**Name** ______________________

**MATH 550 - Midterm EXAM**

Wednesday February 13th, 2013, 14:35PM-15:55PM

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This examination booklet contains 3 problems on 7 sheets of paper including the front cover. Do all of your work in this booklet and show all your computations. Justify your answers.

You can choose which two out of the three problems to work on. You must indicate your choice.

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| Problem | Your choice | Your score |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| Total | | |

**1. Sunflowers.**

Let $(X, \mathcal{F})$ be a set system. A $k$-*sunflower* in $\mathcal{F}$ is a collection of distinct sets $F_1, F_2, \ldots, F_k \in \mathcal{F}$ such that for some $Z \subseteq X$ we have $F_i \cap F_j = Z$ for all $1 \leq i < j \leq k$. (The intersection of every pair of distinct sets in the sunflower is the same.)

Let $c(k, r)$ denote the maximum possible size of a set system $\mathcal{F}$ such that

($*$) $|F| \leq r$ for every $F \in \mathcal{F}$, and $\mathcal{F}$ does not contain a $k$-sunflower.

Suppose that $(X, \mathcal{F})$ satisfies ($*$).

**a)** Show that there exists a set $Y \subseteq X$ with $|Y| \leq (k-1)r$ such that every set in $\mathcal{F}$ contains an element of $Y$.

**b)** Let $\mathcal{F}_y = \{F - y \mid F \in \mathcal{F}, y \in F\}$. Show that $|\mathcal{F}_y| \leq c(k, r-1)$ for every $y$.

**c)** Deduce from a) and b) that

$$c(k, r) \leq (k-1)^r r!$$

**d)** Construct an explicit example of a family $\mathcal{F}$ satisfying ($*$) to show that

$$c(k, r) \geq (k-1)^r.$$

**2. Shadows.**
Let $\mathcal{A} \subseteq \mathbb{N}^{(3)}$ be a 3-graph.

**a)** Show that if $|\mathcal{A}| \geq 46$ then $|\partial\mathcal{A}| \geq 27$.

Let $\mathcal{A} \subseteq \mathbb{N}^{(3)}$ satisfy $|\mathcal{A}| = 50$ and $|\partial\mathcal{A}| = 27$.

**b)** Deduce from a) that every $B \in \partial\mathcal{A}$ is a subset of at least 5 different sets in $\mathcal{A}$.

**c)** Show that $|\cup_{A\in\mathcal{A}} A| = 8$. (*Hint:* If $\partial\mathcal{A}$ is considered as a graph then it has minimum degree 6 by b).)

**d)** Deduce from c) that for some $Z \subseteq X \subseteq \mathbb{N}$ with $|X| = 8, |Z| = 2$, we have

$$\mathcal{A} = \{A \in \mathbb{N}^{(3)} \mid A \subset X, Z \not\subset A\}.$$

**3. Triangles.**

Let $G$ be a graph on $n$ vertices for some $n \geq 3$. Suppose that $|G| = \lfloor \frac{n^2}{4} \rfloor + 1$. A *triangle* is a $K_3$ subgraph of $G$. Our goal is to show that $G$ contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

Let $H \subseteq G$ be the set of all the edges of triangles in $G$. Let $S \subseteq V(G)$ be the set of vertices of these triangles.

**a)** Show that if $|H| < 3\lfloor \frac{n}{2} \rfloor$ then some edge in $G \setminus H$ has an end in $S$. (*Hint:* Assume the contrary. Count the edges with both ends in $S$ and both ends in $V(G) - S$. Apply Turán's theorem.)

**b)** Let $\{u, v\}$ be an edge of $G \setminus H$. Show that at most $n - 1$ edges of $G$ contain either $u$ or $v$.

**c)** Use a), b) and induction on $n$ to deduce that $G$ contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

**d)** Show that the bound in c) is tight: For every $n \geq 3$ there exists a graph $G$ on $n$ vertices with $|G| = \lfloor \frac{n^2}{4} \rfloor + 1$ containing exactly $\lfloor \frac{n}{2} \rfloor$ triangles.