

Assignment #1: Set systems. Due in class on Wednesday, February 17th.

**1. Bollobás 3.9.** Suppose  $\mathcal{A} \subseteq \mathcal{P}([n])$  is an *ideal*, i.e. if  $B \subseteq A$  and  $A \in \mathcal{A}$  then  $B \in \mathcal{A}$ . Use the local LYM inequality to show that the average size of an element of  $\mathcal{A}$  is at most  $n/2$ .

**2.** Let  $n$  be a positive integer. Consider a set  $\mathcal{T}_n = \{0, 1, 2\}^n$  consisting of all sequences  $(a_1, a_2, \dots, a_n)$  with  $a_i \in \{0, 1, 2\}$  for  $i \in [n]$ .

We define a partial order on  $\mathcal{T}_n$  so that  $(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n)$  if and only if  $a_i \leq b_i$  for every  $i \in [n]$ . (For example  $(1, 0, 1) \leq (1, 2, 2)$ , while  $(1, 0, 1)$  and  $(0, 1, 2)$  are incomparable.)

For a sequence  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  define the *weight* of  $\mathbf{a}$  to be  $w(\mathbf{a}) := a_1 + a_2 + \dots + a_n$ . A chain  $\mathcal{C} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k)$  with  $\mathbf{a}_1 < \mathbf{a}_2 < \dots < \mathbf{a}_k$  in  $\mathcal{T}_n$  is called *symmetric* if  $w(\mathbf{a}_{i+1}) = w(\mathbf{a}_i) + 1$  for  $i = 1, 2, \dots, k - 1$  and  $w(\mathbf{a}_1) + w(\mathbf{a}_k) = 2n$ .

a) Show that  $\mathcal{T}_n$  allows a symmetric chain decomposition.

b) Give an example of an antichain in  $\mathcal{T}_n$  which intersects every symmetric chain. Deduce that this antichain is maximum. (An *antichain* is a subset  $\mathcal{A} \in \mathcal{T}_n$  such that for  $\mathbf{a}, \mathbf{b} \in \mathcal{A}$  if  $\mathbf{a} \leq \mathbf{b}$  then  $\mathbf{a} = \mathbf{b}$ , i. e. no two distinct elements of  $\mathcal{A}$  are comparable.)

**3. Hilton, 1974.** Let  $1 \leq g \leq h \leq n$  be integers with  $g + h \leq n$ . Let  $\mathcal{F} \subseteq \mathcal{P}([n])$  be an intersecting family and suppose that  $g \leq |F| \leq h$  for every  $F \in \mathcal{F}$ . Use Erdős-Ko-Rado theorem to show that

$$|\mathcal{F}| \leq \sum_{i=g}^h \binom{n-1}{i-1}.$$

**4.** A *k-sunflower* in a set system  $\mathcal{F}$  on  $X$  is a collection of distinct sets  $F_1, F_2, \dots, F_k \in \mathcal{F}$  such that for some  $Z \subseteq X$  we have  $F_i \cap F_j = Z$  for all  $1 \leq i < j \leq k$ . (I.e. the intersection of every pair of distinct sets in the sunflower is the same.) Let  $c(k, r)$  denote the maximum possible size of a set system  $\mathcal{F}$  such that

(\*)  $|F| \leq r$  for every  $F \in \mathcal{F}$ , and  $\mathcal{F}$  does not contain a  $k$ -sunflower.

Suppose that a set system  $\mathcal{F}$  on  $X$  satisfies (\*).

a) Show that there exists a set  $Y \subseteq X$  with  $|Y| \leq (k-1)r$  such that every set in  $\mathcal{F}$  contains an element of  $Y$ .

b) Let  $\mathcal{F}_y = \{F - y \mid F \in \mathcal{F}, y \in F\}$ . Show that  $|\mathcal{F}_y| \leq c(k, r-1)$  for every  $y$ .

c) Deduce from a) and b) that

$$c(k, r) \leq (k-1)^r r!$$

d) Construct an explicit example of a family  $\mathcal{F}$  satisfying (\*) to show that

$$c(k, r) \geq (k-1)^r.$$

**5.** Let  $\mathcal{A} \subseteq \mathbb{N}^{(3)}$  be a 3-graph such that  $|\mathcal{A}| = 50$  and  $|\partial\mathcal{A}| = 27$ . Show that for some  $Z \subseteq X \subseteq \mathbb{N}$  with  $|X| = 8, |Z| = 2$ , we have

$$\mathcal{A} = \{A \in \mathbb{N}^{(3)} \mid A \subset X, Z \not\subset A\}.$$

**6.** What is the minimum size of compressed  $\mathcal{A} \subseteq \mathbb{N}^{(3)}$  such that  $\{1, 10, 100\}, \{1, 20, 50\} \in \mathcal{A}$ ?