MATH 550: Combinatorics. Winter 2016.
Final exam.
Due electronically at snorine@gmail.com by 5PM on Saturday, April 30th.

1. Intersecting families.

Let $1 \le r \le n/2$. Let $\mathcal{F} \subseteq \mathcal{P}([n])$ be an intersecting family. Suppose that $|A| \le r$ for every $A \in \mathcal{F}$. Show that

$$|\mathcal{F}| \le \binom{n-1}{r-1}.$$

2. Ramsey Theorem.

For a pair of positive integers $n \ge t$, let $d(n,t)$ denote the minimum number of monochromatic copies of K_t among all edge colorings of K_n in two colors.

a) Show that $d(t) := \lim_{n\to\infty} \frac{d(n,t)}{\binom{n}{t}}$ exists for every t and is positive.

b) Show that $d(t) \le 2^{1-\binom{t}{2}}$ for every $t \ge 2$.

c) Show that $d(3) = 1/4$.

d) Show that $d(4) \ge 2^{-11}$.

3. Convexity.

Show that for every set P of n points in $\mathbb{R}^2$ there exist two points $p, q \in \mathbb{R}^2$, such that every convex set C satisfying $|C \cap P| > 4n/7$ contains at least one of p and q.

4. Szemeredi-Trotter theorem.

Let P be a collection of n points in $\mathbb{R}^2$, and let $k \le \sqrt{n}$ be an integer. Let S be the set of pairs $\{p, q\} \subseteq P$, such that the number of points of P on the line joining p and q is between k and $2k$. Prove that

$$|S| \le c\frac{n^2}{k},$$

for some constant c independent of n and k.

5. Combinatorial Nullstellensatz.

Show that any graph G with average degree at least $2p-2$ and maximum degree at most $2p-1$ contains a subgraph H such that $\deg_H(v) = p$ for every $v \in V(H)$.

6. Shannon capacity.

Let $\Theta(C_7)$ denote the Shannon capacity of the cycle of length 7. Show that

$$\sqrt{10} \le \Theta(C_7) \le \frac{7}{2}.$$