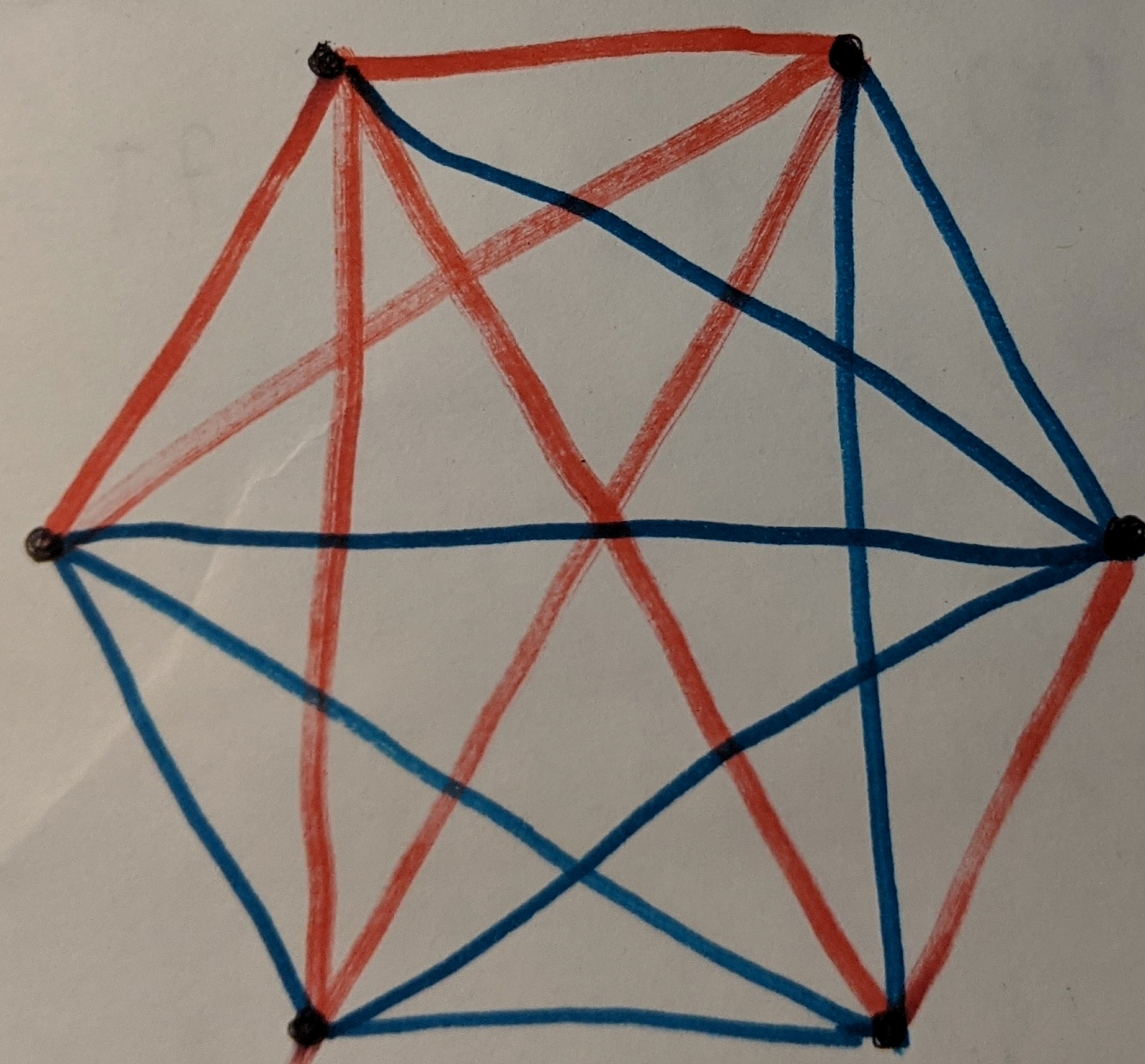
Lecture 1:
Introduction



The probabilistic method -

- to find an object with certain properties choose a random one (from carefully selected distribution).

Pioneered by Paul Erdős.

Lemma 1.1: Every graph with m edges has a bipartite subgraph with $\geq \frac{m}{2}$ edges.

(Graphs are simple:

Graph G consists of vertex set $V(G)$ and edge set $E(G) \subseteq V(G)^{(2)}$ collection of vertex pairs).

