

Problem Seminar. Fall 2016.

Problem Set 6. Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers x_1, x_2, \dots, x_n ,

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \dots + x_n}{n}.$$

2. **Cauchy-Schwarz.** For any real $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$,

$$(x_1 y_1 + x_2 y_2 + \dots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \dots + x_n^2)(y_1^2 + y_2^2 + \dots + y_n^2).$$

3. **Arithmetic-harmonic mean.** For any non-negative real numbers x_1, x_2, \dots, x_n ,

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}} \leq \frac{x_1 + x_2 + \dots + x_n}{n}.$$

4. **Jensen.** For any convex function f and any real x_1, x_2, \dots, x_n ,

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}.$$

Problems.

1. **Putnam 2003. A2.** Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be nonnegative real numbers. Show that

$$(a_1 a_2 \dots a_n)^{1/n} + (b_1 b_2 \dots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \dots (a_n + b_n)]^{1/n}.$$

2. **Putnam 2004. B2.** Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

3. **USA 1997.** A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2 . Show that the largest positive number is at least 2.
4. **IMO 1994.** Let m and n be positive integers. Let a_1, a_2, \dots, a_m be distinct elements of $\{1, 2, \dots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k . Prove that:

$$\frac{a_1 + a_2 + \dots + a_m}{m} \geq \frac{n+1}{2}.$$

5. **IMO 2007. Shortlist.** Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Show that

$$\left(\sum_{k=1}^n \frac{1+x^{2k}}{1+x^{4k}}\right) \left(\sum_{k=1}^n \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

6. **Putnam 2003. A4.** Let a, b, c, A, B, C be real, a, A non-zero such that $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ for all real x . Show that $|b^2 - 4ac| \leq |B^2 - 4AC|$.
7. **Putnam 2013. B4.** For any continuous real-valued function f defined on the interval $[0, 1]$, let

$$\mu(f) = \int_0^1 f(x) dx, \quad \text{Var}(f) = \int_0^1 (f(x) - \mu(f))^2 dx,$$
$$M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0, 1]$, then

$$\text{Var}(fg) \leq 2\text{Var}(f)M(g)^2 + 2\text{Var}(g)M(f)^2.$$