Problem Seminar. Fall 2016.

Problem Set 1. Proofs by contradiction.

Classical results.

1. Prove that there are inifinitely many prime numbers. (Recall that a number $p > 1$ is *prime* if and only its only divisors are 1 and p.)
2. Prove that $\sqrt{2}$ is irrational.
3. Let $\{S_1, S_2, \ldots, S_k\}$ be a family of r element subsets of some set X. Show that if intersection of any $r+1$ (not necessarily distinct) subsets in the family is non-empty then the intersection of all subsets in the family is non-empty.

Problems.

1. **Putnam 2004. A1.** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?
2. **Germany 1985.** Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.
3. **Canada 2002.** Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$xf(y) + yf(x) = (x+y)f(x^2+y^2)$$
for all positive integers x and y.
4. **Putnam 1995. A1.** Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.
5. **Putnam 2010. B1.** Is there an infinite sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$ such that
$$a_1^m + a_2^m + \ldots + a_n^m + \ldots = m.$$
for every positive integer m?
6. **Putnam 1965. B6.** Show that a (simple) graph with $2n$ vertices and n^2+1 edges necessarily contains a cycle of length three, but that we can find a (simple) graph with $2n$ vertices and n^2 edges without a cycle of length three.
7. **Putnam 1998. A3.** Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$
8. **Putnam 1964. B3.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, such that for each $\alpha > 0$, $\lim_{n\to\infty, n\in\mathbb{N}} f(n\alpha) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.