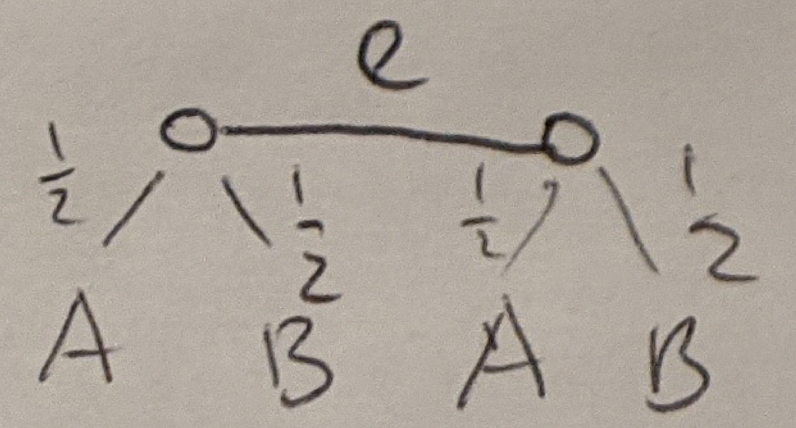
Proof: Let (A, B) be a partition of $V(G)$ chosen uniformly at random.

$P[v \in A] = 1/2$ independently for every $v \in V(G)$.

Let $E' \subseteq E(G)$ consists of edges with one end in A another in B.

$P[e \in E'] = 1/2$.

$$\mathbb{E}[|E'|] = \sum_{e \in E(G)} P[e \in E'] = \tfrac{1}{2}|E| = \underbrace{\tfrac{1}{2}m}.$$



$R(k,l)$ - ramsey number is the minimum integer N s.t. in every coloring of edges of complete graph on N vertices in two colors red & blue there is either a complete subgraph on k vertices with all red edges

or l vertices with all blue edges.

Ramsey theorem: $R(k,l)$ exists for all $k, l \in \mathbb{N}$.

Erdős - Szekeres: $R(k,k) \le \underbrace{\binom{2k-2}{k-1}}$ $\sim c \frac{4^k}{\sqrt{k}}$.

Theorem 1.2 (Erdős, 1947): If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ $(*)$.

then $R(k,k) > n$.

Proof: Color the edges of K_n in red or blue independently at random. uniformly

(there are $2^{\binom{n}{2}}$ colorings, this is the same as choosing one at random).

Assume $V(K_n) = \{1,2,\ldots,n\} = [n]$

For every $X \subseteq [n]$ with $|X| = k$

let $A_X = \begin{cases} 1 & \text{if all edges between vertices of } X \text{ are the same color} \\ 0 & \text{, otherwise.} \end{cases}$

$$P[A_X = 1] = \frac{1}{2^{\binom{k}{2}-1}} = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$E\left[\sum_X A_X\right] = \sum_{X} E[A_X] = \frac{1}{2^{\binom{k}{2}-1}} \cdot \binom{n}{k} \overset{(*)}{<} 1.$$

$$E[A_X] = 1 \cdot P[A_X = 1] + 0 \cdot P[A_X = 0]$$

So $P\left[\sum A_X = 0\right] > 0 \rightarrow$ there exists a coloring with no monochratic K_k.

$R(k,k) \geq 2^{k/2}$

$R(k,k) \geq (1 + o(1)) \frac{k}{e\sqrt{2}} 2^{k/2}$

The best explicit constructions only give colorings with no monochromatic K_k on $\sim 2^{\sqrt{k}}$ vertices.

Lemma 1.3: For all $k, n \in \mathbb{N}$

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}. \quad (**)$$

"Alteration method"

Proof: As seen in 1.2. there is a 2-coloring of K_n with $\leq \binom{n}{k} 2^{1-\binom{k}{2}}$ monochromatic K_k.
Remove a vertex from each of them to obtain a ~~graph~~ coloring of complete graph with $\geq (**)$ vertices and no monochromatic K_k.

$$\downarrow$$

$$R(k,k) \geq (1+o(1)) \frac{k}{e} 2^{k/2}.$$

Best known bound:

$$R(k,k) \geq c \underset{=}{k^2} 2^{k/2}. \qquad k^2 = e^{\underline{2\log k}}$$

