Name ______________________

MATH 550 - Midterm EXAM

Tuesday February 14th, 2012, 14:35PM-16:55PM

This examination booklet contains 3 problems on 6 sheets of paper including the front cover. Do all of your work in this booklet and show all your computations. Justify your answers.

You can choose which two out of the three problems to work on. You must indicate your choice.

Problem	Your choice	Your score
1		
2		
3		
Total		

1. Symmetric chain decompositions.
Let n be a positive integer. Consider a set $\mathcal{T}_n = \{0,1,2\}^n$ consisting of all sequences $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0,1,2\}$ for $i \in [n]$.

We define a partial order on $\mathcal{T}_n$ so that $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if and only if $a_i \leq b_i$ for every $i \in [n]$. (For example $(1,0,1) \leq (1,2,2)$, while $(1,0,1)$ and $(0,1,2)$ are incomparable.)

For a sequence $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$ define the *weight* of $\mathbf{a}$ to be $w(\mathbf{a}) := \mathbf{a_1} + \mathbf{a_2} + \ldots + \mathbf{a_n}$.

a) What is the maximum length of a chain in $\mathcal{T}_n$?.

A chain $\mathcal{C} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k})$ with $\mathbf{a_1} < \mathbf{a_2} < \ldots < \mathbf{a_k}$ in $\mathcal{T}_n$ is called *symmetric* if $w(\mathbf{a_{i+1}}) = \mathbf{w}(\mathbf{a_i}) + \mathbf{1}$ for $i = 1, 2, \ldots, k-1$ and $w(\mathbf{a_1}) + \mathbf{w}(\mathbf{a_k}) = \mathbf{2nx}$.

b) Show that if $\mathcal{C} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k})$ is a symmetric chain in $\mathcal{T}_n$ then

$$\mathcal{C}_1 = (\mathbf{a_1 0}, \mathbf{a_2 0}, \ldots, \mathbf{a_k 0}, \mathbf{a_k 1}, \mathbf{a_k 2}),$$

$$\mathcal{C}_2 = (\mathbf{a_2 1}, \mathbf{a_3 1}, \ldots, \mathbf{a_{k-1} 1}),$$

and

$$\mathcal{C}_3 = (\mathbf{a_1 1}, \mathbf{a_1 2}, \mathbf{a_2 2}, \ldots, \mathbf{a_{k-1} 2})$$

are symmetric chains in $\mathcal{T}_{n+1}$. (Here $\mathbf{a_i 0}$, $\mathbf{a_i 1}$, $\mathbf{a_i 2}$ denote the sequences obtained by appending 0, 1 or 2, respectively to the sequence $\mathbf{a_i}$.)

c) Deduce that $\mathcal{T}_n$ allows a symmetric chain decomposition.

d) What is the maximum size of an antichain in $\mathcal{T}_n$? (An *antichain* is a subset $\mathcal{A} \in \mathcal{T}_n$ such that for $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ if $\mathbf{a} \leq \mathbf{b}$ then $\mathbf{a} = \mathbf{b}$, i. e. no two distinct elements of $\mathcal{A}$ are comparable.)

2. Intersecting families and compression.

Reminder: For a set $A \subset [n]$ and $i, j \in [n]$ we define $R_{ij}(A) = (A \setminus \{j\}) \cup \{i\}$ if $i \notin A$, $j \in A$ and $R_{ij}(A) = A$, otherwise. For $\mathcal{A} \subseteq \mathcal{P}([n])$ we define

$$\tilde{R}_{ij}(\mathcal{A}) = \{R_{ij}(A) \mid A \in \mathcal{A}\} \cup \{A \mid A, R_{ij}(A) \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is *compressed* if $\tilde{R}_{ij}(\mathcal{A}) = \mathcal{A}$ for all $1 \leq i < j \leq n$.

Our goal in this problem is to give a new proof of Erdős-Ko-Rado theorem using compression operators. We will show that if $k \leq n/2$ and $\mathcal{A} \in [n]^{(k)}$ is intersecting then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

We assume that the statement holds for $k = n/2$. The proof proceeds by induction on n.

a) Show that if $\mathcal{A}$ is intersecting then $\tilde{R}_{ij}(\mathcal{A})$ is intersecting.

b) Deduce that it suffices to give a proof for compressed intersecting families.

Let $\mathcal{A}$ be a compressed intersecting family. Define $\mathcal{A}_0 = \{A \in \mathcal{A} \mid n \notin A\}$, $\mathcal{A}_1 = \{A \in \mathcal{A} \mid n \in A\}$ and $\mathcal{A}_1' = \mathcal{A}_1 - \{n\}$.

c) Show that $\mathcal{A}_1'$ is intersecting. (It is necessary to use the facts that $\mathcal{A}$ is compressed and that $2k < n$.)

b) By applying the induction hypothesis to $\mathcal{A}_0$ and $\mathcal{A}_1'$ show that Erős-Ko-Rado theorem holds for $\mathcal{A}$.

3. Turán density. A *diamond* D is the graph obtained by deleting one edge from the complete graph on 4 vertices.

a) Show that the Turán density of the diamond $\pi(D)$ is at least 1/2.

Let $\varepsilon > 0$ be a real number. Let G be a graph on n vertices with

$$|G| \geq \frac{1+\varepsilon}{2}\binom{n}{2}.$$

Let $n_0 < n$ be a positive integer.

b) Show that at least $\frac{\varepsilon}{2}\binom{n}{n_0}$ subsets of $V(G)$ of size n_0 induce a subgraph of G with at least $\frac{1+\varepsilon/2}{2}\binom{n_0}{2}$ edges.

c) Show that there exists a positive integer n_0 independent on n so that every graph G on $n \geq n_0$ vertices contains at least

$$\frac{\frac{\varepsilon}{2}\binom{n}{n_0}}{\binom{n-3}{n_0-3}} \geq \frac{3\varepsilon}{n_0^3}\binom{n}{3}$$

triangles (K_3). (Use Turán's theorem for the triangle.)

d) Show that if n is sufficiently large then graph G, as above, contains an edge belonging to at least 2 triangles. Deduce that $\pi(D) = 1/2$.