MATH 550: Combinatorics. Winter 2018.
Final exam.
Due electronically at snorine@gmail.com by 5PM on Sunday, April 29th.

1. Sperner families. Let $k \geq 1$ be fixed, and let $\mathcal{F} \subseteq \mathcal{P}([n])$ be a Sperner family containing at least one set of size at most k, at least one set of size at least $n-k$, and no sets whose size is strictly between k and $n-k$.

a) Show that there exists a constant c_k such that $|\mathcal{F}| \leq c_k n^{k-1}$.

b) What is the maximum of $|F|$ for $k=2$?

2. Ramsey theorem & subgraph densities.
For a pair of positive integers $n \geq t$, let $d(n,t)$ denote the minimum number of monochromatic copies of K_t among all edge colorings of K_n in two colors.

a) Show that $d(t) := \lim_{n\to\infty} \frac{d(n,t)}{\binom{n}{t}}$ exists for every t and is positive.

b) Show that $d(t) \leq 2^{1-\binom{t}{2}}$ for every $t \geq 2$.

c) Show that $d(3) = 1/4$.

d) Show that $d(4) \geq 2^{-11}$.

3. Convexity & Ramsey theory.
Show that for all positive integers n, d there exists a positive integer $N = N(n,d)$ such that in every set of N points $P \subseteq \mathbb{R}^d$ in general position one can find a subset of size n in convex position. (A set of P is in *general position* in $\mathbb{R}^d$ if no set of $d+1$ points in P is affinely dependent. A set X is in *convex position* if no point of X lies in the convex hull of the remaining points.)

4. Convexity.
Let $K \subseteq \mathbb{R}^d$ be a convex set, and let $C_1, C_2, \ldots, C_n \subseteq \mathbb{R}^d$ for some $n \geq d+1$ be convex sets such that intersection of any $d+1$ of them contains a translated copy of K. Show that $\cap_{i=1}^n C_i$ also contains a translated copy of K.

5. Szemeredi-Trotter theorem.
Let P be a collection of n points in $\mathbb{R}^2$, and let $k \leq \sqrt{n}$ be an integer. Let S be the set of pairs $\{p,q\} \subseteq P$, such that the number of points of P on the line joining p and q is between k and $2k$. Prove that

$$|S| \leq c\frac{n^2}{k},$$

for some constant c independent of n and k.

6. Combinatorial Nullstellensatz.
Let $\mathcal{F}$ be a set system such that $|\cup_{F\in\mathcal{F}} F| > 2|\mathcal{F}|$. Show that there exists $\mathcal{F}' \subseteq \mathcal{F}, \mathcal{F}' \neq \emptyset$ such that for every $x \in X$ the number of elements of $\mathcal{F}'$ containing x is divisible by 3.