Problem Seminar. Fall 2015.

Problem Set 4. Number Theory.

Classical results.

1. **Polignac's formula.** If p is a prime number and n a positive integer, then the exponent of p in $n!$ is
$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots .$$

2. **The Chinese Remainder theorem.** Let $m_1, m_2, \dots, m_k$ be pairwise positive integers greater than 1, such that $\gcd(m_i, m_j) = 1$ for $i \neq j$. Then for any integers $a_1, a_2, \dots, a_k$ the system of congruences
$$\begin{aligned} x &\equiv a_1 \qquad (\operatorname{mod} m_1), \\ x &\equiv a_2 \qquad (\operatorname{mod} m_2), \\ &\dots \\ x &\equiv a_k \qquad (\operatorname{mod} m_k). \end{aligned}$$
has solutions, and any two such solutions are congruent modulo $m = m_1 m_2 \dots m_k$.

3. **Sylvester's theorem.** Let a and b be positive integers with $\gcd(a, b) = 1$. Then $ab - a - b$ is the largest positive integer c for which the equation $ax + by = c$ is not solvable in nonnegative integers.

Problems.

1. Prove that $n!$ is not divisible by 2^n for any positive integer n.

2. The number 2^{29} has 9 distinct digits. Which digit is missing?

3. Prove that for every n, there exist n consecutive integers each of which is divisible by two different primes.

4. **Putnam 1993. B1.** Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which
$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

5. **Putnam 2000. A2.** Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

6. **Putnam 1983. A3.** Let p be an odd prime, and let
$$F(n) = 1 + 2n + 3n^2 + \dots + (p-1)n^{p-2}.$$
Prove that if a and b are not congruent modulo P then $F(a)$ and $F(b)$ are not congruent modulo p.

7. **IMO 2011.** Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

8. **IMO 2014. Shortlist.** A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most 99.5, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

9. **Putnam 1996. A6.** The sequence a_n is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$$

Show that, for all n, a_n is an integer multiple of n.

10. **IMO 2012.** Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \ldots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1.$$