.

Pick $x_i \in V_i$ uniformly at random.



Consider bad event $A_{\boxed{i,j}}$: there is an edge from x_i to x_j.

$\mathbb{P}[A_{i,j}] \overset{?}{\leq} \frac{\Delta}{k} = p$

What is the dependency graph.

$A_{i,j}$ ~~dep~~ is dependent $A_{i',j'}$ as long as $\{i,j\} \cap \{i',j'\}$.

$\underbrace{\leq 2\Delta k}_{= d}$ (only need to consider sets adjacent to V_i or V_j

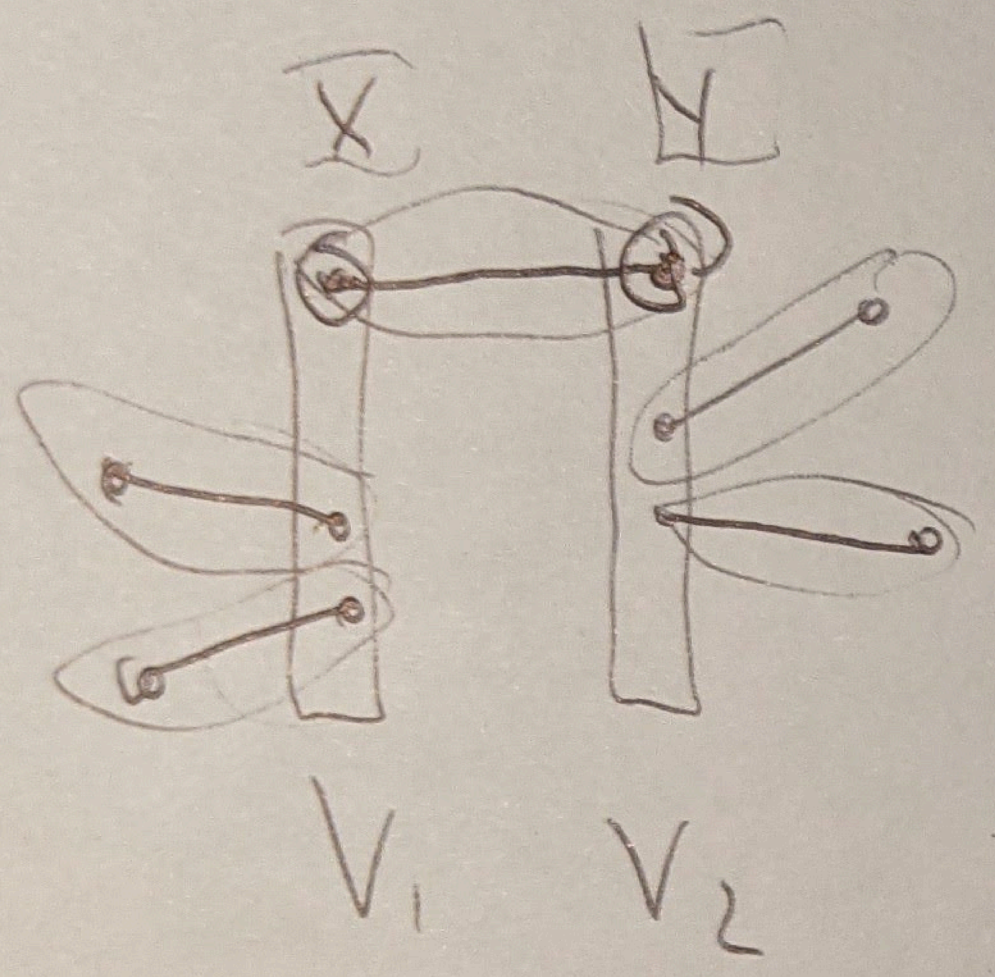
$\underbrace{p \cdot d \gg 1}$

Attempt 2:

Bad event: A_{x_i,x_j}

if an adjacent pair of vertices x_i, x_j is selected.

$$\mathbb{P}[A_{\boxed{x,y}}] = \frac{1}{k^2}$$



$A_{x,y}$ is dependent on $A_{x',y'}$ if $\{x',y'\} \cap (V_1 \cup V_2) \neq \emptyset$.

$x \in V_1$
$y \in V_2$

$\leq \underbrace{2k \cdot \Delta}_{=d} - 1$ other edges.

So by LLL a selection is possible as long as

$$\underbrace{e \cdot \frac{1}{k^2}}_{=p} \cdot \overbrace{2k\Delta}^{=d+1} \leq 1$$

$$\boxed{k \geq 2e\Delta.} \checkmark$$

is optimal

Haxell' 2000: $\underline{2e}$ in 6.5 can be replaced by $\boxed{2}$.

The proof is sophisticated alternating path argument

Corollary 6.6: Let D be a d-regular digraph with no directed cycle of length $< \underbrace{8ed}$.

Then D can be partitioned into $(d+1)$ linear directed forests.

Proof: The edges of D can be partitioned into sets $F_1, F_2, \dots, F_d$ s.t. F_i is a union of directed cycles.