Upper bound:

$$R(k,k) \leq e^{-c(\log k)^2} \binom{2k}{k}$$

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k-uniform hypergraphs

H has vertex set $V(H)$

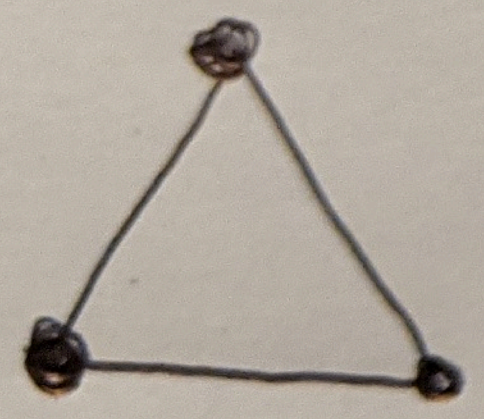
and edge set $E(H) \subseteq \underbrace{V(H)^{(k)}}_{\text{collection of } k \text{ element subsets of } V(H).}$

H is 2-colorable if there exists a 2-coloring of $V(H)$ s.t. every edge contains two vertices of different colors.

What is the minimum number of edges in a non 2-colorable k-uniform hypergraph?

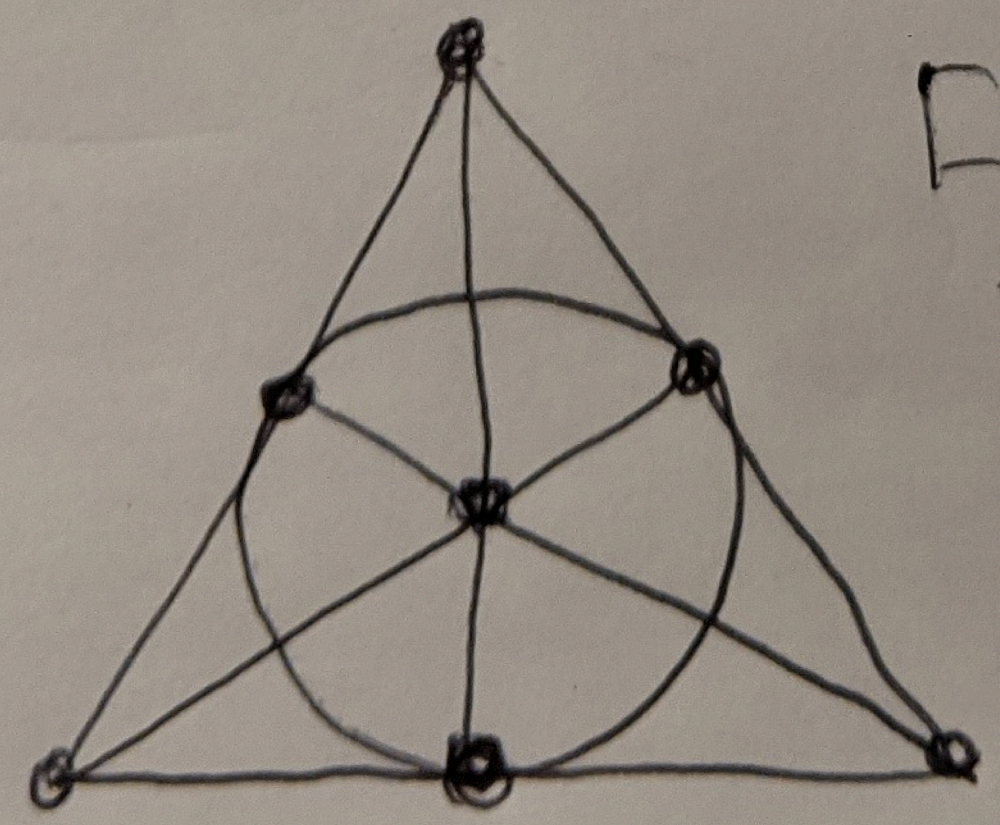
Let $m(k)$ denote the answer.

$m(2) = 3$ $\qquad$ $m(3) = 7$



$K_5^{(3)} \to$ all triples on 5 vertices.

10 edges



Fano plane
3-uniform hypergraph.

7 edges

Lemma 1.4: $m(k) \geq 2^{k-1}$ for all $k \geq 2$.

Proof: If H is k-uniform $|E(H)| \overset{=}{<} 2^{k-1}$ want to show there is a 2-coloring.

Color each vertex white or black indepeutly uniformly at random

$$P[e \text{ is monochromatic}] = 2 \cdot \frac{1}{2^k} = \frac{1}{2^{k-1}}$$

$$E[\#\text{ monochromatic edges}] = |E(H)| \cdot \frac{1}{2^{k-1}} \overset{=}{<} 1.$$

So there is a coloring with no monochromatic edges.

there is a coloring with all edges monochromat

< 1.

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$$m(k) \neq O(k^2 2^k)$$