Problem Seminar. Fall 2015.

Problem Set 6. Inequalities.

Classical results.

1. **AM-GM.** For any non-negative real numbers $x_1, x_2, \ldots, x_n$,
$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$
2. **Cauchy-Schwarz.** For any real $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$,
$$(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \ldots x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2).$$
3. **Arithmetic-harmonic mean.** For any non-negative real numbers $x_1, x_2, \ldots, x_n$,
$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$
4. **Jensen.** For any convex function f and any real $x_1, x_2, \ldots, x_n$,
$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}.$$

Problems.

1. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$
2. **Putnam 2003. A2.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that
$$(a_1a_2 \cdots a_n)^{1/n} + (b_1b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$
3. **Putnam 2004. B2.** Let m and n be positive integers. Show that
$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$
4. **IMO 1994.** Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:
$$\frac{a_1 + a_2 + \ldots + a_m}{m} \geq \frac{n+1}{2}.$$
5. **Putnam 2002. B3.** Show that, for all integers $n > 1$,
$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

6. **IMO 2007. Shortlist.** Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Show that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

7. **Putnam 2013. B4.** For any continuous real-valued function f defined on the interval $[0,1]$, let

$$\mu(f) = \int_0^1 f(x)\,dx,\ \mathrm{Var}(f) = \int_0^1 (f(x)-\mu(f))^2\,dx,$$
$$M(f) = \max_{0\le x\le 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg) \le 2\mathrm{Var}(f)M(g)^2 + 2\mathrm{Var}(g)M(f)^2.$$