Let $\underline{V_1, V_2, \dots, V_r}$ be the edge sets of all these cycles.
$|V_i| \geq \underbrace{8ed}$

By $\underline{6.5}$ we can select $X \subseteq E(D)$ s.t. X is a matching and X contains an edge of each such cycle.

(two edges are adjacent in G if they share an end

$\Delta = 2(2d-1)$
$< \boxed{4d}$

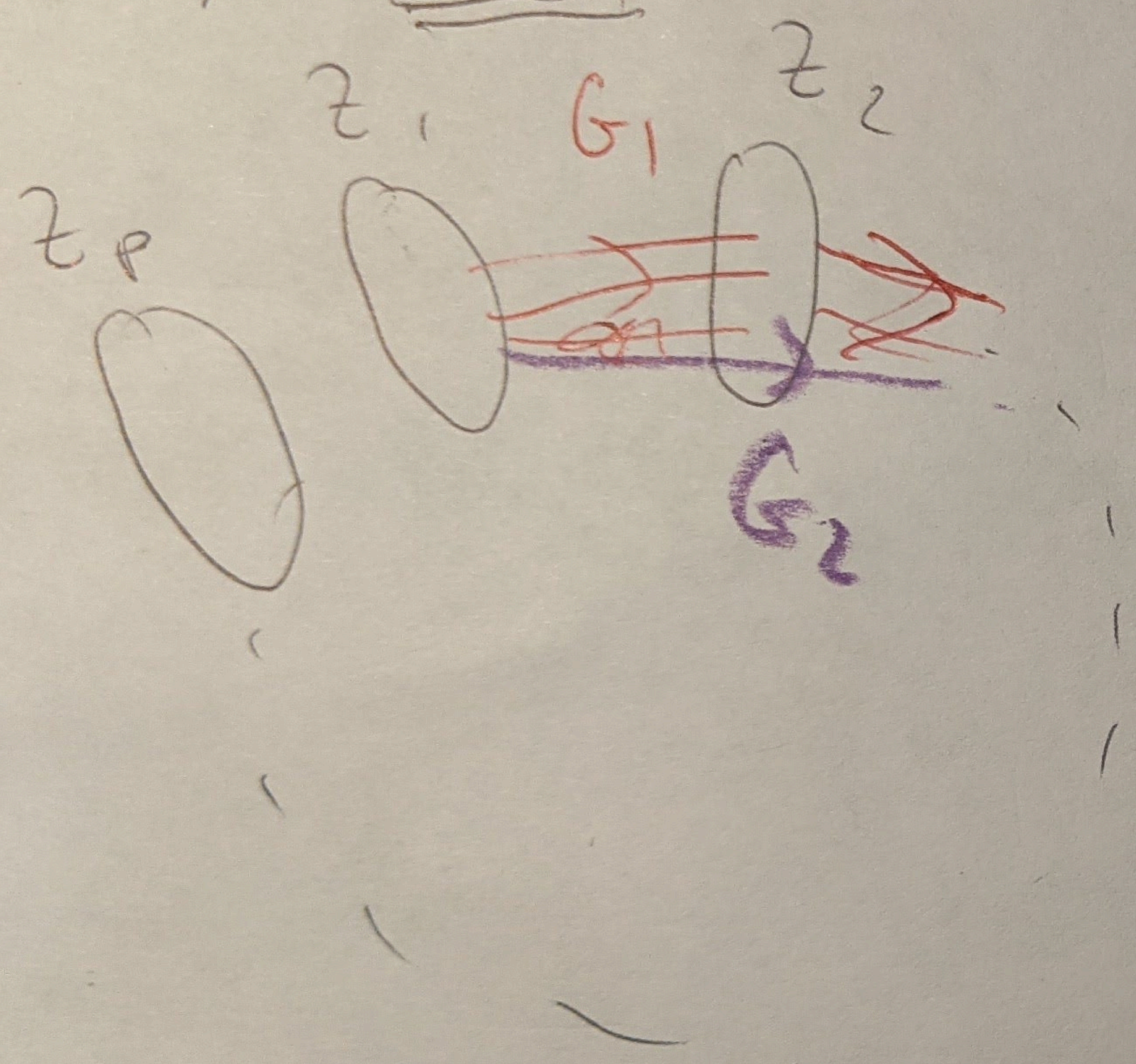
Let $X_i = F_i \setminus X$ then $X_1, X_2, \dots, X_d, X$ are linear directed forests as desired.

Let p be a prime $\quad p \sim C\sqrt{d}$

Divide vertices of G into p sets

$Z_1, Z_2, \ldots, Z_p$ uniformly at random.

By LLL every vertex v has $\sim \frac{d}{p}$ out and in neighbors in each Z_i.



Edges within each Z_i can be easily colored.

The rest can be decomposed into graph $G_1, G_2, \ldots, G_{p-1}$

G_i consists of edges between Z_k and $Z_{k+i \bmod p}$ for every i

Each of G_i has no cycles of length $< p$. So 6.6. applies.

(Sketch of Alon's proof that linear arboricity ~~ko~~ holds asymptotically)