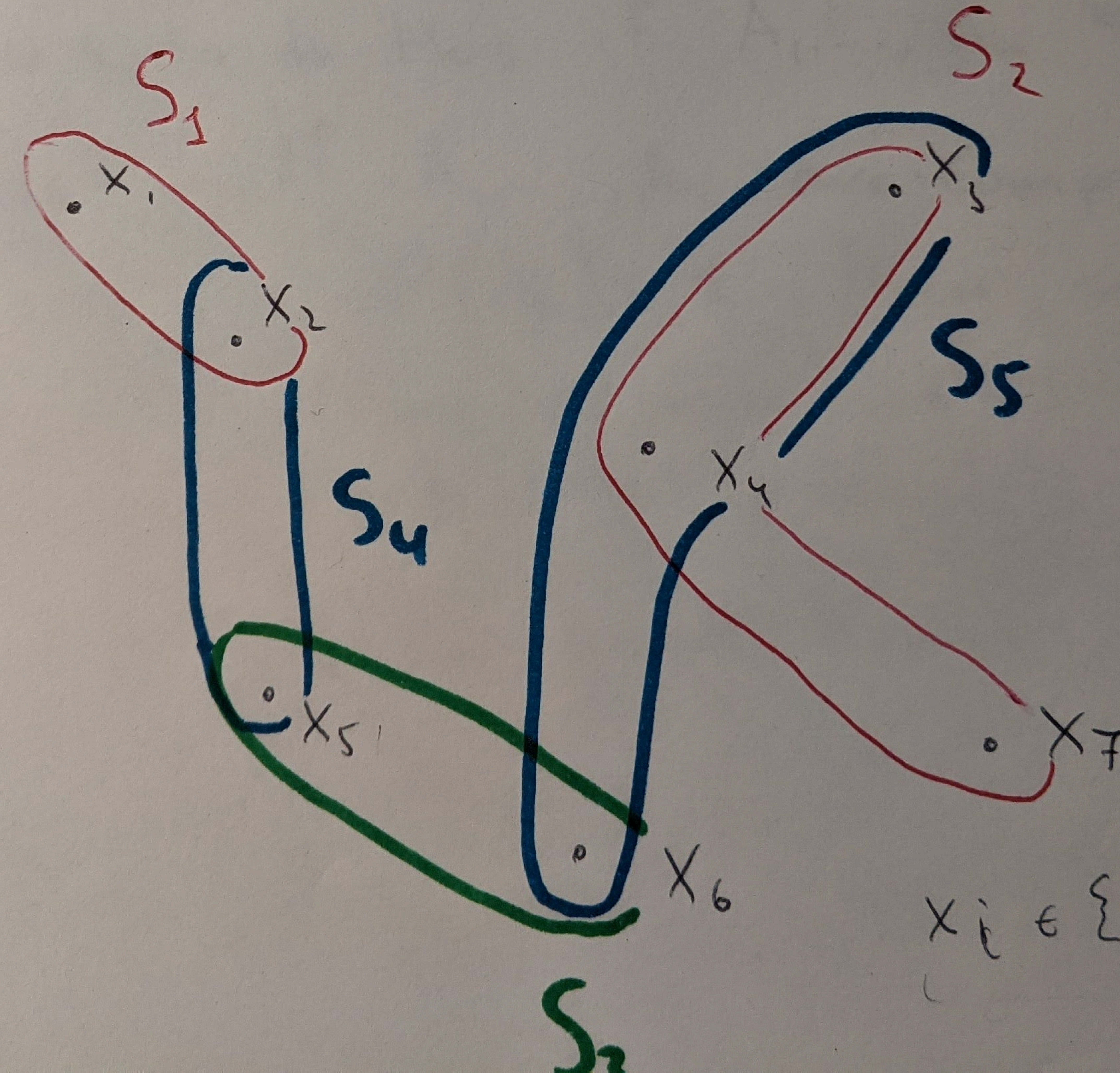
Lecture 11: Lovász Local Lemma



$A_1 = x_1 \wedge x_2$

$A_5 = \bar{x}_3 \wedge x_4 \wedge x_5$

$N(1) = \{4\}$.

