Problem Seminar. Fall 2013.

Problem Set 7. Calculus.

Classical results.

1. Every continuous mapping of a circle into a line carries some pair of diametrically opposite points to the same point.

2. **Mean value theorem.** If $f : [a, b] \to \mathbb{R}$ is a differentiable function then there exists $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3. Compute
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \right)$$

4. **Gaussian integral.**
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Problems.

1. **Putnam 1946. A1.** Let $p(x)$ be a polynomial of degree at most 2 with real coefficients, which satisfies $|p(x)| \le 1$ for all $-1 \le x \le 1$. Show that $|p'(x)| \le 4$ for all $-1 \le x \le 1$.

2. **Putnam 2002. A1.** Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

3. **Putnam 2012. B1.** Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

   (i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

   (ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

   (iii) If $f(x)$ and $g(x)$ are in S and $f(x) \ge g(x)$ for all $x \ge 0$, then the function $f(x) - g(x)$ is in S.

   Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

4. **Putnam 1995. A2.** For what pairs (a, b) of positive real numbers does the improper integral
$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$
converge?

5. **Putnam 2013. A3.** Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$
Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

6. **Putnam 2005. B3.** Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that
$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$
for all $x > 0$.

7. **Putnam 2000. A4.** Show that the improper integral
$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2)\, dx$$
converges.

8. **Putnam 2010. A6.** Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that
$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$$
diverges.