Problem Seminar. Fall 2016.

Problem Set 4. Number theory.

Classical results.

1. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is
$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots.$$

2. **Wilson.**
$$(p-1)! \equiv -1 (\operatorname{mod} p)$$
for any prime p.

3. **Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences
$$\begin{aligned} x &\equiv a_1 \quad (\operatorname{mod} m_1), \\ x &\equiv a_2 \quad (\operatorname{mod} m_2), \\ &\dots \\ x &\equiv a_k \quad (\operatorname{mod} m_k). \end{aligned}$$
has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

4. **Erdős-Ginzburg-Ziv theorem.** Sny collection of $2n-1$ integers contains a subcollection of n integers with the sum divisible by n.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. The number 2^{29} has 9 distinct digits. Which digit is missing?

3. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

4. **Put 1989. A1.** How many primes among the positive integers, written in the usual base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

5. **IMO 1970**. Prove that there are no positive integers n such that the set $\{n+1, n+2, \dots, n+6\}$ can be divided into two sets with the product of elements in one set equal to the product of elements in the other set.

6. **Put 1983. A3.** Let p be an odd prime, and let

$$F(n) = 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}.$$

Prove that if a and b are not congruent modulo P then $F(a)$ and $F(b)$ are not congruent modulo p.

7. **Put 2001. A5.** Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

8. **Put 1996. A6.** The sequence a_n is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

Show that, for all n, a_n is an integer multiple of n.