Problem Seminar. Fall 2016.

Problem Set 5. Algebra.

Classical results.

1. **Vandermonde.** Let
$$V = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$
Then
$$\det(V) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

2. **Lagrange interpolation.** For every positive integer $n$ and every collection of real numbers $a_1, a_2$, $\dots$, $a_n$ there exists a polynomial of degree at most $n$ so that $P(1) = a_1$, $P(2) = a_2, \dots,$ $P(n) = a_n$.

3. **Lucas's theorem.** Show that for any polynomial $P(z)$ over complex numbers, the zeroes of $P'(z)$ lie in the convex hull of zeroes of $P(z)$.

4. **Cayley-Hamilton.** Given an $n \times n$ matrix $A$ *the characteristic polynomial* of $A$ is defined by $P_A(\lambda) = \det(\lambda I_n - A)$, where $I_n$ is the identity matrix. Then $P_A(A) = 0$ for every $A$.

5. **Eisenstein criterion.** Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients and $p$ be a prime so that

   (i) $p$ divides $a_0, a_1, \dots a_{n-1}$;

   (ii) $p$ does not divide $a_n$;

   (iii) $p^2$ does not divide $a_0$.

   Then $P(x)$ can not be expressed as a product of two non-constant polynomials with integer coefficients.

Problems.

1. **Putnam 1959. A1.** Prove that one can find a polynomial $P(y)$ with real coefficients such that $P(x - 1/x) = x^n - 1/x^n$ if and only if $n$ is odd.

2. **Putnam 2004. B1.** Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers
$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\dots,\ c_n r^n + c_{n-1} r^{n-1} + \dots + c_1 r$$
are integers.

3. **Putnam 2012. A2.** Let $*$ be a commutative and associative binary operation on a set $S$. Assume that for every $x$ and $y$ in $S$, there exists $z$ in $S$ such that $x * z = y$. (This $z$ may depend on $x$ and $y$.) Show that if $a, b, c$ are in $S$ and $a * c = b * c$, then $a = b$.

4. **Putnam 2008. A2.** Alan and Barbara play a game in which they take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

5. **Putnam 1994. A4.** Let $A$ and $B$ be $2 \times 2$ matrices with integer entries such that $A, A + B, A + 2B, A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

6. **Putnam 2006. B4.** Let $Z$ denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus $Z$ has $2^n$ elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Let $k$ be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension $k$, of the number of points in $V \cap Z$.

7. **Putnam 2009. B4.** Say that a polynomial with real coefficients in two variables, $x, y$, is *balanced* if the average value of the polynomial on each circle centered at the origin is $0$. The balanced polynomials of degree at most 2009 form a vector space $V$ over $\mathbb{R}$. Find the dimension of $V$.

8. **Putnam 2014. A6.** Let $n$ be a positive integer. What is the largest $k$ for which there exist $n \times n$ matrices $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ with real entries such that for all $i$ and $j$, the matrix product $M_i N_j$ has a zero entry somewhere on its diagonal if and only if $i \neq j$?