Problem Seminar. Fall 2017.

Problem Set 2. Induction.

Classical results.

1. **Fermat's little theorem.** Let p be a prime number, and n a positive integer. Show that $n^p - n$ is divisible by p.

2. An *Hadamard matrix* is an $n \times n$ square matrix, all of whose entries are $+-1$ or 1, such that every pair of distinct rows is orthogonal. In other words, if the rows are considered to be vectors of length n, then the dot product between any two distinct row-vectors is zero. Show that there exist infinitely many Hadamard matrices.

3. **Ramsey's theorem.** Show that for any pair of positive integers (r, s), there exists a positive integer $R(r, s)$ such that if the edges of a complete graph on $R(r, s)$ vertices are coloured red or blue, then either there exists a complete subgraph on r vertices which is entirely blue, or a complete subgraph on s vertices which is entirely red. (A *complete graph* is a graph where every two vertices are connected by an edge.)

Problems.

1. Let n be a positive integer. Prove that
$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} < \frac{3}{2}.$$

2. **Putnam 2008. B2.** Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate
$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

3. **GA 32.** Show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers, then
$$(1 + a_1)(1 + a_2)\ldots(1 + a_n) \geq (1 + \sqrt[n]{a_1 a_2 \ldots a_n})^n.$$

4. **IMO 2001. B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

5. **Putnam 2004. A3.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

6. **Putnam 2015. B2.** Given a list of the positive integers $1, 2, 3, 4, \ldots$, take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: $6, 16, 27, 36, \ldots$. Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2015.

7. **Putnam 1996. A4.** Let S be the set of ordered triples (a, b, c) of distinct elements of a finite set A. Suppose that

   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;

   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$;

   (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S.

   Prove that there exists a one-to-one function g from A to $\mathbb{R}$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

8. **Putnam 1996. A6.** A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.