Problem Solving Seminar. Fall 2017.
Problem Set 9. Probability

Classical results.

1. **Monty Hall problem.** Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 (but the door is not opened), and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

2. What is the probability that three randomly chosen points on a circle form an acute triangle?

3. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

4. **Lower bound on Ramsey numbers.** Show that it is possible to color the edges of the complete graph on $\lfloor 2^{(r-1)/2} \rfloor$ in red and blue colors so that there exists no complete subgraph on r vertices with all edges of the same color.

Problems.

1. **Putnam 1968. B1.** The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

   (i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,

   (ii) $P(y^\circ = 70^\circ)$, and

   (iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

   Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

2. **Putnam 2001. A2.** You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

3. **Putnam 1961. B2.** Two points are selected independently and at random from a segment length α. What is the probability that they are at least distance β $(< \alpha)$ apart?

4. **AIMC 1995.** Find the probability that in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails.

5. **Putnam 1958. A3.** Real numbers are chosen at random from the interval $[0, 1]$. Suppose that after choosing the n-th number the sum of the numbers so chosen first exceeds 1. Show that the expected value of n is e.

6. **Putnam 2014. A4.** Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

7. **Putnam 2002. B4.** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you *must* guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

8. **Putnam 2006. A6.** Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

9. **Putnam 1995. A6.** Suppose that each of n people writes down the numbers 1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.