MATH 550: Combinatorics. Winter 2015.

Assignment # 3: Discrete Geometry.

Due in class on Thursday, April 9th.

1. For $X \subseteq \mathbb{R}^d$ define $S(X)$ as a set of all points which lie on segments with ends in X. Let $S_2(X) := S(S(X))$ and, more generally, $S_{k+1}(X) = S(S_k(X))$. Show that $S_{\lceil \log_2(d+1) \rceil}(X)$ is always convex.

2. Show that for all positive integers n, d there exists a positive integer N such that in every set of N points $P \subseteq \mathbb{R}^d$ in general position one can find a subset of size n in convex position. (A set of P is in *general position* in $\mathbb{R}^d$ if no set of $d+1$ points in P is affinely dependent.)

3. **Matoušek. 1.3.4.** A *strip of width* w is a part of the plane bounded by two parallel lines at distance w. The *width* of a set $X \subseteq \mathbb{R}^2$ is the smallest width of a strip containing X.

(a) Show that every compact convex set of width 1 contains a segment of length 1 in every direction.

(b) Let $C_1, C_2, \dots, C_n$ be closed convex sets in the plane, $n \geq 3$, such that the intersection of every 3 of them has width at least 1. Show that $\cap_{i=1}^n C_i$ has width at least 1.

4.

(a) Prove that if a collection of n convex sets in $\mathbb{R}^2$ has the property that out of every 4 sets some three have a point in common then there is a point that belongs to at least $n/12$ sets in the collection.

(b) Prove that for all positive integers p, d so that $p \geq d+1$ there exists a constant $c = c(d,p) > 0$ so that if a family of $n \geq p$ convex sets in $\mathbb{R}^d$ has the property that among any p sets some $d+1$ have a point in common then some point belongs to at least cn sets in the family.

(c) Prove that for every positive integer d there is a constant $c = c(d)$ such that if a family $\mathcal{F}$ of n convex sets in $\mathbb{R}^d$ has the property that out of any $d+2$ sets in F some $d+1$ have a point in common, then $\mathcal{F}$ can be partitioned into at most $c \log n$ intersecting sub-families.

5. **Matoušek. 4.1.5 (a).** Use the Szemerédi-Trotter theorem to show that n points in the plane determine at most $O(n^{7/3})$ triangles of unit area.

6. **Tao-Vu. 8.2.6.** (Beck's theorem.) Let $P \subseteq \mathbb{R}^2$ be finite. Show that there either exists a line incident with $\Omega(|P|)$ points in P or there exist $\Omega(|P|^2)$ lines incident with at least 2 points in P.