A_1 is independent of $\{A_2, A_3, A_5\}$

$x_i \in \{0,1\}$ $\quad \mathbb{P}[x_i = 0] = 1/2$

6. Lovász local lemma. (LLL)

Erdös & Lovász, 1975

Setup: collection of bad events

$A_1, A_2, \ldots, A_n$

want to "avoid" all of them

show that $\mathbb{P}(\bar{A}_1 \wedge \bar{A}_2 \wedge \ldots \wedge \bar{A}_n) > 0$

(none of them happen).

LLL allows us to do this if $A_1, \ldots, A_n$ have "limited" dependence.

Extreme cases:
- if $A_1, \ldots, A_n$ are completely independent $\mathbb{P}(A_i) < 1$ then we can avoid all
- if we do not control dependence, but total probality $\sum_{i=1}^{n} \mathbb{P}(A_i) < 1$.

  then union bound implies $\sum_{i=1}^{n} \mathbb{P}(A_i) \geq \mathbb{P}(A_1 \vee A_2 \vee \ldots \vee A_n)$.

$A_1, A_2, \ldots, A_n$ are bad events

each A_i is independent of all events A_j except for the set $\boxed{N(i)}$.

A_i is <u>independent</u> of $\{ A_j \mid j \notin N(i) \cup \{i\} \}$

This is ~~so~~ more restrictive then " A_i is independent of A_j " for each such j.

$$\mathbb{P}(A_i) = \mathbb{P}(A_i \mid B_1 \wedge B_2 \wedge \ldots \wedge B_m)$$

$B_k = A_j$ or $\overline{A_j}$ for some $j \notin N(i) \cup \{i\}$.

Is this genuinely more restrictive then pairwise independence?

$A = \{ x_1 = 0 \}$ $\quad B = \{ x_2 = 0 \}$ $\quad C = \{ x_1 + x_2 = 0 \}$

for $x_1, x_2 \in \mathbb{Z}/2\mathbb{Z}$ selected uniformly at random.

A, B, C are pairwise independent, but

$$\mathbb{P}(A \mid B \wedge C) = 1 \neq \mathbb{P}(A).$$

Random variable model:

$X_1, X_2, \dots, X_N$ independent random variables.
(e.g. colors of vertices of a hypergraph).

Each A_i depends only on variables in a (small) set S_i (can be considered as a hypergraph edge).

Then $N(i) = \{ j : S_i \cap S_j \neq \emptyset \}$.
satisfies the requirement.

Theorem 6.1 (LLL, symmetric form)
Let $A_1, \dots, A_n$ be events. $\mathbb{P}[A_i] \leq p$ .. for every i.
Each A_i is independent of all but at most $\boxed{d}$ other events.

If

$$e \cdot p \cdot (d+1) \leq 1$$

then with positive probability we can avoid all A_i

First application:

Theorem 6.2 (Spencer, 1977): If

$$e\cdot\left(\underbrace{\binom{k}{2}\binom{n}{k-2}}+1\right)2^{1-\binom{k}{2}} \le 1$$

then $\underbrace{R(k,k)}_{\text{the Ramsey number}} > n$

(There exists a graph on n vertices with no complete subgraph or independent set of size k,
i.e. ~~K_n can be~~ edges of K_n can be red or blue 2-colored s.t. no K_k is monochromatic).

— 6.2. implies that $R(k,k) \geq \left(\frac{\sqrt{2}}{e}+o(1)\right)k\,2^{k/2}$.
Factor $\sqrt{2}$ better then what we proved in the first lecture using alterations.
This is the best known lower bound

Proof: Consider events $\{A_X : X \subseteq [n],\ |X|=k\}$.

A_X = all edges with both ends have the same color$\}$.

$\mathbb{P}[A_X] = 2^{1-\binom{k}{2}} = p$.

$N(X) = \{Y : |Y|=k, |X\cap Y| \geq 2\}$.

$|N(X)| \leq \binom{k}{2}\binom{n}{k-2} = d$ by 6.1. (symmetric LLL) ~~there~~ with positive probability no A_i happens.

Theorem 6.3 (LLL, general form).

Let $A_1, \dots, A_n$ be events.
For each $i \in [n]$ let $N(i) \subseteq [n]$ be such that
A_i is independent of $\{A_j : j \notin N(i) \cup \{i\}\}$.

If $x_1, x_2, \dots, x_n \in [0,1)$ are s.t.

$$\boxed{\mathbb{P}(A_i) \leq x_i \prod_{j \in N(i)} (1 - x_j)} \quad \forall i \in [n]$$

Then $\mathbb{P}(\bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_n) \geq (1-x_1)(1-x_2) \cdots (1-x_n) > 0.$

Proof 6.1 assuming 6.3: Let $\boxed{x_i = \frac{1}{d+1}}$

Need to check

$$\underline{p \leq \frac{1}{d+1} \cdot \left(1 - \frac{1}{d+1}\right)^d}$$

given $ep(d+1) \leq 1$
$p \leq \frac{1}{e(d+1)}$

I.e. $\frac{1}{e\cancel{(d+1)}} \leq \frac{1}{\cancel{d+1}} \cdot \left(1 - \frac{1}{d+1}\right)^d$ ✓

Proof of 6.3:

By induction on $|S|$ we will show that

$$(*) \quad \mathbb{P}(A_i \mid \bigwedge_{j \in S} \bar{A}_j) \le x_i \quad \text{for every } S \subseteq [n],\ i \in [n].$$

This implies the theorem

$$\mathbb{P}(\bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_n) = \mathbb{P}(\bar{A}_1 \mid \bigwedge_{i \ge 2} \bar{A}_i) \cdot \mathbb{P}(\bar{A}_2 \mid \bigwedge_{i \ge 3} \bar{A}_i) \cdot \ldots \cdot \mathbb{P}(\bar{A}_n)$$

$$\overset{(*)}{\ge} (1-x_1)(1-x_2) \cdots (1-x_n).$$

Base case $|S| = 0$ ✓

Induction step: $i \in S$ then left side of $(*)$ is 0 ✓

$i \notin S$.

If $N(i) \cap S = \emptyset$ then $(*)$ holds.

So we assume $S_1 := N(i) \cap S \neq \emptyset$, define $S_2 = S \setminus S_1$.

$$\mathbb{P}(A_i \mid \bigwedge_{j \in S} \bar{A}_j) = \frac{\mathbb{P}(A_i \wedge (\bigwedge_{j \in S_1} \bar{A}_j) \mid \bigwedge_{j \in S_2} \bar{A}_j)}{\mathbb{P}(\bigwedge_{j \in S_1} \bar{A}_j \mid \bigwedge_{j \in S_2} \bar{A}_j)} \begin{matrix} \boxed{= P_1} \\ \\ \boxed{= P_2} \end{matrix}$$

$$p_1 \le \mathbb{P}\Big(A_i \,\Big|\, \bigwedge_{j\in S_2} \overline{A}_j\Big) = \underbrace{\mathbb{P}(A_i)} \le \underline{x_i \prod_{j\in N(i)} (1-x_j)}.$$

$$p_2 \ge \mathbb{P}\Big(\overline{A}_{j_1} \,\Big|\, \bigwedge_{j\in S\setminus\{j_1\}} \overline{A}_j\Big)\cdot \mathbb{P}\Big(\overline{A}_{j_2} \,\Big|\, \bigwedge_{j\in S\setminus\{j_1,j_2\}} \overline{A}_j\Big)\cdot \cdots\cdots$$

Let $S_1 = \{j_1, j_2, \dots, j_k\}$. $\ge (1-\underbrace{x_{j_1}})\cdot(1-x_{j_2})\cdot \cdots\cdots$

$\cap$ $N(i)$.

by ind hypothesis

$$\ge \underline{\prod_{j\in N(i)} (1-x_j)}.$$

So $\frac{p_1}{p_2} \le x_i$ as desired.

2-coloring k-uniform hypergraphs.

Recall: every such hypergraph with $\leq 2^{k-1}$ edges is 2-colorable.

Theorem 6.4: If every edge in a k-uniform hypergraph intersects $\leq \frac{2^{k-1}}{e} - 1$ other edges then the hypergraph is 2-colorable.

Proof: Color each vertex in one of two colors uniformly, independently at random.

Let A_f = edge f is monochromatic

$\mathbb{P}[A_f] = 2^{-k+1} = p$.

$N(f) = \{ f' : \underbrace{f' \cap f \neq \emptyset} \}$. $|N(f)| \leq \frac{2^{k-1}}{e} - 1 = d$.

$e \cdot p \cdot (d+1) \leq 1$ so 6.1 applies.

Note that if $|f' \cap f| = 1$ then $A_{f'}$ and A_f are independent.

Could $N(f)$ be defined as $\{ f' : |f' \cap f| \geq 2 \}$? No

Not independent of collection of all such events.